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Variation du conducteur de Swan d’un faisceau étale ℓ-adique sur un disque ou une couronne rigide

Variation of the Swan conductor of an ℓ-adic étale sheaf on a rigid disc or annulus

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À la mémoire de Koto Souleymane Toubbou

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Chapitre I

Introduction Générale

Sommaire

Ce mémoire de thèse est constitué de deux parties liées, portant toutes deux sur l'étude de la variation du conducteur de Swan d'un faisceau étale de torsion sur une courbe rigide. La première partie, qui est le chapitre II du texte traite d'un faisceau lisse sur le disque unité fermé rigide. La seconde correspond au chapitre III et porte sur un faisceau lisse sur une couronne fermée rigide, permettant de traiter aussi le cas d'un faisceau ramifié en dehors d'un nombre fini de points. Les parties sont précédées d'une introduction générale.

I.1. Théorie de ramification d'Abbes et Saito.

I.1.1. Soient $\mathcal{O}_K$ un anneau de valuation discrète hensélien, K son corps des fractions, $\mathfrak{m}_K$ son ideal maximal, k son corps résiduel de caractéristique $p > 0$, et π une uniformisante de $\mathcal{O}_K$. Soient $\overline{K}$ une clôture séparable de K, $\mathcal{O}_{\overline{K}}$ la clôture intégrale de $\mathcal{O}_K$ dans $\overline{K}$ et $\overline{k}$ son corps résiduel. On note G_K le groupe de Galois de $\overline{K}$ sur K et $v : \overline{K}^{\times} \to \mathbb{Q}$ la valuation sur $\overline{K}$ normalisée par $v(\pi) = 1$. On pose $S = \mathrm{Spec}(\mathcal{O}_K)$ et on note s le point fermé de S, η son point générique et $\overline{\eta}$ le point géométrique générique $\mathrm{Spec}(\overline{K})$.

I.1.2. Lorsque k est parfait, on a la théorie de ramification classique [**Ser68**, IV-VI] due essentiellement à E. Artin, H. Hasse, C. Arf, J. Herbrand, R. Swan et J.-P. Serre. Elle fournit une filtration décroissante de G_K par des sous-groupes fermés distingués indexée par $\mathbb{Q}_{\geq 0}$, la *filtration de ramification supérieure*, qui se comporte bien par passage au quotient (théorème de Herbrand) et dont les sauts d'indices sont entiers pour les extensions abéliennes (théorème de Hasse-Arf). On dispose aussi de conducteurs, dits d'Artin et de Swan, associés à une représentation de G_K, qui mesurent les effets de la filtration de ramification sur la représentation.

I.1.3. La théorie de ramification géométrique a été initiée par Grothendieck. Le cadre est celui d'une variété X sur un corps de base parfait et, pour un nombre premier ℓ distinct de la caractéristique de ce corps, d'un faisceau ℓ-adique constructible $\mathcal{F}$ sur X, lisse sur un ouvert non vide U de X. On souhaite alors construire des invariants locaux qui mesurent la ramification de $\mathcal{F}$ le long de $X \backslash U$. Le prototype est celui de la formule de Grothendieck-Ogg-Shafarevich qui correspond au cas où X est une courbe projective, lisse, irréductible, géométriquement connexe et de genre g. Celle-ci calcule l'invariant global $\chi(X, \mathcal{F}) - (2-2g)\mathrm{rk}(\mathcal{F})$, où $\chi(X, \mathcal{F})$ est la caractéristique d'Euler-Poincaré de $\mathcal{F}$ et $\mathrm{rk}(\mathcal{F})$ désigne la dimension de la fibre de $\mathcal{F}$ en un point géométrique générique de X, en termes d'invariants locaux aux points de $X\backslash U$ que sont les conducteurs de Swan de $\mathcal{F}$ en ces points.

La généralisation en dimension supérieure de cette formule d'indice pour les courbes a motivé bien des développements ultérieurs de la théorie de ramification. Dans cette direction, développant des idées de Deligne, Laumon traita le cas d'une surface connexe, normale et projective dans sa thèse [Lau83]. Dans cette quête d'invariants et de généralisations en dimension supérieure, l'analogie entre la théorie des D-modules, telle que développée par Kashiwara, Dubson et Malgrange, entre autres, et la théorie des faisceaux ℓ-adiques de Grothendieck et compagnie, a été et demeure un principe directeur fructueux. À tout D-module holonome $\mathcal{M}$ sur une variété analytique complexe, on peut associer, d'une part, sa *variété caractéristique* $\mathrm{Char}(\mathcal{M})$, qui est un fermé conique du fibré cotangent de la variété de base, et, d'autre part, son *cycle caractéristique* $\mathrm{CC}(\mathcal{M})$, qui est une combinaison $\mathbb{Z}$-linéaire des composantes irréductibles de $\mathrm{Char}(\mathcal{M})$. Ce cycle caractéristique fournit une formule d'indice pour $\mathcal{M}$. Plus précisément, Dubson [Dub84] et Kashiwara [Kas85] ont montré que la caractéristique d'Euler-Poincaré s'obtient comme produit d'intersection de $\mathrm{CC}(\mathcal{M})$ avec la section nulle du fibré cotangent de la variété. Dans un travail relativement récent, Beilinson, guidée par l'analogie avec la théorie complexe, a construit le support singulier $\mathrm{SS}(\mathcal{F})$ d'un faisceau ℓ-adique constructible $\mathcal{F}$ [Bei16]. À sa suite, T. Saito, se basant sur ce résultat crucial, ainsi que sur des travaux antérieurs de K. Kato et ses propres travaux en collaboration avec A. Abbes (voir ci-dessous), a construit le cycle caractéristique $\mathrm{CC}(\mathcal{F})$ en dimension quelconque et établi une formule d'indice dans ce cadre. L'incarnation de ce cycle caractéristique en codimension 1 (I.1.7.5) joue un rôle important dans ce mémoire.

Au cours de ces développements, la théorie de la ramification s'est avérée un outil précieux pour le calcul de la dimension de l'espace des cycles évanescents d'une courbe relative sur un trait. Ainsi, Deligne et Laumon établirent une formule pour cette dimension en termes de conducteurs de Swan dont ils déduirent la semi-continuité des dits conducteurs. Leur formule fut généralisée par Kato (voir ci-dessous). Cette généralisation joue aussi un rôle important dans ce mémoire.

I.1.4. Reprenons les notations de I.1.1. Soient $\mathfrak{X}$ une courbe relative sur S, x un point fermé de sa fibre spéciale $\mathfrak{X}_s$ tel que $\mathfrak{X}-\{x\}$ soit lisse sur S, X la localisation stricte de $\mathfrak{X}$ en x et U un ouvert non vide de la fibre générique X_η de X d'injection canonique $j: U \hookrightarrow X_\eta$. Soient $\ell \neq p$ un nombre premier et $\mathcal{F}$ un faisceau étale de $\mathbb{F}_\ell$-modules sur U qui est lisse, i. e. localement constant et constructible. L'espace des cycles proches de $\mathcal{F}$ en x

$$\Psi_x^i(j_!\mathcal{F}) = H^i_{\text{ét}}(X_{\overline{\eta}}, j_!\mathcal{F}) \quad (i \geq 0) \tag{I.1.4.1}$$

est nul pour $i \geq 2$ [SGA 7, I, Théorème 4.2]. La généralisation par Kato de la formule de Deligne et Laumon s'écrit [Kat87a, Theorem 6.7]

$$\dim_{\mathbb{F}_\ell} \Psi_x^0(j_!\mathcal{F}) - \dim_{\mathbb{F}_\ell} \Psi_x^1(j_!\mathcal{F}) = \varphi_s(\mathcal{F}) - \varphi_\eta(\mathcal{F}) - 2\delta_x \mathrm{rk}(\mathcal{F}), \tag{I.1.4.2}$$

où $\varphi_s(\mathcal{F})$ et $\varphi_\eta(\mathcal{F})$ sont des entiers définis à partir des conducteurs de Swan de $\mathcal{F}$ respectivement au point générique de X_s et aux points de $X_{\overline{\eta}} - U_{\overline{\eta}}$, et δ_x est un entier qui mesure le défaut de normalité de X. La formule de Deligne et Laumon correspond au cas particulier où $\mathcal{F}$ est non ramifié au point générique de X_s.

Une variante de l'entier $\varphi_s(\mathcal{F})$, définie à partir du cycle caractéristique de $\mathcal{F}$ (I.2.1.3) est au coeur de ce texte.

I.1.5. En dimension supérieure, on est naturellement confronté à des anneaux de valuations discrètes henséliens à corps résiduels imparfaits. Les généralisations évoquées ci-dessus (I.1.3, (I.1.4.2)) ont été aussi rendues possibles par l'élaboration progressive d'une théorie de ramification qui accommode les corps résiduels imparfaits. C'est K. Kato qui, dans les années 80, a initié cette étude pour les caractères de rang 1 [**Kat87a**, **Kat87b**]. Dans les années 2000, A. Abbes et T. Saito ont développé, par des méthodes géométriques, une théorie de ramification dans la généralité convenable [**AS02**, **AS03**, **AS11**]. Plus précisément, ils construisent une filtration décroissante $(G^r_{K,\log})_{r\in\mathbb{Q}_{\geq 0}}$ par des sous-groupes distingués fermés de G_K, dite *filtration de ramification logarithmique*, qui coïncide avec la filtration de ramification supérieure classique lorsque k est parfait et possède, en outre, les propriétés suivantes. Pour un rationnel $r\geq 0$, posant

(I.1.5.1) $$G^{r+}_{K,\log}=\overline{\cup_{s>r}G^s_{K,\log}},$$

(I.1.5.2) $$\mathrm{Gr}^r_{\log}G_K=G^r_{K,\log}/G^{r+}_{K,\log},$$

on trouve que $G^0_{K,\log}=I_K$ est le sous-groupe d'inertie de G_K tandis que $G^{0+}_{K,\log}$ coïncide avec le sous-groupe d'inertie sauvage P_K, l'unique p-Sylow de I_K. La filtration de ramification logarithmique se comporte bien par extension modérée de K. Plus précisément, pour toute extension finie séparable K' de K, d'indice de ramification e', on a une inclusion

(I.1.5.3) $$G^{e'r}_{K',\log}\subseteq G^r_{K,\log},\quad \text{pour tout } r\in\mathbb{Q}_{>0},$$

qui est une égalité lorsque l'extension K'/K est modérée. Enfin, comme dans la théorie classique, les gradués $\mathrm{Gr}^r_{\log}G_K$ de la filtration sont abéliens et tués par p ([**Sai09**, 1.24], [**Sai12**, Theorem 2] et [**Sai20**, Theorem 4.3.1]).

I.1.6. Soient $r\geq 0$ un rationnel, $\mathfrak{m}^r_{\overline{K}}$ (resp. $\mathfrak{m}^{r+}_{\overline{K}}$) l'ensemble des éléments x de $\overline{K}$ tels que $v(x)\geq r$ (resp. $v(x)>r$). Supposons que le corps résiduel k est de type fini sur un sous-corps parfait k_0 et notons $\Omega^1_k(\log)$ le k-espace vectoriel des 1-formes différentielles logarithmiques défini par

(I.1.6.1) $$\Omega^1_k(\log)=(\Omega^1_{k/k_0}\oplus(k\otimes_{\mathbb{Z}}K^\times))/(\mathrm{d}\overline{a}-\overline{a}\otimes a, a\in\mathcal{O}_K^\times).$$

Généralisant au cas non logarithmique une construction de Kato pour les caractères de G_K de degré 1 [**Kat89**, Theorem 0.1], ainsi qu'un de ses travaux avec Abbes, [**AS09**, §9], Takeshi Saito ([**Sai09**, 1.24], [**Sai12**, Theorem 2], [**AS11**, Theorem 6.13]), établit l'existence d'un homomorphisme injectif, dit *conducteur de Swan raffiné*,

(I.1.6.2) $$\mathrm{rsw}:\mathrm{Hom}(\mathrm{Gr}^r_{\log}G_K,\mathbb{F}_p)\to\mathrm{Hom}_{\overline{k}}(\mathfrak{m}^r_{\overline{K}}/\mathfrak{m}^{r+}_{\overline{K}},\Omega^1_k(\log)\otimes_k\overline{k}),$$

qui exprime le $\mathbb{F}_p$-dual du gradué $\mathrm{Gr}^r_{\log}G_K$ en termes de formes différentielles logarithmiques.

I.1.7. Soient ℓ un nombre premier différent de p, Λ un corps fini de caractéristique ℓ et $\psi:\mathbb{F}_p\to\Lambda^\times$ un caractère non trivial fixé. Notons L/K une extension finie galoisienne, G son groupe de Galois et M un Λ-espace vectoriel de dimension finie muni d'une action linéaire de G. La filtration $(G^r_{K,\log})_{r\in\mathbb{Q}_{\geq 0}}$ induit, par [**Ka88**, 1.1], une unique décomposition de M en somme directe de sous-modules P_K-stables, dite *décomposition en pentes*,

(I.1.7.1) $$M=\bigoplus_{r\in\mathbb{Q}_{\geq 0}}M^{(r)},$$

telle que $M^{(0)}=M^{P_K}$ et, pour tout nombre rationnel $r>0$, on ait

(I.1.7.2) $$(M^{(r)})^{G^r_{K,\log}}=0\quad\text{et}\quad(M^{(r)})^{G^{r+}_{K,\log}}=M^{(r)}.$$

Les termes de la somme sont nuls sauf pour un nombre fini d'indices r, les *pentes* de M. Le *conducteur de Swan* $\mathrm{sw}_G^{\mathrm{AS}}(M)$ de M est alors défini comme

$$\mathrm{sw}_G^{\mathrm{AS}}(M) = \sum_{r \in \mathbb{Q}_{\geq 0}} r \cdot \dim_\Lambda M^{(r)}. \tag{I.1.7.3}$$

Cet invariant mesure l'action des sous-groupes de la filtration sur M ; par exemple, $\mathrm{sw}_G^{\mathrm{AS}}(M) = 0$ si et seulement si P_K agit trivialement sur M. Pour voir l'effet des gradués de la filtration sur M, il nous faut examiner les termes de la filtration (I.1.7.1). Si $r > 0$ est une pente de M, $M^{(r)}$ admet aussi une décomposition, dite *décomposition en caractères centraux*, [AS11, 6.7]

$$M^{(r)} = \bigoplus_\chi M_\chi, \tag{I.1.7.4}$$

paramétrée par un nombre fini de *caractères centraux* $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda_\chi^\times$, où Λ_χ est une extension finie séparable de Λ. Comme $\mathrm{Gr}^r_{\log} G_K$ est tué par p, l'existence de ψ fournit une factorisation $\mathrm{Gr}^r_{\log} G_K \xrightarrow{\overline{\chi}} \mathbb{F}_p \xrightarrow{\psi} \Lambda^\times$ de χ. Alors, H. Hu [Hu15] définit le *cycle caractéristique* $\mathrm{CC}_\psi(M)$ de M par

$$\mathrm{CC}_\psi(M) = \bigotimes_{r \in \mathbb{Q} > 0} \bigotimes_{\chi \in X(r)} (\mathrm{rsw}(\overline{\chi})(\pi^r))^{\otimes (\dim_\Lambda M_\chi^{(r)})} \in (\Omega^1_k(\log) \otimes_k \overline{k})^{\otimes m}, \tag{I.1.7.5}$$

où $m = \dim_\Lambda M/M^{(0)}$. Cette forme différentielle logarithmique dépend du choix de π mais pas de celui des π^r. En effet, notant b le dénominateur commun des pentes de M, les π^r sont définis au choix d'une racine b-ième de l'unité près ; un autre choix modifie le terme de droite de (I.1.7.5) par un facteur $\zeta^{b\,\mathrm{sw}_G^{\mathrm{AS}}(M)}$, où ζ est une racine b-ième de l'unité, qui disparaît parce que $\mathrm{sw}_G^{\mathrm{AS}}(M)$ est un entier ([Xia10, 4.4.3], [Xia12, 4.5.14] et [Sai20, 4.3.1]). Ceci lève toute ambiguité dans la définition de $\mathrm{CC}_\psi(M)$.

