Groups and Symmetry

Dave Bayer
Notes for MATH V1010, Fall 2001
Sunday, 17 September 2001, 7:48 AM

1 Introduction

A *group* is a set of actions, together with a well-behaved rule for combining two actions into a single action. We ask that this rule has certain properties, in order to say that a given set is a group. Groups are of interest because these properties capture the essential behavior of sets of actions that can be physically carried out in our world, and because many years of study have revealed a beautiful structure to rules with these properties. The interplay between these motivations allows us to apply an appreciation of this beauty to draw conclusions about practical problems.

Before describing these properties which define a group, we give some examples of groups:

Example 1.1 (The permutation group on two balls) Imagine a stack of two balls. Let $\rightrightarrows$ be the name of the action that does nothing, leaving both balls alone. Let $\rtimes$ be the name of the action that swaps the balls with each other. Then $G = \{\rightrightarrows, \rtimes\}$ is a set of actions.

Actions change an environment. Here, the environment is a stack of two balls, which are rearranged, or *permuted*, by these actions. These actions are called *permutations*. We can represent the environment as a stack consisting of the numbers 1 and 2, initially arranged as $\begin{smallmatrix}1\\2\end{smallmatrix}$. The permutations of G then act as follows:

$$\begin{smallmatrix}1\\2\end{smallmatrix} \rightrightarrows \begin{smallmatrix}1\\2\end{smallmatrix} \qquad\qquad \begin{smallmatrix}1\\2\end{smallmatrix} \rtimes \begin{smallmatrix}2\\1\end{smallmatrix}$$

If the balls are already out of order, the permutations nevertheless act in the same way, blindly rearranging balls by position without regard to their identities. Thus, the permutations of G also act as follows:

$$\begin{smallmatrix}2\\1\end{smallmatrix} \rightrightarrows \begin{smallmatrix}2\\1\end{smallmatrix} \qquad\qquad \begin{smallmatrix}2\\1\end{smallmatrix} \rtimes \begin{smallmatrix}1\\2\end{smallmatrix}$$

Our rule for combining two actions into a single action is to do the first action, then do the second action, viewing the combined effect of these two actions as the effect of a single action.

$$\begin{smallmatrix}1\\2\end{smallmatrix} \rightrightarrows \begin{smallmatrix}1\\2\end{smallmatrix} \rightrightarrows \begin{smallmatrix}1\\2\end{smallmatrix} = \begin{smallmatrix}1\\2\end{smallmatrix} \rightrightarrows \begin{smallmatrix}1\\2\end{smallmatrix} \qquad\qquad \begin{smallmatrix}1\\2\end{smallmatrix} \rightrightarrows \begin{smallmatrix}1\\2\end{smallmatrix} \rtimes \begin{smallmatrix}2\\1\end{smallmatrix} = \begin{smallmatrix}1\\2\end{smallmatrix} \rtimes \begin{smallmatrix}2\\1\end{smallmatrix}$$

$$\begin{smallmatrix}1\\2\end{smallmatrix} \rtimes \begin{smallmatrix}2\\1\end{smallmatrix} \rightrightarrows \begin{smallmatrix}2\\1\end{smallmatrix} = \begin{smallmatrix}1\\2\end{smallmatrix} \rtimes \begin{smallmatrix}2\\1\end{smallmatrix} \qquad\qquad \begin{smallmatrix}1\\2\end{smallmatrix} \rtimes \begin{smallmatrix}2\\1\end{smallmatrix} \rtimes \begin{smallmatrix}1\\2\end{smallmatrix} = \begin{smallmatrix}1\\2\end{smallmatrix} \rightrightarrows \begin{smallmatrix}1\\2\end{smallmatrix}$$

We write combinations of actions from left to right as if we are multiplying numbers together, and we suppress the environment on which they act. Thus,

$\rightrightarrows\rightrightarrows = \rightrightarrows$ “doing nothing twice has the effect of doing nothing”

$\rightrightarrows\times = \times$ “doing nothing then swapping has the effect of swapping”

$\times\rightrightarrows = \times$ “swapping then doing nothing has the effect of swapping”

$\times\times = \rightrightarrows$ “swapping twice has the effect of doing nothing”

This last rule is familiar in many contexts: Toggling a light switch twice has the effect of leaving it alone. Turning a shirt inside-out twice has the effect of leaving it alone. An odd number plus an odd number is an even number.

