Take-Home Exam 1

Modern Algebra II, Dave Bayer, September 23, 2008

Name: ______________________________

[1] (6 pts)	[2] (6 pts)	[3] (6 pts)	[4] (6 pts)	[5] (6 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

[1] Let $\mathbb{Z}[x]$ be the ring of polynomials in x with coeficients in $\mathbb{Z}$. Give an example of a maximal ideal $I \subset \mathbb{Z}[x]$. Give an example of an ideal I which is prime but not maximal. Are there any ideals I such that the quotient $\mathbb{Z}[x]$ is a field not of the form $\mathbb{Z}/p\mathbb{Z}$ for a prime p?

[2] Let A be an $n \times n$ matrix with entries in $\mathbb{R}$, satisfying the polynomial relation

$$(x+1)(x+2) \;=\; 0$$

Find a formula for e^{At} as a polynomial expression in A. Give an example of a matrix A for which this is the minimal polynomial relation, and check your formula using this matrix.

[3] Let A be an $n \times n$ matrix with entries in $\mathbb{R}$, satisfying the polynomial relation

$$x^2(x+1) \;=\; 0$$

Find a formula for e^{At} as a polynomial expression in A. Give an example of a matrix A for which this is the minimal polynomial relation, and check your formula using this matrix.

[4] Construct the finite field $\mathbb{F}_9$ as an extension of $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$, by finding an irreducible polynomial of degree 2 with coefficients in $\mathbb{F}_3$. Which elements of the multiplicative group $\mathbb{F}_9^*$ are primitive (have order 8)?

[5] Let L be a partially-ordered set for which every pair of elements a, b has a least upper bound c. In other words, we have an order relation $\leqslant$ with which we can sometimes compare elements of L, and for any two elements $a, b \in L$ there exists a unique least element $c \in L$ such that $a \leqslant c$ and $b \leqslant c$. (The positive integers, ordered by divisibility, is an example of such a set.) Let $\mathbb{R}[L]$ be the vector space over $\mathbb{R}$ with basis L, and give $\mathbb{R}[L]$ a multiplicative structure by defining the product of two elements of L to be their least upper bound. Under what circumstances is $\mathbb{R}[L]$ a commutative ring with identity?