Une extension finie séparable de K est dite de type (II) si son indice de ramification vaut 1 et son corps résiduel est une extension radicielle et monogène sur k. Sous les hypothèses que p n'est pas une uniformisante de $\mathcal{O}_K$ et que L est de type (II) sur une sous-extension non ramifiée sur K, Haoyu Hu établit en [Hu15, 10.5] que le cycle caractéristique est non logarithmique, c'est-à-dire que $\mathrm{CC}_\psi(M) \in (\Omega^1_k)^{\otimes m}$, ce qui est une sorte de théorème de Hasse-Arf. Il déduit ce résultat d'un théorème de comparaison qu'il prouve qui, sous les hypothèses ci-dessus, identifie $\mathrm{CC}_\psi(M)$ à un autre cycle caractéristique dû à Kato [Kat87b, 4.4], [Hu15, (3.17.1)], et par là redonne une preuve de la formule raffinée de Deligne et Kato pour la dimension de l'espace des cycles proches d'une courbe relative (I.1.4.2).

Enfin, notons aussi que $\mathrm{CC}_\psi(M)$ est l'incarnation en codimension 1 du cycle caractéristique caractéristique $\mathrm{CC}(\mathcal{F})$ associé à un faisceau ℓ-adique $\mathcal{F}$ défini par T. Saito [Sai17].

Pour une représentation galoisienne comme en I.1.7, quelle relation y a-t-il entre ses deux invariants que sont son conducteur de Swan et son cycle caractéristique ?

Dans ce mémoire, on donne *une* réponse variationnelle à cette question pour une représentation associée à un faisceau de torsion sur un disque ou une couronne rigide.

I.2. Faisceau lisse sur un disque rigide.

I.2.1. On suppose que le corps de valuation discrète K est complet et que son corps résiduel k est algébriquement clos. Notons D le disque unité rigide fermé sur K et, pour $t \in \mathbb{Q}_{\geq 0}$, $D^{(t)}$ son sous-disque de rayon $|\pi|^t$. Notons aussi $o \to D$ un point géométrique en l'origine de D et $\pi_1^{\text{ét}}(D, o)$

le groupe fondamental algébrique de D au point base o, tels que définis par de Jong [deJ95, §2]. Soit $\mathcal{F}$ un faisceau lisse de Λ-modules sur D. Par [deJ95, 2.10], $\mathcal{F}$ correspond à la donnée d'un revêtement (fini) étale, galoisien, connexe $f : X \to D$ et d'une représentation continue de dimension finie $\rho_{\mathcal{F}}$ de $\pi_1^{\text{ét}}(D, o)$ à coefficients dans Λ se factorisant à travers le quotient $G = \text{Aut}(X/D)$ de $\pi_1^{\text{ét}}(D, o)$. Pour $t \in \mathbb{Q}_{\geq 0}$, il existe une extension finie K'/K, avec $t \in v(K')$, telle que les espaces rigides $D_{K'}^{(t)}$ et $X_{K'}^{(t)} = f^{-1}(D_{K'}^{(t)})$ soient les fibres génériques respectives, au sens de Raynaud, de schémas formels $\mathfrak{D}_{K'}^{(t)}$ et $\mathfrak{X}_{K'}^{(t)}$ sur $\text{Spf}(\mathcal{O}_{K'}) = \{s'\}$ dont les fibres spéciales respectives $\mathfrak{D}_{s'}^{(t)}$ et $\mathfrak{X}_{s'}^{(t)}$ sont géométriquement réduites (II.4.9). Visuellement, cela donne le diagramme

$$\begin{array}{ccccc} \mathfrak{X}_{K'}^{(t)} & \leftarrow\cdots & X^{(t)} & \longrightarrow & X \\ \downarrow & & \downarrow & \square & \downarrow f \\ \mathfrak{D}_{K'}^{(t)} & \leftarrow\cdots & D^{(t)} & \longrightarrow & D, \end{array}$$

où les flèches horizontales du carré cartésien sont des immersions ouvertes. Ces modèles, qui sont normaux et stables par toute extension finie de K', sont dits *modèles entiers normalisés de* $D^{(t)}$ et $X^{(t)}$ définis sur $\mathcal{O}_{K'}$. Soient $\mathfrak{p}^{(t)}$ le point générique de $\mathfrak{D}_{s'}^{(t)}$ et $\overline{\mathfrak{p}}^{(t)} \to \mathfrak{D}_{s'}^{(t)}$ un point géométrique générique. L'ensemble $\mathfrak{X}_{K'}^{(t)}(\overline{\mathfrak{p}}^{(t)})$ des points géométriques génériques de $\mathfrak{X}_{s'}^{(t)}$ au-dessus de $\overline{\mathfrak{p}}^{(t)}$ est muni d'une action transitive de G induite par l'action naturelle à droite de G sur $X_{K'}^{(t)}$. Pour tout $\overline{\mathfrak{q}}^{(t)} \in \mathfrak{X}_{K'}^{(t)}(\overline{\mathfrak{p}}^{(t)})$, on dispose d'un homomorphisme fini d'anneaux locaux (II.2.12)

$$A_{\overline{\mathfrak{p}}^{(t)}} = \mathcal{O}_{\mathfrak{D}_{K'}^{(t)}, \overline{\mathfrak{p}}^{(t)}} \to A_{\overline{\mathfrak{q}}^{(t)}} = \mathcal{O}_{\mathfrak{X}_{K'}^{(t)}, \overline{\mathfrak{q}}^{(t)}}, \tag{I.2.1.1}$$

pour les topologies étales formelles, qui est une extension d'anneaux de valuation discrètes henséliens. L'extension des corps de fractions induite est galoisienne de groupe le stabilisateur $G_{\overline{\mathfrak{q}}^{(t)}}$ de $\overline{\mathfrak{q}}^{(t)}$ sous l'action de G (II.7.4). La restriction $M_{\overline{\mathfrak{q}}^{(t)}} = \rho_{\mathcal{F}}|G_{\overline{\mathfrak{q}}^{(t)}}$ est justifiable de I.1.7 ; son conducteur de Swan $\text{sw}^{\text{AS}}_{G_{\overline{\mathfrak{q}}^{(t)}}}(M_{\overline{\mathfrak{q}}^{(t)}})$ et son cycle caractéristique $\text{CC}_{\psi}(M_{\overline{\mathfrak{q}}^{(t)}})$ sont indépendants à la fois du choix de K', extension suffisamment grande de K sur laquelle sont définis les modèles entiers normalisés de $D^{(t)}$ et $X^{(t)} = f^{-1}(D^{(t)})$, et de $\overline{\mathfrak{q}}^{(t)} \in \mathfrak{X}_{K'}^{(t)}(\overline{\mathfrak{p}}^{(t)})$. On dispose ainsi d'une fonction

$$\text{sw}_{\text{AS}}(\mathcal{F}, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}, \quad t \mapsto \text{sw}^{\text{AS}}_{G_{\overline{\mathfrak{q}}^{(t)}}}(M_{\overline{\mathfrak{q}}^{(t)}}), \tag{I.2.1.2}$$

qui est bien définie et quantifie la ramification de $\mathcal{F}$ le long de la fibre spéciale du modèle entier normalisé de $D^{(t)}$.
Par ailleurs, le corps résiduel $\kappa(\overline{\mathfrak{p}}^{(t)})$ de $A_{\overline{\mathfrak{p}}^{(t)}}$ coïncide avec $\mathcal{O}_{\mathfrak{D}_{s'}^{(t)}, \overline{\mathfrak{p}}^{(t)}}$. Soit $\text{ord}_{\overline{\mathfrak{p}}^{(t)}} : \kappa(\overline{\mathfrak{p}}^{(t)})^{\times} \to \mathbb{Z}$ la valuation normalisée définie par l'origine de $\mathfrak{D}_{s'}^{(t)}$; notons de même son unique extension multiplicative à $(\Omega^1_{\kappa(\overline{\mathfrak{p}}^{(t)})})^{\otimes(\dim_{\Lambda}(M_{\overline{\mathfrak{q}}^{(t)}}/M^{(0)}_{\overline{\mathfrak{q}}^{(t)}}))}$ définie par la relation $\text{ord}_{\overline{\mathfrak{p}}^{(t)}}(b\text{d}a) = \text{ord}_{\overline{\mathfrak{p}}^{(t)}}(b)$, pour tout $a, b \in \kappa(\overline{\mathfrak{p}}^{(t)})^{\times}$ tels que $\text{ord}_{\overline{\mathfrak{p}}^{(t)}}(a) = 1$. On peut alors définir la fonction

$$\varphi_s(\mathcal{F}, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}, \quad t \mapsto -\text{ord}_{\overline{\mathfrak{p}}^{(t)}}(\text{CC}_{\psi}(M_{\overline{\mathfrak{q}}^{(t)}})) - \dim_{\Lambda}(M_{\overline{\mathfrak{q}}^{(t)}}/M^{(0)}_{\overline{\mathfrak{q}}^{(t)}}). \tag{I.2.1.3}$$

qui, elle aussi, est indépendante de $\overline{\mathfrak{q}}^{(t)} \in \mathfrak{X}_{K'}^{(t)}(\overline{\mathfrak{p}}^{(t)})$. Contrairement à $\text{CC}_{\psi}(M_{\overline{\mathfrak{p}}^{(t)}})$, la fonction $\varphi_s(\mathcal{F}, \cdot)$ ne dépend pas non plus du choix d'une uniformisante fixée π' de $\mathcal{O}_{K'}$ ni du caractère ψ. Le résultat principal de cette partie, qui fournit le lien cherché entre nos deux invariants, est le suivant.

Théorème I.2.2 (Théorème II.1.9)**.** *La fonction* $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F},\cdot)$ *est continue, affine par morceaux et possède un nombre fini de pentes, toutes entières. Sa dérivée à droite est la fonction* $\varphi_s(\mathcal{F},\cdot)$*. En particulier, cette dernière est localement constante.*

Remarquons, vue la normalisation $v(K)=\mathbb{Z}$, que le groupe des valeurs de $A_{\overline{\mathfrak{p}}^{(t)}}$ est canoniquement isomorphe au sous-groupe $\frac{1}{e(K'/K)}\mathbb{Z}$ de $\mathbb{Q}$, $e(K'/K)$ désignant l'indice de ramification de K'/K. Ainsi, $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F},t)$ n'est, en général, pas entier.

Notons aussi que, au regard de la formule de Deligne et Kato pour la dimension de l'espace des cycles proches I.1.4.2, notamment de sa reformulation par H. Hu dans le langage d'Abbes et Saito [**Hu15**, 11.9], l'entier $\varphi_s(\mathcal{F},t)$ se comprend, à un terme correctif près, comme la dimension de l'espace total des cycles proches $\Psi_o(\mathcal{F}|D^{(t)})$ en l'origine de la fibre spéciale $\mathfrak{D}^{(t)}_{s'}$. Ainsi, le théorème I.2.2 exprime cette dimension comme la dérivée à droite de $\mathrm{sw}(\mathcal{F},\cdot)$ en t.

Signalons également que, dans le cadre de l'étude des équations différentielles à singularités irrégulières sur les courbes p-adiques, F. Baldassarri [**Bal10**] et, à sa suite, A. Pulita dans [**Pul15**] puis avec J. Poineau [**PP15**] ainsi que K. Kedlaya [**Ked15**], ont établi des résultats de variation analogues à I.2.2 pour le rayon de convergence des équations susmentionnées.

Cependant, dans notre contexte, c'est l'étude par W. Lütkebohmert du discriminant associé à un revêtement du disque D, cruciale pour sa démonstration du théorème d'existence de Riemann p-adique [**Lüt93**], et notamment le résultat de variation de ce discriminant en fonction du rayon, qui vont nous servir.

I.2.3. Soient X un K-espace rigide lisse et $f : X \to D$ un morphisme fini, plat qui est étale au-dessus d'un ouvert admissible de D contenant l'origine o. Lütkebohmert associe à f une fonction discriminant $\partial_f : \mathbb{Q}_{\geq 0} \to \mathbb{R}_{\geq 0}$ de la variable t, qui mesure la ramification le long de la spéciale du morphisme entre modèles entiers normalisés induit par la restriction $X^{(t)} \to D^{(t)}$ de f. Il montre qu'elle est continue et affine par morceaux, et en calcule explicitement les pentes. D'après le théorème de préparation de Weierstrass, si une fonction g sur une couronne $A(\rho,\rho')=\{x\in\overline{K} \mid \rho\geq v(x)\geq\rho'\}$ de coordonnée ξ est inversible, alors elle se met sous la forme

$$g(\xi)=c\xi^d(1+h(\xi)),\quad \text{avec}\quad h(\xi)=\sum_{i\in\mathbb{Z}-\{0\}}h_i\xi^i, \tag{I.2.3.1}$$

où $c\in K^\times$, h est une fonction sur $A(\rho,\rho')$ satisfaisant $|h|_{\sup}<1$ et d, appelé *ordre de* g, est un entier.