In a sense, these situations are all the same, and only the language which describes them is different. To see this, write out an addition table stating the rule for adding even and odd numbers, and write out a multiplication table stating the rule for combining rearrangements of two balls:

+	even	odd
even	even	odd
odd	odd	even

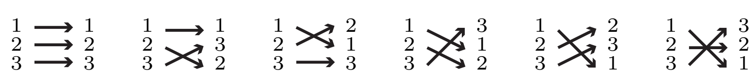
$\cdot$	$\rightrightarrows$	$\times$
$\rightrightarrows$	$\rightrightarrows$	$\times$
$\times$	$\times$	$\rightrightarrows$

If we rewrite addition as multiplication, “even” as $\rightrightarrows$, and “odd” as $\times$, we see that these two arithmetic tables are identical. We call such a rewriting an *isomorphism*, and we say that the two groups described by these tables are *isomorphic*.

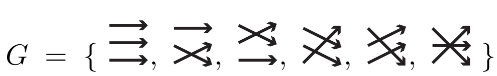
Example 1.2 (The symmetric group on three balls) Now imagine a stack of three balls. There are six ways of rearranging three balls:

$$123, \quad 132, \quad 213, \quad 312, \quad 231, \quad 321.$$

The first ball can end up in any of three positions, the second ball can end up in either of the remaining two positions, and the third ball must then end up in the remaining position. This gives six possibilities. Therefore, there are six permutations on three balls, taking an initial arrangement of the balls to any of these six arrangements:

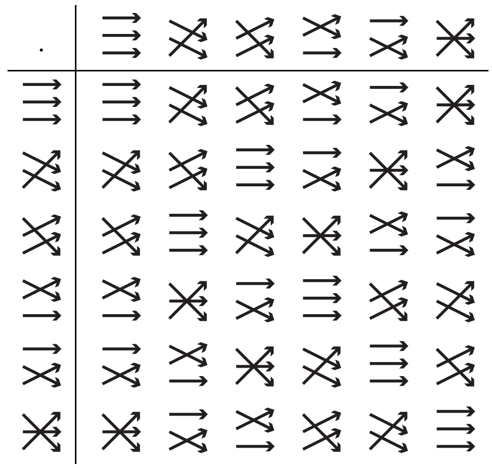


We can write

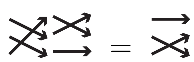
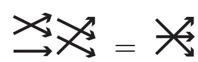


as the group of these six actions.

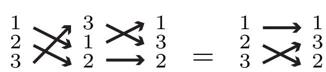
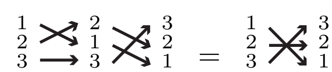
G has the following multiplication table, where the row label gives the first action to carry out, and the column label gives the second action to carry out. We have rearranged G to make its algebraic structure more apparent:



Note that the order of multiplication often matters. For example,

 is different from 

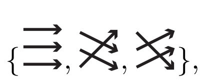
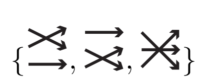
We can check this using stacks of numbers:

 is different from 

However, it is just as easy to multiply permutations by tracing the paths of the arrows, without writing out the stacks of numbers. Imagine that after hooking up the arrows head to tail, they snap tight like rubber bands.

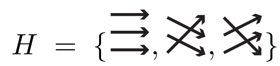
For physical actions in the real world, order also often matters: Cracking open an egg and then frying it has a different effect than frying an egg then cracking it open. However, order doesn't matter for addition or multiplication of numbers: Because $a+b = b+a$ and $ab = ba$ for any two numbers a and b, we say that addition and multiplication of numbers is *commutative*. One could view the the fact that groups can fail to be commutative as a weakness, but it is in fact a strength; groups for which the order of multiplication matters can be used as models to study real world sets of actions for which the order of events matters. In this sense, groups are more powerful algebraic structures than our familiar number systems, because they can capture more of the complexity of reality.

A great deal of structure can be seen in the multiplication rule for G. For now, we note one feature of G: Just as the integers can be separated into even and odd numbers, G can be separated into "even" and "odd" actions:

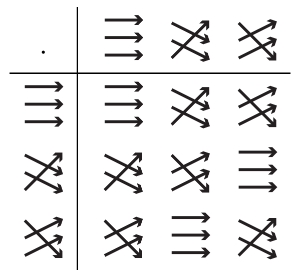
"even" = , "odd" = .

We saw a hint of this pattern in Example 1.1, where the swap behaved like an odd integer. Here, the three swaps behave like odd integers.

The "even" actions

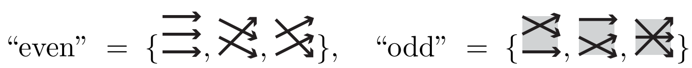


form a group by themselves, just as even integers stay to themselves when they are added. The multiplication table for H,

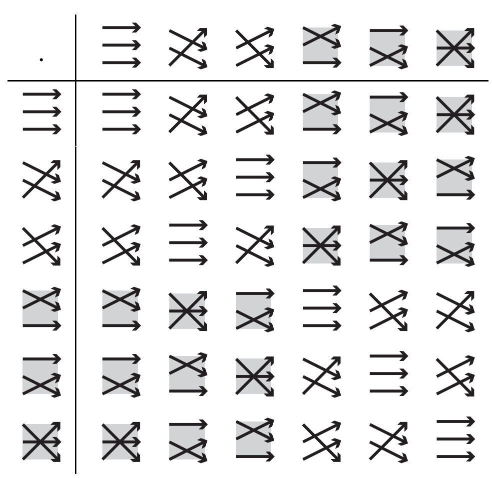


sits inside the multiplication table for G, as its upper left corner. We say that the subset H of G is a *subgroup* of G. G has other subgroups, but they are not so easily spotted, because their multiplication tables are not contiguous inside the multiplication table for G.