Lorsque $X=A(r/d,r'/d)$ pour un entier d et des rationnels $r\geq r'\geq 0$, et $f : A(r/d,r'/d)\to A(r,r')\subset D$ est fini étale d'ordre d, un calcul direct donne une formule explicite pour ∂_f II.4.18 dont se déduit l'expression suivante pour sa dérivée à droite en $t\in[r',r[\cap\mathbb{Q}$

$$\frac{d}{dt}\partial_f(t+)=\sigma-d+1, \tag{I.2.3.2}$$

σ désignant l'ordre de la dérivée de f.

Cet énoncé se prolonge aux couronnes ouvertes, et l'énoncé général de Lütkebohmert se ramène à ce cas particulier. En effet, le théorème de réduction semi-stable fournit une partition finie $0=r_{n+1}<r_n<\cdots<r_1<r_0=\infty$ telle que l'image inverse par f de chaque couronne ouverte $A^\circ(r_{i-1},r_i)=\{x\in\overline{K}\mid r>v(x)>r'\}$ se décompose en réunion disjointe de $\delta_f(i)$ couronnes ouvertes sur lesquelles f est étale, de la forme (I.2.3.1). La dérivée à droite de ∂_f en $t\in[r_i,r_{i-1}[$ est alors

$$\frac{d}{dt}\partial_f(t+)=\sigma_i-d+\delta_f(i), \tag{I.2.3.3}$$

où σ_i est l'ordre total de la dérivée de la restriction de f au-dessus de $A^\circ(r_{i-1}, r_i)$.

I.2.4. Pour passer du résultat de variation du discriminant ∂_f ci-dessus au théorème I.2.2, il nous faut une interprétation de la dérivée (I.2.3.3) de nature cohomologique, pour ainsi dire. Celle-ci nous viendra de la théorie de ramification de Kato pour les $\mathbb{Z}^2$-anneaux de valuation [**Kat87a**, §1-3], c'est-à-dire les anneaux de valuation dont le groupe des valeurs est isomorphe à $\mathbb{Z}^2$ muni de l'ordre lexicographique. Si V est un anneau de valuation hensélien de ce type, de corps des fractions $\mathbb{K}$, et $\mathbb{L}/\mathbb{K}$ une extension finie galoisienne de groupe $\mathbb{G}$ telle que la clôture intégrale W de V dans $\mathbb{L}$ soit monogène, Kato définit une filtration de $\mathbb{G}$ indexée par le groupe des valeurs Γ_V de la valuation de V, généralisant la filtration de ramification classique [**Ser68**, IV] à une situation où l'extension résiduelle de W/V en les idéaux premiers de hauteur 1 n'est plus supposée séparable. Après une normalisation qui identifie Γ_V à $\mathbb{Z}^2$, la filtration induit, par projections au premier et second facteur de $\mathbb{Z}^2$, des fonctions centrales d'Artin $a^{\alpha}_{\mathbb{G}} : \mathbb{G} \to \mathbb{Z}$ et de Swan $\mathrm{sw}^{\beta}_{\mathbb{G}} : \mathbb{G} \to \mathbb{Z}$. Notant $\mathfrak{p}$ (resp. $\mathfrak{q}$) l'idéal premier de hauteur 1 de V (resp. W), $a^{\alpha}_{\mathbb{G}}$ correspond à l'extension d'anneaux de valuations discrètes $V_{\mathfrak{p}} \to W_{\mathfrak{q}}$.

I.2.5. Dans le contexte de I.2.3, à la différence de Kato qui utilise sa théorie de ramification dans un cadre algébrique [**Kat87a**, §5, 6], on l'applique quant à nous à des anneaux locaux issus de la géométrie formelle. Plus précisément, avec les notations de I.2.1, si $o^{(t)}$ est le point géométrique en l'origine de $\mathfrak{D}^{(t)}_{s'}$ et $\overline{x}^{(t)}_j \to \mathfrak{X}^{(t)}_{s'}$ est un point géométrique au-dessus de $o^{(t)}$, on pose $A^{(t)} = \mathcal{O}_{\mathfrak{D}^{(t)}_{K'}, o^{(t)}}$ et $B^{(t)}_j = \mathcal{O}_{\mathfrak{X}^{(t)}_{K'}, \overline{x}^{(t)}_j}$ (*cf.* II.2.5.1). Ces anneaux locaux pour la topologie étale formelle sont normaux, henséliens et de dimension 2. Si $\mathfrak{p}^{(t)}$ et $\mathfrak{q}^{(t)}$ sont des idéaux premiers de hauteur 1 de $A^{(t)}$ et $B^{(t)}_j$ respectivement, au-dessus de $\mathfrak{m}_{K'}$, ils fournissent des $\mathbb{Z}^2$-anneaux de valuation V_t et $W_{j,t}$ tels que $A^{(t)} \subset V_t \subset A^{(t)}_{\mathfrak{p}^{(t)}}$ et $B^{(t)}_j \subset W_{j,t} \subset (B^{(t)}_j)_{\mathfrak{q}^{(t)}}$, et le morphisme $\widehat{f}^{(t)} : \mathfrak{X}^{(t)}_{K'} \to \mathfrak{D}^{(t)}_{K'}$ induit une extension $V_t \to W_{j,t}$. Après hensélisation de cette extension, induction à $G = \mathrm{Aut}(X/D)$ des fonctions centrales obtenues en I.2.4 et renormalisation, on arrive à des fonctions centrales $\widetilde{a}^{\alpha}_f(t)$ et $\widetilde{\mathrm{sw}}^{\beta}_f(t)$ sur G, à valeurs dans $\mathbb{Q}$ et $\mathbb{Z}$ respectivement.

Par ailleurs, le quotient $B^{(t)}_{j,s'} = B^{(t)}_j / \mathfrak{m}_{K'} B^{(t)}_j$ est réduit ; notons $\widetilde{B^{(t)}_{j,s'}}$ sa clôture intégrale dans son anneau total des fractions et posons $\delta^{(t)}_j = \dim_k(\widetilde{B^{(t)}_{j,s'}}/B^{(t)}_{j,s'})$. Les anneaux $A^{(t)}_{K'} = A^{(t)} \otimes_{\mathcal{O}_{K'}} K'$ et $B^{(t)}_{j,K'} = B^{(t)}_j \otimes_{\mathcal{O}_{K'}} K'$ sont de Dedekind. Notons $T^{(t)}_j$ le K'-homomorphisme déterminant induit par la forme bilinéaire trace $B^{(t)}_{j,K'} \times B^{(t)}_{j,K'} \to A^{(t)}_{K'}$ et $d^{(t)}_j$ la dimension sur K' du conoyau de $T^{(t)}_j$.

L'interprétation cohomologique recherchée de la dérivée du discriminant ∂_f est la *formule des cycles proches* suivante qui, à bien des égards, est le résultat clé de cette première partie.

Proposition I.2.6 (Proposition II.4.28)**.** *Avec les notations de* I.2.3 *et* I.2.5*, supposons que le faisceau canonique de X soit trivial. Alors, pour tout $i = 1, \ldots, n$ et tout $t \in]r_i, r_{i-1}[\cap \mathbb{Q}$, on a l'égalité*

(I.2.6.1)
$$\sum_j \left(d^{(t)}_j - 2\delta^{(t)}_j + |P^{(t)}_j| \right) = \sigma_i + \delta_f(i),$$

où, j parcourt les indices des points géométriques de $\mathfrak{X}^{(t)}_{s'}$ au-dessus de $o^{(t)}$, $P^{(t)}_j$ est l'ensemble des idéaux premiers de hauteur 1 de $B^{(t)}_j$ au-dessus de $\mathfrak{p}^{(t)}$, $|P^{(t)}_j|$ son cardinal et $\delta_f(i)$ est le nombre de composantes connexes de l'image inverse par f de la couronne ouverte $A^\circ(r_{i-1}, r_i)$.

La trivialité du diviseur canonique de X est une condition technique qui fait marcher la preuve en permettant, au rayon t, de calculer le degré de la forme différentielle $\mathrm{d}f$ (II.2.21). Il serait intéressant de savoir si on peut se passer de cette condition dans I.2.6 (et I.2.7 ci-dessous).

Pour établir la proposition I.2.6, on commence par construire une compactification explicite $\mathfrak{Y}^{(t)}_{K'}$ du modèle entier normalisé $\mathfrak{X}^{(t)}_{K'}$. Ensuite, on l'algébrise en une courbe relative propre $Y^{(t)}_{K'} \to \mathrm{Spec}(\mathcal{O}_{K'})$ par le théorème d'algébrisation de Grothendieck, puis on approxime le morphisme $\widehat{f}^{(t)} : \mathfrak{X}^{(t)}_{K'} \to \mathfrak{D}^{(t)}_{K'}$ par (la fibre générique du complété d') un morphisme algébrique $g^{(t)} : Y^{(t)}_{K'} \to \mathbb{P}^1_{\mathcal{O}_{K'}}$ grâce à une version rigide du théorème de Runge due à Michel Raynaud. Enfin, on déduit l'identité (I.2.6.1) d'un calcul du degré du diviseur défini par la forme différentielle $\mathrm{d}g^{(t)}$ via Riemann-Roch et une formule de Kato qui exprime $2\delta_f(i) - |P^{(t)}_j| + 1$ comme la dimension de cycles proches relativement au morphisme $Y^{(t)}_{K'} \to \mathrm{Spec}(\mathcal{O}_{K'})$ (d'où l'appellation "formule des cycles proches" accolée à (I.2.6.1) plus haut).

Dès lors, le lien avec la fonction discriminant ∂_f, très légèrement simplifié dans cette introduction, est le suivant.

Théorème I.2.7 (Théorème II.7.12)**.** *On note $\langle \cdot, \cdot \rangle$ l'accouplement usuel entre fonctions centrales sur G et r_G le caractère de la représentation régulière de G.*

(i) *Pour tout $t \in \mathbb{Q}_{\geq 0}$, on a l'identité*

$$\langle \widetilde{a}^{\alpha}_f(t), r_G \rangle = \partial_f(t). \tag{I.2.7.1}$$

(ii) *On suppose que le faisceau canonique de X est trivial. Alors, pour tout $i = 1, \ldots, n$ et tout $t \in [r_i, r_{i-1}[$, on a*

$$\langle \widetilde{\mathrm{sw}}^{\beta}_f(t), r_G \rangle = \frac{d}{dt}\partial_f(t+) = \sigma_i - d + \delta_f(i). \tag{I.2.7.2}$$

L'égalité (I.2.7.1) n'est pas surprenante ; elle est une incarnation géométrique de l'égalité classique entre le discriminant d'une extension finie galoisienne de corps de valuations discrètes et le caractère d'Artin du groupe de Galois de cette extension évalué en le neutre [**Ser68**, IV, §2, Prop. 4]. L'égalité (I.2.7.2) est plus subtile et délicate à établir ; c'est là qu'intervient la formule des cycles proches I.2.6.1.

On déduit de cette proposition et du résultat de variation de ∂_f de Lütkebohmert (I.2.3) que la fonction

$$\widetilde{a}^{\alpha}_f(r_G, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}, t \mapsto \langle \widetilde{a}^{\alpha}_f(t), r_G \rangle \tag{I.2.7.3}$$

est continue et affine par morceaux avec un nombre fini de pentes, toutes entières. De plus sa fonction dérivée à droite est

$$\widetilde{\mathrm{sw}}^{\beta}_f(r_G, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}, t \mapsto \langle \widetilde{\mathrm{sw}}^{\beta}_f(t), r_G \rangle. \tag{I.2.7.4}$$

Signalons que L. Ramero [**Ram05**, 3.3.10] a le premier démontré un résultat analogue à I.2.7 en utilisant une filtration de ramification due à R. Huber, suivi en cela par S. Wewers [**Wew05**], sans toutefois que la relation entre sa fonction discriminant et celle de Lütkebohmert ait été établie explicitement (à notre connaissance). Bien que notre preuve de I.2.7 soit différente de celle de Ramero en substance, son travail nous a bien-sûr suggéré la forme de l'énoncé ainsi que le passage suivant.

Par induction de Brauer et un raisonnement par récurrence, l'énoncé de variation donné ci-dessus se généralise du caractère r_G à tout élément χ du groupe de Grothendieck $R_{\overline{\mathbb{Q}}_\ell}(G)$. Plus précisément, pour un tel χ, la fonction

$$\widetilde{a}^{\alpha}_{f}(\chi,\cdot):\mathbb{Q}_{\geq 0}\to\mathbb{Q}, t\mapsto\langle\widetilde{a}^{\alpha}_{f}(t),\chi\rangle \tag{I.2.7.5}$$

est continue et affine par morceaux avec un nombre fini de pentes, toutes entières. De plus sa fonction dérivée à droite est

$$\widetilde{\mathrm{sw}}^{\beta}_{f}(\chi,\cdot):\mathbb{Q}_{\geq 0}\to\mathbb{Q}, t\mapsto\langle\widetilde{\mathrm{sw}}^{\beta}_{f}(t),\chi\rangle. \tag{I.2.7.6}$$

Rappelons que Λ est un corps fini de caractéristique $\ell\neq p$ et notons $\Lambda_{\mathbb{Q}_\ell}$ le corps des fractions de son anneau des vecteurs de Witt $W(\Lambda)$. Alors, par l'homomorphisme de Cartan $d_G : R_{\Lambda_{\mathbb{Q}}}(G)\to R_{\Lambda}(G)$ [Ser98, §15.2, 16.1], cet énoncé de variation se prolonge comme tel aux éléments de $R_{\Lambda}(G)$.

I.2.8. Le dernier pont qui mène à la preuve I.2.2 est le résultat de H. Hu, auquel on a déjà fait allusion à la fin de I.1.7, identifiant, avec les notations de ce numéro, la forme différentielle $\mathrm{CC}_{\psi}(M_{\overline{\mathfrak{q}}^{(t)}})$ à un cycle caractéristique construit à partir d'un conducteur de Swan à valeurs différentielles dû à Kato [Hu15, 3.17]. L'énoncé qui s'en déduit et qui nous sert à conclure est le suivant (valable pour tout M comme en I.1.7).

Proposition I.2.9 (Proposition II.8.25)**.** *Avec les notations de* I.2.1 *et* I.2.4*, pour tout $t\in\mathbb{Q}_{\geq 0}$, on a*

$$\widetilde{a}^{\alpha}_{f}(\chi_{\mathcal{F}},t)=\mathrm{sw}_{\mathrm{AS}}(\mathcal{F},t), \tag{I.2.9.1}$$

$$\widetilde{\mathrm{sw}}^{\beta}_{f}(\chi,t)=\varphi_s(\mathcal{F},t). \tag{I.2.9.2}$$

I.3. Faisceau lisse sur une couronne rigide

I.3.1. On garde les notations de I.2.1. On veut étendre le Théorème I.2.2 à un faisceau étale $\mathcal{F}$ sur une couronne dans le disque D qui est ramifié en au plus un nombre fini de points rigides de la couronne. Excluant les rayons des points de ramification de $\mathcal{F}$, on est ramené à généraliser le Théorème I.2.2 pour un faisceau lisse sur une couronne rigide.

I.3.2. Soient $0\leq r\leq r'$ deux nombres rationnels et $A(r',r)$ la couronne fermée dans D de rayons $|\pi|^{r'}\leq|\pi|^{r}$. Un faisceau étale de Λ-modules $\mathcal{F}$ sur $A(r,r')$ est dit *méromorphe* s'il est lisse sur l'ouvert complémentaire d'un nombre fini de points rigides de $A(r,r')$. Soient $\mathcal{F}$ un tel faisceau et C son ouvert de lissité dans $A(r,r')$. Alors, la restriction $\mathcal{F}|C$ correspond à la donnée d'un revêtement étale connexe galoisien $f : X\to C$ et d'une représentation continue de dimension finie $\rho_{\mathcal{F}}$ de $G=\mathrm{Aut}(X/C)$, à coefficients dans Λ. À un rationnel $r\leq t\leq r'$, les mêmes constructions qu'en I.2.1, avec cette fois la couronne $C^{[t]}$ de rayon $|\pi|^{t}$ et d'épaisseur nulle et son image inverse $X^{[t]}$ par f à la place de $D^{(t)}$ et $X^{(t)}$ respectivement, associent un conducteur de Swan $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F},t)\in\mathbb{Q}$ et l'ordre $\varphi_s(\mathcal{F},t)\in\mathbb{Q}$ du cycle caractéristique de $\mathcal{F}$ en t.

Théorème I.3.3 (Théorème III.1.2)**.** *Soient $\mathcal{F}$ un faisceau étale méromorphe de Λ-modules sur $A(r,r')$ et $\{r_1>\ldots>r_n\}$ l'ensemble ordonné des valuations de ses points de ramifications sur $A(r,r')$. Alors, la fonction*

$$\mathrm{sw}_{\mathrm{AS}}(\mathcal{F},\cdot):[r,r']\cap\mathbb{Q}_{\geq 0}\to\mathbb{Q}, \tag{I.3.3.1}$$

est continue, affine par morceaux sur $\mathbb{Q}-\{r_1,\ldots,r_n\}$ et a un nombre fini de pentes, toutes entières. De plus, si $r_i<t<t'<r_{i-1}$ sont deux rationnels, la différence des dérivées à droite et à gauche de $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F},\cdot)$ en t et t' respectivement est

$$\frac{d}{dt}\mathrm{sw}_{\mathrm{AS}}(\mathcal{F},t+)-\frac{d}{dt}\mathrm{sw}_{\mathrm{AS}}(\mathcal{F},t'-)=\varphi_s(\mathcal{F},t)-\varphi_s(\mathcal{F},t'). \tag{I.3.3.2}$$

Ce résultat ne découle pas du Théorème I.2.2, sauf si $\mathcal{F}$ est la restriction d'un faisceau lisse sur D tout entier. Il est moins précis que I.2.2, ne donnant que le changement de pentes entre deux rayons distincts comme différence des ordres des cycles caractéristiques de $\mathcal{F}$ en t et t'. Il se peut que $\varphi_s(\mathcal{F},\cdot)$ soit en effet la dérivée à droite de $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F},\cdot)$, mais des techniques autres que celles que nous employons ici seraient probablement requises pour établir cela.

Vu l'énoncé qu'on veut établir, on voit bien qu'en excluant les rayons des points de ramifications de $\mathcal{F}$, on se ramène à traiter le cas d'un faisceau lisse sur une couronne rigide, et donc à considérer les morphismes étales vers une couronne.

I.3.4. Dans ce qui suit $C = A(r,r')$ est une couronne fermée rigide. La stratégie de démonstration du Théorème I.3.3 est la même que celle employée pour établir le Théorème I.2.2. La complication supplémentaire vient de ce qu'il faut tenir compte des deux rayons de la couronne C.

Soient X un K-espace affinoïde lisse et $f : X \to C$ un morphisme fini, plat et génériquement étale. Si f est étale au-dessus des couronnes d'épaisseur nulle $A(r,r)$ et $A(r',r')$ et les images inverses $f^{-1}(A(r,r))$ et $f^{-1}(A(r',r'))$ se décomposent en

$$f^{-1}(A(r,r)) = \coprod_{i=1}^{\delta_f} A(r/d_i, r/d_i) \quad \text{et} \quad f^{-1}(A(r',r')) = \coprod_{j=1}^{\delta'_f} A(r'/d'_j, r'/d'_j), \tag{I.3.4.1}$$

conditions satisfaites en dehors d'un nombre fini de rayons, par la réduction semi-stable (III.4.8), alors la formule des cycles proches pour f (I.2.6.1) s'écrit (III.4.7)

$$\sum_j (d_j - 2\delta_j + |P_j|) = \sigma + \delta_f - (\sigma' + \delta'_f), \tag{I.3.4.2}$$

où d_j, δ_j et P_j sont définis comme en I.2.5 et I.2.6, relativement au modèle entier normalisé $\mathfrak{X}_{K'} \to \mathscr{C}_{K'}$ de $f : X \to C$ sur une extension finie K'/K et à l'origine o de la fibre spéciale $\mathscr{C}_{s'}$ de $\mathscr{C}_{K'}$, la somme portant sur les points géométriques de $\mathfrak{X}_{K'}$ au-dessus de o ; l'entier σ (resp. σ') est l'ordre total de la restriction de f au dessus de $A(r,r)$ (resp. $A(r',r')$).

L'identité (I.3.4.2) se prouve de la même façon que (I.2.6.1) à la seule différence qu'il faut compactifier les modèles formels des images inverses par f des deux bords $A(r,r)$ et $A(r',r')$; ceci, en tenant compte des orientations, induit la différence dans le terme de droite de (I.3.4.2). Celle-ci se propage aux énoncés de variation du conducteur de Swan et est à l'origine de (I.3.3.2).

Le terme de $2\delta_j - |P_j|$ de (I.3.4.2) s'interprète comme la dimension de la fibre au point géométrique d'indice j (au-dessus de o) du faisceau des cycles proches $R^1\Psi(\Lambda)$, relatif à (une algébrisation de) la compactification de $\mathfrak{X}_{K'} \to \mathscr{C}_{K'}$. Si f est étale, les d_j sont nuls ; alors, la non-annulation des fibres susmentionnées de $R^1\Psi(\Lambda)$ permet de déduire de (I.3.4.2) le résultat de convexité suivant.

Proposition I.3.5. *On suppose que $f : X \to C$ est étale. Alors, la fonction discriminant ∂_f est convexe.*

I.3.6. La théorie de ramification de Kato pour les $\mathbb{Z}^2$-anneaux de valuation joue le même rôle de pont que dans la partie II.

Soit $\mathbb{G}$ un groupe fini muni d'une action à droite sur X telle que f soit $\mathbb{G}$-équivariant. Pour un rationnel $r \leq t \leq r'$, la théorie de Kato fournit une fonction centrale d'Artin $\widetilde{a}_f(t)$ sur $\mathbb{G}$ à valeurs dans $\mathbb{Q}$. Cependant, pour la raison expliquée ci-dessus, le bon analogue à introduire pour la fonction centrale $\widetilde{\mathrm{sw}}^{\beta}_f(t)$ est $\widetilde{\mathrm{sw}}^{\beta}_f([t,t'])$, pour un intervalle $[t,t']$ dans $[r,r']$ tel que $t \neq t'$, qui correspond aux deux branches $A(t,t)$ et $A(t',t')$. Notant $r_{\mathbb{G}}$ le caractère de la représentation régulière de $\mathbb{G}$, on a les identités suivantes (III.5.10).

$$\partial_f(t) = \langle \widetilde{a}_f(t), r_{\mathbb{G}} \rangle, \tag{I.3.6.1}$$

$$\frac{d}{dt}\partial_f(t+) - \frac{d}{dt}\partial_f(t'-) = \langle \widetilde{\mathrm{sw}}_f^{\beta}([t,t']), r_{\mathbb{G}} \rangle. \tag{I.3.6.2}$$

La preuve de (I.3.6.2) réquiert, outre la formule des cycles proches (I.3.4.2), une formule de Hurwitz (III.3.4), due à Kato, pour les homomorphismes locaux comme $\mathcal{O}_{\mathscr{C}_{K'},o} \to \mathcal{O}_{\mathfrak{X}_{K'},\overline{x}_j}$ en I.2.5, où $\overline{x}$ est un point géométrique de la fibre spéciale $\mathfrak{X}_{s'}$ de $\mathfrak{X}_{K'}$ au dessus de l'origne o de $\mathscr{C}_{s'}$. Là où on pouvait se contenter d'un cas particulier de cette formule de Kato-Hurwitz pour établir I.2.7, ne travaillant qu'avec des sous-disques de D, la formule générale s'impose du fait de la non régularité de la fibre spéciale du modèle entier normalisé d'une couronne d'épaisseur non-nulle.

On déduit de (I.3.6.1) et (I.3.6.2) le résultat de variation suivant.

Théorème I.3.7 (Théorème III.5.16)**.** *Soit $\overline{\chi}$ un élément du groupe de Grothendieck $R_\Lambda(G)$. Alors, la fonction*

$$\widetilde{a}_f(\overline{\chi}, \cdot) : \ [r, r'] \cap \mathbb{Q}_{\geq 0} \to \mathbb{Q}, \quad t \mapsto \langle \widetilde{a}_f(t), \overline{\chi} \rangle \tag{I.3.7.1}$$

est continue, affine par morceaux et a un nombre fini de pentes, toutes entières. De plus, si $r < t < t' < r'$ sont deux rationnels, alors

$$\frac{d}{dt}\widetilde{a}_f(\overline{\chi}, t+) - \frac{d}{dt}\widetilde{a}_f(\overline{\chi}, t'-) = \langle \widetilde{\mathrm{sw}}_f^{\beta}([t,t']), \overline{\chi} \rangle. \tag{I.3.7.2}$$

Le Théorème I.3.3 se déduit de I.3.7 grâce à (un analogue de) I.2.9 (voir III.6.3).

La convexité de ∂_f I.3.5 équivaut à la convexité de la fonction $\widetilde{a}_f(\overline{r_G}, \cdot)$ ((I.3.6.1), (I.3.6.2)) et, avec III.6.3, laisse penser que la fonction $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$ aussi est convexe.

Chapitre II

Variation of the Swan conductor of an $\mathbb{F}_\ell$-sheaf on a rigid disc

Sommaire

II.1. Introduction.

II.1.1. Let $\mathcal{O}_K$ be a henselian discrete valuation ring, K its field of fractions, $\mathfrak{m}_K$ its maximal ideal, k its residue field of characteristic $p > 0$, and π a uniformizer of $\mathcal{O}_K$. Let also $\overline{K}$ be a separable closure of K, $\mathcal{O}_{\overline{K}}$ the integral closure of $\mathcal{O}_K$ in $\overline{K}$, $\overline{k}$ its residue field, G_K the Galois group of $\overline{K}$ over K, and $v : \overline{K}^\times \to \mathbb{Q}$ the valuation of $\overline{K}$ normalized by $v(\pi) = 1$.

II.1.2. When k is perfect, the classical ramification theory [**Ser68**, IV-VI] gives a filtration of G_K by closed normal subgroups indexed by $\mathbb{Q}_{\geq 0}$ and studies the action of theses subgroups on representations of G_K, producing numerical measures such as the Artin and Swan conductors.

II.1.3. The geometric study of ramification theory was initiated by Grothendieck. The setting is that of a variety X over a perfect base field and, for ℓ a prime different from the characteristic of the base field, a constructible ℓ-adic sheaf $\mathcal{F}$ on X which is lisse on a non-empty open subset U. One wishes first to construct local invariants, such as Swan conductors, to measure the ramification of $\mathcal{F}$ at the points of $X\backslash U$ and, second, one wants to use these invariants to produce an index formula computing the Euler-Poincaré characteristic $\chi(X, \mathcal{F})$ of $\mathcal{F}$.

When X is a smooth, projective and geometrically connected curve of genus g and $\mathrm{rk}(\mathcal{F})$ denotes the dimension of the stalk of $\mathcal{F}$ at a geometric generic point of X, the wish formulated above is achieved by the Grothendieck-Ogg-Shafarevich formula. The latter computes a global invariant, $\chi(X, \mathcal{F}) - (2 - 2g)\mathrm{rk}(\mathcal{F})$ in terms of finite local data at the points of $X\backslash U$, the Swan conductors of $\mathcal{F}$ at these points.

II.1.4. The generalization of this index formula to higher dimensions was a driving force in much subsequent works in the field of ramification theory. In this direction, following suggestions by Deligne, Laumon treated in his thesis [**Lau83**] the case of connected, normal and projective surfaces over an algebraically closed fied. Deligne and Laumon also proved a formula for the dimension of the space of vanishing cycles of a relative curve in terms of Swan conductors [**Lau81**, Theorem 5.1.1], which was later refined by Kato [**Kat87a**, Theorem 6.7], and deduced from it the lower semi-continuity of Swan conductors [**Lau81**, Theorem 2.1.1]. In the quest for higher dimensional invariants and a higher dimensional index formula, the analogy between the theory of D-modules and the theory of ℓ-adic sheaves was a guiding principle and remains so today. To any holonomic D-module $\mathcal{M}$ on a complex analytic variety, one can associate a *characteristic variety* $\mathrm{Char}(\mathcal{M})$, which is a closed conical subset of the cotangent bundle of the variety, and a *characteristic cycle* $\mathrm{CC}(\mathcal{M})$, which is a linear combination of the connected components of $\mathrm{Char}(\mathcal{M})$. Then, the Euler-Poincaré characteristic of $\mathcal{M}$ was shown, by Dubson [**Dub84**] and Kashiwara [**Kas85**] to be the intersection of $\mathrm{CC}(\mathcal{M})$ with the zero-section of the cotangent bundle of the variety. Recently, exploiting the aforementioned analogy, Beilinson constructed the singular support $\mathrm{SS}(\mathcal{F})$ of a constructible ℓ-adic sheaf $\mathcal{F}$ [**Bei16**]. Building on this crucial result, on earlier work of K. Kato and on his joint work with A. Abbes (see below), T. Saito [**Sai17**] constructed the characteristic cycle $\mathrm{CC}(\mathcal{F})$ in arbitrary dimension and established the generalization of the index formula.