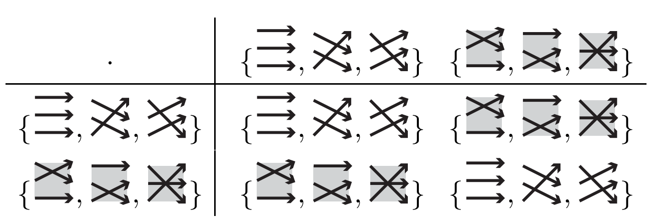
We now shade the "odd" actions gray:



This reveals that the multiplication table for G



has the overall structure



We call such an overall structure a *quotient* of G. Rewriting multiplication as addition, rewriting {…} as even, and rewriting {…} as odd, we say that G has a quotient isomorphic to the rule for adding even and odd integers.

Not every subgroup of G corresponds in this way to a quotient of G. Subgroups that do correspond to quotients are called *normal subgroups*, and play a special role in group theory.

Example 1.3 (The symmetry group of the triangle) Now, let our environment be a two-sided equalateral triangle, which looks the same on both sides. There are six actions which preserve the appearance of such a triangle:

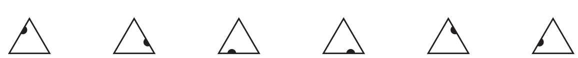
1	"do nothing"
	"rotate one-third turn clockwise"
	"rotate one-third turn counterclockwise"
	"flip across an upward-sloping diagonal axis"
	"flip across a vertical axis"
	"flip across a downward-sloping diagonal axis"

We can write

$$G = \{ 1, \; , \; , \; , \; , \; \}$$

as the group of these six actions.

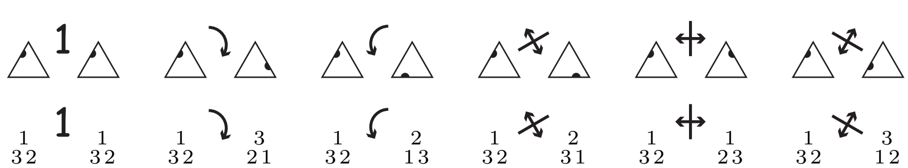
One way to track the effects of these actions is to place a mark on the triangle that moves to different positions under different actions. For example, if we mark one end of one edge, then the mark can end up on either end of any edge, or in six possible positions:



Another way to track the effects of these actions is to number the corners of the triangle 1, 2, 3. Any numbering scheme is as good as any other; we will adopt the convention of starting with the top corner, and numbering clockwise. Under different actions, these numbers can end up in any of six arrangements:

1 3 2	3 2 1	2 1 3	2 3 1	1 2 3	3 1 2

Now we can annotate each action by its effect on a marked or numbered triangle:

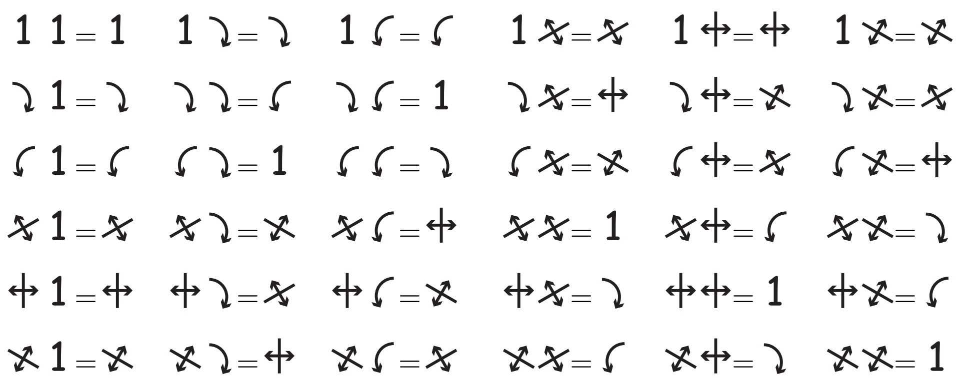


For example, the computations

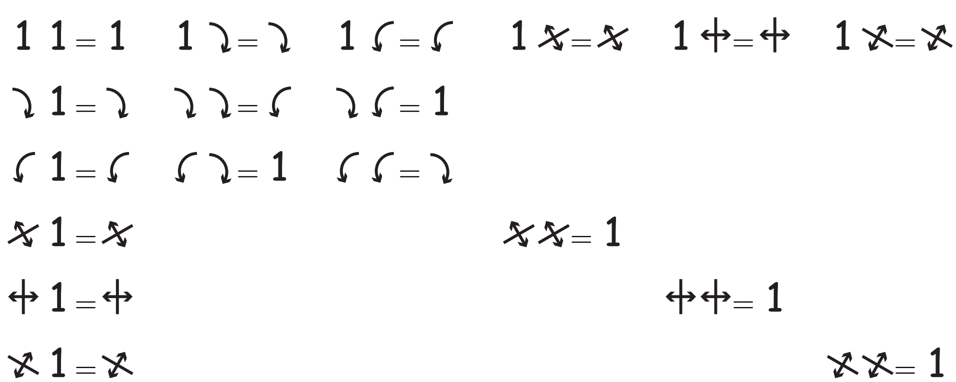
can be carried out as

or

To write out the full multiplication table for G, we need to make the following computations:



Some of these rules are familiar from physical experience. Doing nothing before or after doing any action has the same effect as doing that action. After flipping a sheet of paper to look at the other side, repeating the same flip will return the sheet of paper to its original state. Turning a knob a little bit, several times, has the same effect as turning the knob more, once. We have an innate mastery of these rules that allows us to read, keeps us from getting scalded in showers, and so forth. This explains the following rules:



The remaining computations are intrinsically no more difficult, but they are less familiar to us because they do not arise frequently in our physical experience. They can be explained by a few basic rules.

A rotation *after* a flip has the effect of twisting the axis of the flip *forward* by half the angle of the rotation. A rotation *before* a flip has the effect of twisting the axis of the flip *backward* by half the angle of the rotation.

⤡↷ = ⤢   ⤡↶ = ↔

↔↷ = ⤡   ↔↶ = ⤢

⤢↷ = ↔   ⤢↶ = ⤡

↷⤡ = ↔   ↷↔ = ⤢   ↷⤢ = ⤡

↶⤡ = ⤢   ↶↔ = ⤡   ↶⤢ = ↔

Finally, two flips have the same effect as a rotation by twice the angle between the axis of the flips:

⤡↔ = ↶   ⤡⤢ = ↷

↔⤡ = ↷   ↔⤢ = ↶

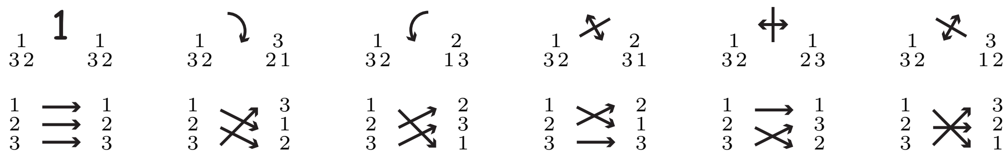
⤢⤡ = ↶   ⤢↔ = ↷

Interestingly, the rules where order of multiplication matters are the rules that feel less familiar from experience. These rules hold for triangles, squares, or for that matter for arbitrary rotations and flips of a circle. It is helpful to practice these rules on an actual disk cut out from a piece of cardboard, until they feel more familiar.

The full multiplication table for G is:

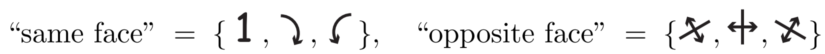
·	1	↷	↶	⤡	↔	⤢
1	1	↷	↶	⤡	↔	⤢
↷	↷	↶	1	↔	⤢	⤡
↶	↶	1	↷	⤢	⤡	↔
⤡	⤡	⤢	↔	1	↶	↷
↔	↔	⤡	⤢	↷	1	↶
⤢	⤢	↔	⤡	↶	↷	1

This table looks very familiar, and it is: The symmetry group of a triangle is isomorphic to the permutation group on three balls. This could laboriously be seen by finding a rewriting rule that translates one multiplication table into the other. However, it is more enlightening to observe that each action of G corresponds to a permutation of the corners of the triangle, as is revealed when we number the corners. This makes the isomorphism apparent:



We have simply been using two different notations for the same group.

It can nevertheless be helpful to have these different ways of looking at G. For example, the "even" and "odd" structure which we observed in Example 1.2 has an interpretation in terms of the two faces of the triangle: Thinking of doing nothing as a rotation by zero degrees, each rotation leaves the same face of the triangle showing, and each flip leaves the opposite face of the triangle showing:



Thus, the rotations form a subgroup of G, and G has a quotient which includes the rule "two flips is a rotation." This is yet another manifestation of the principle "odd plus odd is even."