II.1.5. In the higher dimensional setting, one has to deal with henselian discrete valuation rings with imperfect residue fields. Progress in the study of a ramification theory of K that allows its residue field k to be imperfect contributed to the aforementioned generalizations. In the eighties, K. Kato initiated such a study for rank one characters [**Kat87b, Kat87a**]. In the 2000's, A. Abbes and T. Saito, through geometric methods, produced a compelling ramification theory that accommodates an imperfect residue field [**AS02, AS03, AS11**]. More precisely, they defined a decreasing filtration $(G^r_{K,\log})_{r\in\mathbb{Q}_{\geq 0}}$ of G_K by closed normal subgroups, the logarithmic ramification filtration, which coincides with the classical ramification filtration when k is perfect and is such that, for $r\in\mathbb{Q}_{\geq 0}$, if we put

(II.1.5.1) $$G^{r+}_{K,\log}=\overline{\cup_{s>r}G^s_{K,\log}},$$

(II.1.5.2) $$\mathrm{Gr}^r_{\log}G_K=G^r_{K,\log}/G^{r+}_{K,\log},$$

then $G^0_{K,\log}=I_K$ is the inertia subgroup of G_K and $G^{0+}_{K,\log}$ coincides with the wild inertia subgroup P_K, the unique p-Sylow subgroup of I_K. Moreover, this filtration behaves well under a tame extension of K[**AS02**, 3.15]. As in the classical setting, the graded pieces $\mathrm{Gr}^r_{\log}G_K$ are abelian and killed by p ([**Sai09**, 1.24], [**Sai12**, Theorem 2] and [**Sai20**, Theorem 4.3.1]).

II.1.6. For $r\in\mathbb{Q}$, we let $\mathfrak{m}^r_{\overline{K}}$ (resp. $\mathfrak{m}^{r+}_{\overline{K}}$) be the set of elements x of $\overline{K}$ satisfying $v(x)\geq r$ (resp. $v(x)>r$). Assuming that k is of finite type over a perfect sub-field k_0, we let $\Omega^1_k(\log)$ be the k-vector space of logarithmic differential 1-forms

(II.1.6.1) $$\Omega^1_k(\log)=(\Omega^1_{k/k_0}\oplus(k\otimes_{\mathbb{Z}}K^\times))/(\mathrm{d}\overline{a}-\overline{a}\otimes a, a\in\mathcal{O}_K^\times).$$

Generalizing a construction of Kato for characters of G_K of degree one [**Kat89**, Theorem 0.1] and previous work with Abbes [**AS09**, §9], T. Saito ([**Sai09**, 1.24], [**Sai12**, Theorem 2], [**AS11**, Theorem 6.13]), shows that there is an injective homomorphism, the *refined Swan conductor*

(II.1.6.2) $$\mathrm{rsw}:\mathrm{Hom}(\mathrm{Gr}^r_{\log}G_K,\mathbb{F}_p)\to\mathrm{Hom}_{\overline{k}}(\mathfrak{m}^r_{\overline{K}}/\mathfrak{m}^{r+}_{\overline{K}},\Omega^1_k(\log)\otimes_k\overline{k}).$$

II.1.7. Let Λ be a finite field of characteristic $\ell \neq p$ and fix a nontrivial character $\psi : \mathbb{F}_p \to \Lambda^\times$. Let $L \subset \overline{K}$ be a finite Galois extension of K of group G. Let M be a finite dimensional Λ-vector space with a linear action of G. Then, the filtration $(G^r_{K,\log})_{r\in\mathbb{Q}_{\geq 0}}$ induces a canonical *slope decomposition* of M into P_K-stable sub-modules (II.8.13)

$$M = \bigoplus_{r\in\mathbb{Q}_{\geq 0}} M^{(r)}, \tag{II.1.7.1}$$

where $M^{(0)} = M^{P_K}$. The Swan conductor of M is defined as

$$\mathrm{sw}^{\mathrm{AS}}_G(M) = \sum_{r\in\mathbb{Q}_{\geq 0}} r \cdot \dim M^{(r)}. \tag{II.1.7.2}$$

It is readily seen that $\mathrm{sw}^{\mathrm{AS}}_G(M) = 0$ if and only if P_K acts trivially on M. Each non-vanishing piece $M^{(r)}$, for $r > 0$, has in turn a *central character decomposition* (II.8.16)

$$M^{(r)} = \bigoplus_{\chi} M^{(r)}_\chi, \tag{II.1.7.3}$$

indexed by a finite number of characters $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda^\times_\chi$, where Λ_χ is a finite separable extension of Λ. As $\mathrm{Gr}^r_{\log} G_K$ is killed by p, the existence of ψ ensures that χ factors as $\mathrm{Gr}^r_{\log} G_K \xrightarrow{\overline{\chi}} \mathbb{F}_p \xrightarrow{\psi} \Lambda^\times$. Then, H. Hu [**Hu15**] defines the Abbes-Saito *characteristic cycle* $\mathrm{CC}_\psi(M)$ of M as

$$\mathrm{CC}_\psi(M) = \bigotimes_{r\in\mathbb{Q}>0} \bigotimes_{\chi\in X(r)} (\mathrm{rsw}(\overline{\chi})(\pi^r))^{\otimes(\dim_\Lambda M^{(r)}_\chi)} \in (\Omega^1_k(\log) \otimes_k \overline{k})^{\otimes m}, \tag{II.1.7.4}$$

where $m = \dim_\Lambda M/M^{(0)}$ and $X(r)$ is the set of characters $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda^\times_\chi$ in (II.1.7.3). Note, in relation to π^r appearing in (II.1.7.4), that $\mathrm{CC}_\psi(M)$ is unambiguously defined (see below (II.8.21.5)). A finite separable extension of K is said to be of *type* (II) if its ramification index over K is one and its residue field is a purely inseparable and monogenic extension of k. Under the assumption that p is not a uniformizer of K and that L is of type (II) over a sub-extension which is unramified over K, it is shown in [**Hu15**, 10.5] that $\mathrm{CC}_\psi(M) \in (\Omega^1_k)^{\otimes m}$. The characteristic cycle $\mathrm{CC}_\psi(M)$ is the codimension one incarnation of $\mathrm{CC}(\mathcal{F})$ for a constructible ℓ-adic étale sheaf $\mathcal{F}$ (II.1.4).

II.1.8. In this paper, we establish a new relationship between $\mathrm{sw}^{\mathrm{AS}}_G(M)$ and $\mathrm{CC}_\psi(M)$ in the following setting. Assume that K is complete and k is algebraically closed. Let D be the rigid unit disc over K, $D^{(t)}$ its subdisc of radius $|\pi|^t$, for $t \in \mathbb{Q}_{\geq 0}$, and $\mathcal{F}$ a lisse sheaf of Λ-modules on D. Let $\overline{0} \to D$ be a geometric point above the origin 0 of D and $\pi_1^{\text{ét}}(D, \overline{0})$ the algebraic fundamental group of D with base point $\overline{0}$ [**deJ95**, §2]. By [**deJ95**, 2.10], $\mathcal{F}$ corresponds to the data of a finite Galois étale connected cover $f : X \to D$ and a finite dimensional continuous Λ-representation $\rho_{\mathcal{F}}$ of $\pi_1^{\text{ét}}(D, \overline{0})$ factoring through the finite quotient $G = \mathrm{Aut}(X/D)$ of $\pi_1^{\text{ét}}(D, \overline{0})$. There exists a finite extension K'/K such that $t \in v(K')$ (II.1.1), $D^{(t)}_{K'}$ and $X^{(t)}_{K'} = f^{-1}(D^{(t)}_{K'})$ have integral models $\mathfrak{D}^{(t)}_{K'}$ and $\mathfrak{X}^{(t)}_{K'}$ over $\mathrm{Spf}(\mathcal{O}_{K'}) = \{s'\}$ with geometrically reduced special fibers $\mathfrak{D}^{(t)}_{s'}$ and $\mathfrak{X}^{(t)}_{s'}$ respectively (II.4.9). Let $\overline{\mathfrak{p}}^{(t)}$ be a geometric point of $\mathfrak{D}^{(t)}_{s'}$ above its generic point $\mathfrak{p}^{(t)}$ and $\overline{\mathfrak{q}}^{(t)}$ a codimension one geometric point of $\mathfrak{X}^{(t)}_{s'}$ above $\overline{\mathfrak{p}}^{(t)}$. Then, these geometric points define an extension of henselian discrete valuation rings $\mathcal{O}_{\mathfrak{D}^{(t)}_{K'},\overline{\mathfrak{p}}^{(t)}} \subset \mathcal{O}_{\mathfrak{X}^{(t)}_{K'},\overline{\mathfrak{q}}^{(t)}}$, where $\mathcal{O}_{\mathfrak{D}^{(t)}_{K'},\overline{\mathfrak{p}}^{(t)}}$ (resp. $\mathcal{O}_{\mathfrak{X}^{(t)}_{K'},\overline{\mathfrak{q}}^{(t)}}$) is the formal étale local ring of $\mathfrak{D}^{(t)}_{K'}$ (resp. $\mathfrak{X}^{(t)}_{K'}$) at $\overline{\mathfrak{p}}^{(t)}$ (resp. $\overline{\mathfrak{q}}^{(t)}$) (II.2.5.1). The induced extension of fields of fractions is Galois of group the stabilizer $G_{\overline{\mathfrak{q}}^{(t)}}$ of $\overline{\mathfrak{q}}^{(t)}$ under the natural action of G on the set of codimension one geometric points of $\mathfrak{X}^{(t)}_{s'}$ above $\overline{\mathfrak{p}}^{(t)}$ (II.7.3). We complete this extension and get a representation $M_{\overline{\mathfrak{q}}^{(t)}} = \rho_{\mathcal{F}}|G_{\overline{\mathfrak{q}}^{(t)}}$ of its Galois group for which we can compute the Swan conductor $\mathrm{sw}^{\mathrm{AS}}_{G_{\overline{\mathfrak{q}}^{(t)}}}(M_{\overline{\mathfrak{q}}^{(t)}})$ (II.1.7.2) and the characteristic cycle $\mathrm{CC}_\psi(M_{\overline{\mathfrak{q}}^{(t)}})$ (II.1.7.4), both independent of the choice of both K' large enough and $\overline{\mathfrak{q}}^{(t)}$ in the set of codimension one geometric points of $\mathfrak{X}^{(t)}_{s'}$ above $\overline{\mathfrak{p}}^{(t)}$ (see II.9.2).

We note also that the residue field $\kappa(\overline{\mathfrak{p}}^{(t)})$ of $\mathcal{O}_{\mathfrak{D}^{(t)}_{K'},\overline{\mathfrak{p}}^{(t)}}$ coincide with $\mathcal{O}_{\mathfrak{D}^{(t)}_{s'},\overline{\mathfrak{p}}^{(t)}}$. Let $\mathrm{ord}_{\overline{\mathfrak{p}}^{(t)}} : \kappa(\overline{\mathfrak{p}}^{(t)})^\times \to \mathbb{Z}$ be the normalized discrete valuation map defined by the origin of $\mathfrak{D}^{(t)}_{s'}$; we denote still by $\mathrm{ord}_{\overline{\mathfrak{p}}^{(t)}}$ its unique multiplicative extension to $(\Omega^1_{\kappa(\overline{\mathfrak{p}}^{(t)})})^{\otimes(\dim_\Lambda(M_{\overline{\mathfrak{q}}^{(t)}}/M^{(0)}_{\overline{\mathfrak{q}}^{(t)}}))}$ (where $M^{(0)}_{\overline{\mathfrak{q}}^{(t)}}$ is the tame part of $M_{\overline{\mathfrak{q}}^{(t)}}$) defined by the relation $\mathrm{ord}_{\overline{\mathfrak{p}}^{(t)}}(b\mathrm{d}a) = \mathrm{ord}_{\overline{\mathfrak{p}}^{(t)}}(b)$, for $a, b \in \kappa(\overline{\mathfrak{p}}^{(t)})^\times$ such that $\mathrm{ord}_{\overline{\mathfrak{p}}^{(t)}}(a) = 1$.

The main result of this article is the following.

Theorem II.1.9. *The function*

$$\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}, \quad t \mapsto \mathrm{sw}^{\mathrm{AS}}_{G_{\overline{\mathfrak{q}}^{(t)}}}(M_{\overline{\mathfrak{q}}^{(t)}}) \tag{II.1.9.1}$$

is continuous and piecewise linear, with finitely many slopes which are all integers. Its right derivative is the locally constant function

$$\varphi_s(\mathcal{F}, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}, \quad t \mapsto -\mathrm{ord}_{\mathfrak{p}^{(t)}}(\mathrm{CC}_\psi(M_{\overline{\mathfrak{q}}^{(t)}})) - \dim_\Lambda(M_{\overline{\mathfrak{q}}^{(t)}}/M^{(0)}_{\overline{\mathfrak{q}}^{(t)}}). \tag{II.1.9.2}$$

We remark that, as $v(K) = \mathbb{Z}$, the induced value group of $\mathcal{O}_{\mathfrak{D}^{(t)}_{K'},\overline{\mathfrak{p}}^{(t)}}$ is canonically isomorphic to the subgroup $\frac{1}{e_{K'/K}}\mathbb{Z}$ of $\mathbb{Q}$, where $e_{K'/K}$ is the ramification index of K'/K. Thus, with this normalization, $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, t)$ is not an integer in general.

We note also that, following [**Hu15**, §11], by the theorem of Deligne and Kato on the dimension of the space of nearby cycles ([**Kat87a**, 6.7], [**Hu15**, 11.9]), $\varphi_s(\mathcal{F}, t)$ is equal to the total dimension of $\Psi_0(\mathcal{F}_{|D^{(t)}})$, up to a correction term (notation of *loc. cit*). Hence, one can vaguely interpret the second half of Theorem II.1.9 as saying that the derivative of the function $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$ at t is the dimension of the nearby cycles of $\mathcal{F}_{|D^{(t)}}$ at the origin of $D^{(t)}$.

Theorem II.1.9 is proved in section II.9 as Theorem II.9.4. We expect $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$ to be a decreasing and convex function, and we also expect the theorem to hold even when $\mathcal{F}$ has horizontal ramification. We will come back to these questions in a forthcoming work.

II.1.10. For a cover f as in II.1.8, a variational result for a discriminant associated to f, analogous to Theorem II.1.9, was established by Lütkebohmert and played a key role in his proof of the p-adic Riemann existence theorem [**Lüt93**].

II.1.11. Let X be a smooth K-rigid space and $f : X \to D$ a finite flat morphism which is étale over an admissible open subset of D containing 0. Lütkebohmert associates to f a discriminant function $\partial^\alpha_f : \mathbb{Q}_{\geq 0} \to \mathbb{R}^+$ of the variable t and shows that it is continuous and piecewise linear while also making its slopes explicit (II.4.23). By the Weierstrass preparation theorem, if a function g on an annulus $A(\rho, \rho') = \{x \in \overline{K} \mid \rho \geq v(x) \geq \rho'\}$ ($\rho, \rho' \in \mathbb{Q}$) is invertible, it can be written in the form

$$g(\xi) = c\xi^d(1 + h(\xi)), \quad \text{with} \quad h(\xi) = \sum_{i \in \mathbb{Z} - \{0\}} h_i \xi^i, \tag{II.1.11.1}$$

where $c \in K^\times$, $d \in \mathbb{Z}$ and h is a function on $A(\rho, \rho')$ such that $|h|_{\sup} < 1$; then, the integer d is called the *order of* g. When $X = A(r/d, r'/d)$, for rational numbers $r \geq r' \geq 0$, and $f : A(r/d, r'/d) \to$

$A(r,r') \subset D$ is finite étale of order d, Lütkebohmert computes ∂_f^α explicitly and finds that its right derivative at $t \in [r', r[\cap \mathbb{Q}$ is

$$\frac{d}{dt}\partial_f^\alpha(t+) = \sigma - d + 1, \tag{II.1.11.2}$$

where σ is the order of the derivative of f (II.4.18). Lütkebohmert's general statement reduces to this case thanks to the semi-stable reduction theorem which gives a finite sequence of rational numbers $0 = r_{n+1} < r_n < \cdots < r_1 < r_0 = \infty$ such that, over each open annulus $A^\circ(r_{i-1}, r_i) = \{x \in \overline{K} \mid r > v(x) > r'\}$, f is a sum of étale morphisms on annuli (hence of the form (II.1.11.1)).

II.1.12. Theorem II.1.9 will ultimately be deduced from the aforementioned variational result of Lütkebohmert II.4.23. The bridge between the two is provided by Kato's ramification theory for a $\mathbb{Z}^2$-valuation ring. The latter is a valuation ring whose value group is isomorphic to $\mathbb{Z}^2$ endowed with the lexicographic order. The theory partly rests on an important theorem of Epp on elimination of wild ramification [**Epp73**]. Let $\overline{x}^{(t)}$ be a geometric point at the origin of $\mathfrak{D}_{s'}^{(t)}$. We put $A^{(t)} = \mathcal{O}_{\mathfrak{D}_{K'}^{(t)}, \overline{x}^{(t)}}$ (II.2.5.1); and with the generic point $\mathfrak{p}^{(t)}$, we cook up a henselian $\mathbb{Z}^2$-valuation ring V_t^h, which is a henselization of a $\mathbb{Z}^2$-valuation ring V_t such that $A^{(t)} \subset V_t \subset A_{\mathfrak{p}^{(t)}}^{(t)}$ (II.3.18)). The restriction of f to $X^{(t)} = f^{-1}(D^{(t)})$ has a "normalized" integral model $\widehat{f}^{(t)} : \mathfrak{X}_{K'}^{(t)} \to \mathfrak{D}_{K'}^{(t)}$ (II.2.19, II.4.10) (possibly after enlarging K') which induces monogenic extensions of $\mathbb{Z}^2$-valuations rings $V_t^h \to W_{j,t}^h$, where $W_{j,t}^h$ is the henselization of a $\mathbb{Z}^2$-valuation ring $W_{j,t} \supset A_j^{(t)} = \mathcal{O}_{\mathfrak{X}_{K'}^{(t)}, \overline{x}_j}$ and the $\overline{x}_j$ are the geometric points of the special fiber $\mathfrak{X}_{s'}^{(t)}$ of $\mathfrak{X}_{K'}^{(t)}$, above $\overline{x}^{(t)}$. Applied to these extensions, Kato's theory yields characters $\widetilde{a}_f^\alpha(t)$ and $\widetilde{\mathrm{sw}}_f^\beta(t)$ of $G = \mathrm{Aut}(X/D)$ with values in $\mathbb{Q}$ and $\mathbb{Z}$ respectively.

Theorem II.1.13 (Corollary II.7.20)**.** *We assume that X has trivial canonical sheaf and let $\chi \in R_\Lambda(G)$. We denote by $\langle \cdot, \cdot \rangle_G$ the usual pairing of class functions on G. Then, the map*

$$\widetilde{a}_f^\alpha(\chi, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}, \quad t \mapsto \langle \widetilde{a}_f^\alpha(t), \chi \rangle_G \tag{II.1.13.1}$$

is continuous and piecewise linear, with finitely many slopes which are all integers. Its right derivative at $t \in \mathbb{Q}_{\geq 0}$ is

$$\frac{d}{dt}\widetilde{a}_f^\alpha(\chi, t+) = \langle \widetilde{\mathrm{sw}}_f^\beta(t), \chi \rangle_G. \tag{II.1.13.2}$$

In [**Ram05**, §3 and §4], L. Ramero proved a similar result for $f : X \to D$ an *étale* morphism between adic spaces in the sense of Huber and with a somewhat ad hoc ramification filtration due to R. Huber. Although our proofs are independent of his, we took inspiration from his work to arrive at this statement. We should also mention that F. Baldassarri [**Bal10**], and later A. Pulita [**Pul15**] along with J. Poineau [**PP15**], as well as K. Kedlaya [**Ked15**], proved analogous continuity and piecewise linearity results for the radii of convergence of differential equations with irregular singularities on p-adic analytic curves.

II.1.14. By Brauer induction, we reduce Theorem II.1.13 to the case of the character $\chi = r_G$ of the regular representation of G. Then, the link to Lütkebohmert's discriminant is through the following identities. For $t \in \mathbb{Q}_{\geq 0}$, we have

$$\langle \widetilde{a}_f^\alpha(t), r_G \rangle_G = \partial_f^\alpha(t). \tag{II.1.14.1}$$

Whence, the function $t \mapsto \langle \widetilde{a}_f^\alpha(t), r_G\rangle_G$ is piecewise linear. Let $0 = r_{n+1} < r_n < \cdots < r_1 < r_0 = \infty$ be the partition of $\mathbb{Q}_{\geq 0}$ given by the semi-stable reduction theorem as in II.1.11. Then, assuming also that X *has trivial canonical sheaf*, for $t \in [r_i, r_{i-1}[$, we have

(II.1.14.2) $$\langle \widetilde{\mathrm{sw}}_f^\beta(t), r_G\rangle_G = \frac{d}{dt}\partial_f^\alpha(t+) = \sigma_i - d + \delta_f(i),$$

where σ_i is the total order of the derivative of the restriction of f over $A^\circ(r_{i-1}, r_r)$ (II.4.3, II.4.5) and $\delta_f(i)$ the number of connected components of the inverse image of $A^\circ(r_{i-1}, r_i)$ by f.

While the identity (II.1.14.1) is an incarnation of the classical equality of the valuation of the different with the value of the Artin character at 1 [**Ser68**, IV, §2, Prop. 4], the identity (II.1.14.2) is new and more subtle. Let us explain how it is established. With the notation of II.1.12, the quotient ring $A_{j,0}^{(t)} = A_j^{(t)}/\mathfrak{m}_{K'}A_j^{(t)}$ is reduced. Let $\widetilde{A_{j,0}^{(t)}}$ be its normalization in its total ring of fractions and put $\delta_j^{(t)} = \dim_k(\widetilde{A_{j,0}^{(t)}}/A_{j,0}^{(t)})$. Set $A_{K'}^{(t)} = A^{(t)} \otimes_{\mathcal{O}_{K'}} K'$ and $A_{j,K'}^{(t)} = A_j^{(t)} \otimes_{\mathcal{O}_{K'}} K'$. Let $T_j^{(t)}$ be the determinant K'-linear homomorphism induced by the bilinear trace map $A_{j,K'}^{(t)} \times A_{j,K'}^{(t)} \to A_{K'}^{(t)}$ and set $d_j^{(t)}$ to be the K'-dimension of the cokernel of $T_j^{(t)}$. Then, (II.1.14.2) is deduced from the following key result, which is interpreted (and proved) as a nearby cycle formula.

Proposition II.1.15 (Proposition II.4.28)**.** *Assume that X has trivial canonical sheaf. Then, for each $i = 1, \ldots, n$ and each $t \in]r_i, r_{i-1}[\cap \mathbb{Q}$, we have the following equality*

(II.1.15.1) $$\sum_j \left(d_j^{(t)} - 2\delta_j^{(t)} + |P_j^{(t)}|\right) = \sigma_i + \delta_f(i),$$

where $P_j^{(t)}$ is the set of height 1 prime ideals of $A_j^{(t)}$ above $\mathfrak{m}_K$ and $|\cdot|$ denotes cardinality.

Proposition II.1.15 (with slightly modified notation) is proved at the end of section II.4 in the following way. We first construct a compactification $\mathfrak{Y}_{K'}^{(t)}$ of the integral model $\mathfrak{X}_{K'}^{(t)}$, which is then algebraized as $Y_{K'}^{(t)}$, via Grothendiek's algebraization theorem. We then approximate $\widehat{f}^{(t)}$ by an algebraic function $g^{(t)}$ on $Y_{K'}^{(t)}$ thanks to a rigid Runge theorem due to Raynaud. The identity (II.1.15.1) follows from the computation of the degree of the divisor defined by the differential $dg^{(t)}$ using the Riemann-Hurwitz formula and an expression for $2\delta_j^{(t)} - |P_j^{(t)}|+1$ in the form of a nearby cycles formula for $Y_{K'}^{(t)} \to \mathrm{Spec}(\mathcal{O}_{K'})$ due to Kato.

II.1.16. The last bridge from this result to Theorem II.1.9 is provided by work of Hu [**Hu15**, Theorem 10.5] which identifies $\mathrm{CC}_\psi(M)$ to another characteristic cycle $\mathrm{KCC}_{\psi(1)}(\chi_M)$ constructed from Kato's Swan conductor with differential values [**Kat87b**]. The latter is not hard to compare with $\langle \widetilde{\mathrm{sw}}_f^\beta(t), \chi_M\rangle_G$. Such a comparison is carried out in Proposition II.8.25 and allows us to conclude.

II.1.17. The text is organized as follows. In section II.2, we gather some constructions about formal schemes, mainly the *formal étale local ring* of a formal scheme at a geometric point and its relative behaviour. It also contains a recollection of normalized integral models and a *maximum principle*-like result of Bosch and Lütkebohmert computing the degree of the divisor defined by the differential of a function in terms of the orders of its zeros. Section II.3 is devoted to $\mathbb{Z}^2$-valuation rings and how they arise in algebraic and formal geometric settings. In section II.4, we recall Lütkebohmert's work and relate it to Kato's by proving the nearby cycle formula of Proposition (II.4.28). We also recall (and expand a little) the ramification theory of $\mathbb{Z}^2$-valuation (sections

II.5 and II.6) developed in [Kat87a]. The first variational result for the conductors given by this theory are stated and proved in section II.7. Kato's Swan conductor with differential values, the ramification theory of Abbes and Saito and the link between the two established by H. Hu are gathered in section II.8, as well as the comparison of the conductors coming from the two ramification theories. Then, we have everything at hand to deduce Theorem II.1.9 in section II.9.

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II.2. Preliminaries on formal schemes.

II.2.1. Let $\mathcal{O}_K$ be a complete discrete valuation ring with field of fractions K, maximal ideal $\mathfrak{m}_K$ and residue field k. We fix a uniformizer π of $\mathcal{O}_K$ and assume, except in II.2.15-II.2.19, that k is algebraically closed. We put $\mathcal{S} = \mathrm{Spf}(\mathcal{O}_K)$ and denote by s its unique point. In the whole section, all formal schemes are assumed to be locally noetherian.

II.2.2. Let $\mathfrak{X}$ be an adic formal scheme over $\mathcal{S}$ [Abb10, §2.1] and $x \in \mathfrak{X}$ a point. Recall that the local ring $\mathcal{O}_{\mathfrak{X},x}$ of $\mathfrak{X}$ at x is defined as

$$\mathcal{O}_{\mathfrak{X},x} = \varinjlim_{x \in \mathcal{U}} \mathcal{O}_{\mathfrak{X}}(\mathcal{U}), \tag{II.2.2.1}$$

where $\mathcal{U}$ runs over affine formal open subschemes of $\mathfrak{X}$ containing x. It is indeed a local ring with residue field isomorphic to the residue field of the local ring $\mathcal{O}_{\mathfrak{X}_s,x}$ of the special fiber $\mathfrak{X}_s$ of $\mathfrak{X}$ [EGA I, 10.1.6]. The formal scheme $\mathfrak{X}$ is said to be *normal* at x if $\mathcal{O}_{\mathfrak{X},x}$ is normal. We say that $\mathfrak{X}$ is normal if it is normal at all its points.

Lemma II.2.3 ([Con99, Lemma 1.2.1])**.** *Assume that $\mathfrak{X} = \mathrm{Spf}(A)$ is affine. Then, $\mathfrak{X}$ is normal if and only if A is normal.*

Lemma II.2.4. *Assume that $\mathfrak{X} = \mathrm{Spf}(A)$, x corresponds to an open prime ideal $\mathfrak{p}$ of A and $\widehat{A_{\mathfrak{p}}}$ denotes the π-adic completion of $A_{\mathfrak{p}}$. Then, we have canonical flat local homomorphisms $A_{\mathfrak{p}} \to \mathcal{O}_{\mathfrak{X},x} \to \widehat{A_{\mathfrak{p}}}$ which induce an isomorphism of π-adic completions $\widehat{\mathcal{O}_{\mathfrak{X},x}} \xrightarrow{\sim} \widehat{A_{\mathfrak{p}}}$. Hence, the canonical projection $\mathcal{O}_{\mathfrak{X},x} \to \mathcal{O}_{\mathfrak{X}_s,x}$ yields an isomorphism*

$$\mathcal{O}_{\mathfrak{X},x}/\pi\mathcal{O}_{\mathfrak{X},x} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}_s,x}. \tag{II.2.4.1}$$

Proof. The construction of the local homomorphisms $A_{\mathfrak{p}} \to \mathcal{O}_{\mathfrak{X},x} \to \widehat{A_{\mathfrak{p}}}$ is clear from

$$\mathcal{O}_{\mathfrak{X},x} = \varinjlim_{f \in A-\mathfrak{p}} \varprojlim_{n} A[\frac{1}{f}]/(\pi^{n+1}), \tag{II.2.4.2}$$

with the composition being the canonical completion homomorphism $A_{\mathfrak{p}} \to \widehat{A_{\mathfrak{p}}}$. Since the latter is flat and $\mathcal{O}_{\mathfrak{X},x} \to \widehat{A_{\mathfrak{p}}}$ is faithfully flat [EGA I, Chap. 0, 7.6.18], $A_{\mathfrak{p}} \to \mathcal{O}_{\mathfrak{X},x}$ is also faithfully flat. From (II.2.4.2), we see that $\mathcal{O}_{\mathfrak{X},x}/(\pi^{n+1}) \xrightarrow{\sim} A_{\mathfrak{p}}/(\pi^{n+1})$ since the filtered colimit functor is exact and thus commutes with quotients. Taking the projective limit gives the desired isomorphism. ☐

II.2.5. Let $\overline{x} \to \mathfrak{X}_s$ be a geometric point with image x. The k-schemes $\mathfrak{X}_s$ and $\overline{x}$ have natural structures of $\mathcal{S}$-formal schemes which make $\mathfrak{X}_s$ into a formal closed subscheme of $\mathfrak{X}$ and make $\overline{x} \to \mathfrak{X}_s$ into an $\mathcal{S}$-morphism. We say that the composition $\overline{x} \to \mathfrak{X}_s \hookrightarrow \mathfrak{X}$ is a *geometric point of* $\mathfrak{X}$. We denote by $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(\mathfrak{X})$ the following category. The objects of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(\mathfrak{X})$ are all triples $(\mathcal{U}, \mathcal{U} \to \mathfrak{X}, \varphi_{\mathcal{U}} : \overline{x} \to \mathcal{U})$, simply denoted $\mathcal{U}$, where $\mathcal{U}$ is a formal affine scheme, $\mathcal{U} \to \mathfrak{X}$ is a formal étale morphism and $\varphi_{\mathcal{U}}$ is an $\mathfrak{X}$-morphism. A morphism of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(\mathfrak{X})$ is an $\mathfrak{X}$-morphism $f : \mathcal{U} \to \mathcal{V}$ such that $f \circ \varphi_{\mathcal{U}} = \varphi_{\mathcal{V}}$. The étale local ring of $\mathfrak{X}$ at $\overline{x}$ is defined as

(II.2.5.1)
$$\mathcal{O}_{\mathfrak{X},\overline{x}} = \varinjlim_{\mathcal{U} \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(\mathfrak{X})} \mathcal{O}_{\mathfrak{X}}(\mathcal{U}).$$

The ring $\mathcal{O}_{\mathfrak{X},\overline{x}}$ is local and henselian by Proposition II.2.8 below.

Lemma II.2.6. *Let X be a scheme, X_0 a closed subscheme of X and $\overline{x}$ a geometric point of X_0. Denote by $\mathrm{Nbh}_{\overline{x}}(X)$ (resp. $\mathrm{Nbh}_{\overline{x}}(X_0)$) the category of étale neighborhoods of $\overline{x}$ in X (resp. in X_0) and let $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X)$ (resp. $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X_0)$) be its full subcategory of affine objects. Let $\varphi : \mathrm{Nbh}_{\overline{x}}(X) \to \mathrm{Nbh}_{\overline{x}}(X_0)$ (resp. $\varphi_{\mathrm{aff}} : \mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X) \to \mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X_0)$) be the functor $U \mapsto U_0 = U \times_X X_0$. Then, φ_{aff} and φ are cofinal.*

Proof. We can assume that $X = \mathrm{Spec}(A)$ is affine and $X_0 = \mathrm{Spec}(A_0)$ with $A_0 = A/J$ for some ideal J. Let $\mathrm{Spec}(B_0) = U_0 \to X_0$ be an object of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X_0)$. We prove that there exists an affine étale neighborhood $V \to X$ of $\overline{x}$ with a map $V_0 \to U_0$. By [**Ray70**, Chap. V, Théorème 1], we can assume that B_0 is a standard étale algebra over A_0, i.e. $B_0 = (A_0[X]/\overline{P})[\frac{1}{Q}]$ for some $P, Q \in A[X]$ such that $\overline{P}' = P' \bmod J$ is invertible in B_0. Set $B = (A[X]/P)[\frac{1}{QP'}]$. Then, $V = \mathrm{Spec}(B)$ is étale over X and $V_0 = V \times_X X_0 = U_0$. Hence, V, with the composition $\overline{x} \to U_0 \hookrightarrow V$ is the desired object of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X)$. Now suppose that U_0 is an object of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X_0)$, V is an object of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X)$ and $f_0, g_0 : \varphi_{\mathrm{aff}}(V) \rightrightarrows U_0$ are morphisms of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X_0)$. Since $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X_0)$ is a filtered category, we can find an equalizer of f_0 and g_0, i.e a morphism $h_0 : W_0 \to V_0$ in $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X_0)$ such that $f_0 h_0 = g_0 h_0$. The first part of the proof, applied to V and V_0 instead of X and X_0, gives an object W' of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(V)$ with a morphism $W'_0 \to W_0$; hence, composing with $V \to X$, W' is an object of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X)$ with a morphism $h' : W' \to V$. The morphism $\varphi_{\mathrm{aff}}(h')$ is then an equalizer of f_0, g_0. This proves that φ_{aff} is cofinal [**SGA 4**, I, 8.1.3]. It implies that φ is cofinal since the inclusion functors $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X) \to \mathrm{Nbh}_{\overline{x}}(X)$ and $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X_0) \to \mathrm{Nbh}_{\overline{x}}(X)$ are cofinal. □

II.2.7. We keep the notation of II.2.5. The functor $\mathcal{U} \mapsto \mathcal{U}_s$ induces an equivalence of the étale sites $\mathfrak{X}_{\text{ét}}$ and $\mathfrak{X}_{s,\text{ét}}$ (*cf.* [**EGA IV**, 18.1.2] and [**Abb10**, 2.4.8]). Therefore, the colimit in (II.2.5.1) can be taken in the category $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(\mathfrak{X}_s)$ as defined in II.2.6

(II.2.7.1)
$$\mathcal{O}_{\mathfrak{X},\overline{x}} = \varinjlim_{\mathcal{U}_s \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(\mathfrak{X}_s)} \mathcal{O}_{\mathfrak{X}}(\mathcal{U}).$$

We thus get a canonical surjective homomorphism

(II.2.7.2)
$$\mathcal{O}_{\mathfrak{X},\overline{x}} \to \mathcal{O}_{\mathfrak{X}_s,\overline{x}}.$$

When $\mathfrak{X} = \mathrm{Spf}(A)$ is affine, denoting $X = \mathrm{Spec}(A)$, we get from Lemma II.2.6 above

(II.2.7.3)
$$\mathcal{O}_{\mathfrak{X},\overline{x}} \cong \varinjlim_{\mathrm{Spec}(B) \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(X)} \widehat{B}.$$

Proposition II.2.8. *We keep the notation of II.2.5. We assume that $\mathfrak{X} = \mathrm{Spf}(A)$ is affine, $\mathfrak{p}$ is the open prime ideal of A corresponding to x and denote by $A_{\mathfrak{p}}^{\mathrm{sh}}$ the strict henselization of $A_{\mathfrak{p}}$ with respect to the separably closed field $\kappa(\overline{x})$ defining $\overline{x}$. Then, $\mathcal{O}_{\mathfrak{X},\overline{x}}$ is a noetherian henselian local ring whose residue field is $\kappa(\overline{x})$, and we have canonical local homomorphisms of local rings*

$$A_{\mathfrak{p}}^{\mathrm{sh}} \to \mathcal{O}_{\mathfrak{X},\overline{x}} \to \widehat{A_{\mathfrak{p}}^{\mathrm{sh}}} \tag{II.2.8.1}$$

inducing an isomophism $\widehat{\mathcal{O}_{\mathfrak{X},\overline{x}}} \xrightarrow{\sim} \widehat{A_{\mathfrak{p}}^{\mathrm{sh}}}$ between π-adic completions. Hence, the canonical surjection (II.2.7.2) induces an isomorphism

$$\mathcal{O}_{\mathfrak{X},\overline{x}}/\pi\mathcal{O}_{\mathfrak{X},\overline{x}} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}_s,\overline{x}}. \tag{II.2.8.2}$$

Proof. We use the description (II.2.7.3) of $\mathcal{O}_{\mathfrak{X},\overline{x}}$ and run the argument given in [**EGA I**, Chap. 0, 7.6.17] for the proof of the analogous statement for $\mathcal{O}_{\mathfrak{X},x}$. For $B \in \mathrm{Nbh}_{\overline{x}}^{\mathrm{aff}}(X)$, we let $\mathfrak{p}_B$ be the the image in $\mathrm{Spec}(B)$ of the geometric point $\overline{x}$ and set $\mathfrak{p}_{\overline{x}} = \varinjlim_B \mathfrak{p}_B$. We show that any element in $\mathcal{O}_{\mathfrak{X},\overline{x}} - \mathfrak{p}_{\overline{x}}$ is invertible. Indeed such an element is the image in $\mathcal{O}_{\mathfrak{X},\overline{x}}$ of an element $z = (z_n)$ of some $\widehat{B}$, and it is enough to show that z is invertible modulo π [**EGA I**, Chap. 0, 7.1.12]. Since $\widehat{B}/\pi\widehat{B} \cong B/\pi B = B_0$, it is enough to show that z_0 is invertible in some C_0 such that $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ is a morphism in $\mathrm{Nbh}_{\overline{x}}^{\mathrm{aff}}(X)$. Since z_0 is not in $\mathfrak{p}_B$, the latter is a prime ideal of the étale A_0-algebra $C_0 = B_0[\frac{1}{z_0}]$. By Lemma II.2.6, the latter algebra lifts to some C with $\mathrm{Spec}(C) \in \mathrm{Nbh}_{\overline{x}}^{\mathrm{aff}}(X)$ and above $\mathrm{Spec}(B)$. Hence, the image of z in $\widehat{C}$ is invertible. This proves that $\mathcal{O}_{\mathfrak{X},\overline{x}}$ is a local ring with maximal ideal $\mathfrak{p}_{\overline{x}}$. To prove that $\mathcal{O}_{\mathfrak{X},\overline{x}}$ is henselian, we must show that for every étale homomorphism $\varphi : \mathcal{O}_{\mathfrak{X},\overline{x}} \to A'$, every section of $\varphi \otimes \kappa(\overline{x})$ descends to a section of φ. Since $\mathcal{O}_{\mathfrak{X},\overline{x}}$ is colimit over $\mathcal{U}$, by [**EGA IV**, 17.7.8], φ is the base change to $\mathcal{O}_{\mathfrak{X},\overline{x}}$ of an étale homomorphism $\varphi_{\mathcal{U}} : A_{\mathcal{U}} \to A_1$ for some $\mathcal{U} = \mathrm{Spf}(A_{\mathcal{U}})$ in $\mathrm{Nbh}_{\overline{x}}^{\mathrm{aff}}(\mathfrak{X})$. Taking the π-adic completion, this homomorphism extends to an adic étale homomorphism $A_{\mathcal{U}} \to \widehat{A}_1$. The composition $A \to A_{\mathcal{U}} \to \widehat{A}_1$ is thus étale, hence $\mathcal{U}_1 = \mathrm{Spf}(\widehat{A}_1)$ is in $\mathrm{Nbh}_{\overline{x}}^{\mathrm{aff}}(\mathfrak{X})$. Hence, we have the following commutative diagram

$$\begin{array}{ccccc} A_1 & \longrightarrow & \longrightarrow & \longrightarrow & A' \\ \uparrow & \searrow & & & \uparrow \\ A_{\mathcal{U}} & \longrightarrow & \widehat{A}_1 & \longrightarrow & \mathcal{O}_{\mathfrak{X},\overline{x}} \end{array} \tag{II.2.8.3}$$

Since the square in this diagram is co-Cartesian, the composition $A_1 \to \widehat{A}_1 \to \mathcal{O}_{\mathfrak{X},\overline{x}}$ induces a section $A' \to \mathcal{O}_{\mathfrak{X},\overline{x}}$ of φ.

Now, on the one hand, since, for any $f \in A - \mathfrak{p}$, $D(f)$ is an étale neighborhood of $\overline{x}$, using (II.2.7.3), we get a canonical local homomorphism

$$A_{\mathfrak{p}} = \varinjlim_{\mathfrak{p} \in D(f)} A_f \to \mathcal{O}_{\mathfrak{X},\overline{x}} \tag{II.2.8.4}$$

which extends to $A_{\mathfrak{p}}^{\mathrm{sh}} \to \mathcal{O}_{\mathfrak{X},\overline{x}}$ by the universal property of the strict henselization of $A_{\mathfrak{p}}$ with respect to the residue extension $\kappa(\mathfrak{p}) \to \kappa(\overline{x})$. On the other hand, every $\mathrm{Spec}(B) \in \mathrm{Nbh}_{\overline{x}}^{\mathrm{aff}}(X)$ induces an X-morphism $X_{(\overline{x})} \to \mathrm{Spec}(B)$, i.e an A-homomorphism $B \to A_{\mathfrak{p}}^{\mathrm{sh}}$. Completing it π-adically and taking the limit in (II.2.7.3) give a local homomorphism $\mathcal{O}_{\mathfrak{X},\overline{x}} \to \widehat{A_{\mathfrak{p}}^{\mathrm{sh}}}$, whence the homomorphisms (II.2.8.1). The π-adic completion of the homomorphism $\mathcal{O}_{\mathfrak{X},\overline{x}} \to \widehat{A_{\mathfrak{p}}^{\mathrm{sh}}}$ is an isomorphism because, being exact, the filtered colimit functor commutes with quotients and $\widehat{B}/\pi^n\widehat{B} \xrightarrow{\sim} B/\pi^n B$. This implies the claim on the residue fields. As A is noetherian, it also implies that $\widehat{\mathcal{O}_{\mathfrak{X},\overline{x}}}$ is noetherian; hence, by the faithful flatness of $\mathcal{O}_{\mathfrak{X},\overline{x}} \to \widehat{\mathcal{O}_{\mathfrak{X},\overline{x}}}$, $\mathcal{O}_{\mathfrak{X},\overline{x}}$ is also noetherian. The last statement of the proposition follows since $\mathcal{O}_{\mathfrak{X}_s,\overline{x}} = (A/(\pi))^{\mathrm{sh}}_{\mathfrak{p}}$. □

Lemma II.2.9. *Let $X \to Y$ be a finite morphism of schemes. Let $\overline{y}$ be a geometric point of Y and $\overline{x}_1, \ldots, \overline{x}_n$ be the geometric points of X above $\overline{y}$. Then, there exist an étale neighborhood $Y' \to Y$ of $\overline{y}$ and, for each i, an étale neighborhood $X'_i \to X$ of $\overline{x}_i$ such that*

$$Y' \times_Y X \xrightarrow{\sim} \coprod_{i=1}^{n} X'_i. \tag{II.2.9.1}$$

Proof. We can assume $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$, with B a finite A-algebra. Then, $B \otimes_A \mathcal{O}_{Y,\overline{y}}$ is a finite algebra over the henselian local ring $\mathcal{O}_{Y,\overline{y}}$ and thus it is isomorphic to the product of its localizations at its maximal ideals. These maximal ideals are in bijection with the points of the fiber $X_{\overline{y}} = \{\overline{x}_1, \ldots, \overline{x}_n\}$ (harmless abuse of notation here) and we denote them by $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$. By [Fu11, 2.8.20], we have $(B \otimes_A \mathcal{O}_{Y,\overline{y}})_{\mathfrak{q}_i} \cong \mathcal{O}_{X,\overline{x}_i}$ for each i. Hence, the aforementioned decomposition of $B \otimes_A \mathcal{O}_{Y,\overline{y}}$ yields a decomposition into connected components

$$Y_{(\overline{y})} \times_Y X \cong \coprod_{i=1}^{n} X_{(\overline{x}_i)}, \tag{II.2.9.2}$$

where $Y_{(\overline{y})}$ and $X_{(\overline{x}_i)}$ denote strict localizations. Let $e_1, \ldots, e_n$ be the idempotent elements of

$$\Gamma(Y_{(\overline{y})} \times_Y X, \mathcal{O}_{Y_{(\overline{y})} \times_Y X}) = \varinjlim_{Y' \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{y}}(Y)} \Gamma(Y' \times_Y X, \mathcal{O}_{Y' \times_Y X}), \tag{II.2.9.3}$$

corresponding to the decomposition (II.2.9.2). Then, there exists $Y' \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{y}}(Y)$ such that $e_1, \ldots, e_n$ form a complete orthogonal set of idempotent elements of $\Gamma(Y' \times_Y X, \mathcal{O}_{Y' \times_Y X})$. This gives a decomposition of $Y' \times_Y X$ into

$$Y' \times_Y X \xrightarrow{\sim} \coprod_{i=1}^{n} X'_i \tag{II.2.9.4}$$

such that the fiber product with $Y_{(\overline{y})} \to Y'$ gives back (II.2.9.2). Since $Y' \times_Y X$ is étale over X, X'_i is an étale neighborhood of $\overline{x}_i$, for each i. This proves the lemma. □

Lemma II.2.10. *Let $\mathfrak{X} \to \mathfrak{Y}$ be a finite adic morphism of locally noetherian $\mathcal{S}$-formal schemes. Let $\overline{y}$ be a geometric point of $\mathfrak{Y}$ and let $\overline{x}_1, \ldots, \overline{x}_n$ be the geometric points of $\mathfrak{X}$ above $\overline{y}$. Then, there exist an étale neighborhood $\mathfrak{Y}' \to \mathfrak{Y}$ of $\overline{y}$ and, for each i, an étale neighborhood $\mathfrak{X}'_i \to \mathfrak{X}$ of $\overline{x}_i$, such that*

$$\mathfrak{Y}' \times_{\mathfrak{Y}} \mathfrak{X} \xrightarrow{\sim} \coprod_{i=1}^{n} \mathfrak{X}'_i. \tag{II.2.10.1}$$

Proof. From II.2.9, there exist an étale neighborhood $Y' \to \mathfrak{Y}_s$ of $\overline{y}$ and, for each i, an étale neighborhood $X'_i \to \mathfrak{X}_s$ of $\overline{x}_i$ such that

$$Y' \times_{\mathfrak{Y}_s} \mathfrak{X}_s \xrightarrow{\sim} \coprod_{i=1}^{n} X'_i. \tag{II.2.10.2}$$

From the equivalence of étale sites recalled in II.2.7, there exist formal étale neighborhoods $\mathfrak{Y}' \to \mathfrak{Y}$ of $\overline{y}$ and $\mathfrak{X}'_i \to \mathfrak{X}$ of $\overline{x}_i$ such that $Y' = \mathfrak{Y}'_s$ and $X'_i = \mathfrak{X}'_{i,s}$. Now (II.2.10.2) becomes an $\mathfrak{X}_s$-isomorphism of the special fibers of formal étale $\mathfrak{X}$-schemes

(II.2.10.3) $$(\mathfrak{Y}' \times_{\mathfrak{Y}} \mathfrak{X})_s \xrightarrow{\sim} (\coprod_{i=1}^{n} \mathfrak{X}'_i)_s,$$

which then lifts to (II.2.10.1) by the aforementioned equivalence of étales sites. □

Lemma II.2.11. *Let* $\mathfrak{X} \to \mathfrak{Y}, \overline{y}$ *and* $\overline{x}_1, \dots, \overline{x}_n$ *be as in* II.2.10*. Then,*

(II.2.11.1) $$\varinjlim_{\mathfrak{Y}' \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{y}}(\mathfrak{Y})} \mathcal{O}_{\mathfrak{X}}(\mathfrak{Y}' \times_{\mathfrak{Y}} \mathfrak{X}) \xrightarrow{\sim} \prod_{i=1}^{n} \mathcal{O}_{\mathfrak{X},\overline{x}_i}.$$

Proof. We can assume that $\mathfrak{Y} = \mathrm{Spf}(A)$ and $\mathfrak{X} = \mathrm{Spf}(B)$ are affine. By II.2.10, we can also assume that $n = 1$ and put $\overline{x}_1 = \overline{x}$. By II.2.7, the canonical homomorphism (II.2.11.1) can also be written as

(II.2.11.2) $$\varinjlim_{\mathfrak{Y}'_s \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{y}}(\mathfrak{Y}_s)} \mathcal{O}_{\mathfrak{X}}(\mathfrak{Y}' \times_{\mathfrak{Y}} \mathfrak{X}) \to \mathcal{O}_{\mathfrak{X},\overline{x}} = \varinjlim_{\mathfrak{X}'_s \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(\mathfrak{X}_s)} \mathcal{O}_{\mathfrak{X}}(\mathfrak{X}').$$

To prove that this is an isomorphism, it is enough to show that the right-hand side colimit of (II.2.11.2) can be taken in the full subcategory of objects of the form $\mathfrak{Y}'_s \times_{\mathfrak{Y}_s} \mathfrak{X}_s$, where $\mathfrak{Y}'_s$ is an object of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{y}}(\mathfrak{Y}_s)$. Let φ be the functor $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{y}}(\mathfrak{Y}_s) \to \mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(\mathfrak{X}_s)$, $\mathfrak{Y}'_s \mapsto \mathfrak{Y}'_s \times_{\mathfrak{Y}_s} \mathfrak{X}_s$. It is enough to show that φ is cofinal. We let $\mathfrak{X}''_s$ be an object of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(\mathfrak{X}_s)$ and look for an object $\mathfrak{Y}'_s \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{y}}(\mathfrak{Y}_s)$ and a morphism $h : \mathfrak{Y}'_s \times_{\mathfrak{Y}_s} \mathfrak{X}_s \to \mathfrak{X}''_s$. We have already seen II.2.9.2 that

(II.2.11.3) $$\varprojlim_{\mathfrak{Y}'_s \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{y}}(\mathfrak{Y}_s)} \mathfrak{Y}'_s \times_{\mathfrak{Y}_s} \mathfrak{X}_s = (\mathfrak{Y}_s)_{(\overline{y})} \times_{\mathfrak{Y}_s} \mathfrak{X}_s \xrightarrow{\sim} (\mathfrak{X}_s)_{(\overline{x})} = \varprojlim_{\mathfrak{X}'_s \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(\mathfrak{X}_s)} \mathfrak{X}'_s,$$

where the first identification stems from the fact that colimits commute with tensor product. Then, by [**EGA IV**, 8.8.2], there exists an object $\mathfrak{Y}'_s$ of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{y}}(\mathfrak{Y}_s)$ and an isomorphism $\mathfrak{Y}'_s \times_{\mathfrak{Y}_s} \mathfrak{X}_s \xrightarrow{\sim} (\mathfrak{X}_s)_{(\overline{x})}$ inducing (II.2.11.3). Post-composing the latter with the canonical projection $(\mathfrak{X}_s)_{(\overline{x})} \to \mathfrak{X}''_s$ yields the desired morphism h. Now suppose that $\mathfrak{Y}''_s$ is an object of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{y}}(\mathfrak{Y}_s)$, and $f_0, g_0 : \varphi_{\mathrm{aff}}(\mathfrak{Y}''_s) \rightrightarrows \mathfrak{X}''_s$ are morphisms of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(\mathfrak{X}_s)$. Then, mutatis mutandis, the same argument given in the proof of II.2.6 yields an equalizer $\varphi(h')$ of f_0 and g_0, for some morphism $h' : Z' \to \mathfrak{Y}''_s$ of $\mathrm{Nbh}^{\mathrm{aff}}_{\overline{y}}(\mathfrak{Y}_s)$. This proves that φ is cofinal [**SGA 4**, I, 8.1.3] and establishes the Lemma. □

Lemma II.2.12. *We keep the assumptions of* II.2.10 *and further assume that* $\mathfrak{Y} = \mathrm{Spf}(A)$ *and* $\mathfrak{X} = \mathrm{Spf}(B)$*. Then,*

(II.2.12.1) $$\mathcal{O}_{\mathfrak{Y},\overline{y}} \otimes_A B \xrightarrow{\sim} \prod_{i=1}^{n} \mathcal{O}_{\mathfrak{X},\overline{x}_i}.$$

Proof. From II.2.11, we see that

(II.2.12.2) $$\varinjlim_{\mathrm{Spf}(A') \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{y}}(\mathfrak{Y})} A' \widehat{\otimes}_A B \xrightarrow{\sim} \prod_{i=1}^{n} \mathcal{O}_{\mathfrak{X},\overline{x}_i}.$$

Since B is a finite algebra over A, $A'\otimes B$ is a finite algebra over the adic ring $\widehat{A'}$, hence π-adically complete [**Abb10**, 1.8.25.4 and 1.8.29]. It then follows from [**EGA I**, Chap. 0, 7.7.1] that

$$A'\widehat{\otimes}_A B \xrightarrow{\sim} A'\otimes_A B. \tag{II.2.12.3}$$

Since colimits commute with tensor products, (II.2.12.2) and (II.2.12.3) yield (II.2.12.1). □

Lemma II.2.13. *Let $A\to B$ be a finite homomorphism of adic rings over $\mathcal{O}_K$. Assume that A is flat over $\mathcal{O}_K$ and that $B\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{\overline{K}}}$ is free of finite rank over $A\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{\overline{K}}}$. Then, there exists a finite extension $K'\subseteq\overline{K}$ of K such that $B\otimes_{\mathcal{O}_K}\mathcal{O}_{K'}$ is free of finite rank over $A\otimes_{\mathcal{O}_K}\mathcal{O}_{K'}$.*

Proof. Let $b_1,\dots,b_m$ be a basis of $B\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{\overline{K}}}$ over $A\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{\overline{K}}}$. As colimits commute with quotient and tensor product, it is straightforwardly seen that the π-adic completion of $\varinjlim_{K'} B\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{K'}$ is canonically isomorphic to $B\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\overline{K}}\cong B\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{\overline{K}}}$. It follows that the canonical map

$$\varphi:\varinjlim_{K'} B\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{K'}\to B\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{\overline{K}}} \tag{II.2.13.1}$$

has dense image. Hence, there exist a finite extension K' of K and elements $b'_i\in B\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{K'}$ such that $|b_i-\varphi(b'_i)|<|\pi|$. Then, $(\varphi(b'_1),\dots,\varphi(b'_m))$ is also a basis of $B\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{\overline{K}}}$ over $A\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{\overline{K}}}$. Let

$$\oplus_{i=1}^m\,(A\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{\overline{K}}})\varphi(b'_i)\xrightarrow{\sim} B\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{\overline{K}}} \tag{II.2.13.2}$$

be the canonical isomorphism thus defined. We claim that the induced canonical homomorphism

$$\oplus_{i=1}^m\,(A\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{K'})b'_i\to B\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{K'} \tag{II.2.13.3}$$

is an isomorphism. To prove injectivity, it is enough to show that the canonical homomorphism

$$A\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{K'}\to A\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{\overline{K}}} \tag{II.2.13.4}$$

is injective. It is enough to show that it is injective mod π^n for any $n\geq 1$. Since A is flat over $\mathcal{O}_K$, A/π^nA is flat over $\mathcal{O}_K/(\pi^n)$. Moreover, $\mathcal{O}_{K'}/(\pi^n)\to\mathcal{O}_{\widehat{\overline{K}}}/(\pi^n)=\mathcal{O}_{\overline{K}}/(\pi^n)$ is injective since $\mathcal{O}_{\overline{K}}\cap K'=\mathcal{O}_{K'}$. Thus (II.2.13.3) is injective mod π^n for any $n\geq 1$, hence (II.2.13.2) is injective. To prove surjectivity, it is enough to show that (II.2.13.2) is surjectif mod π [**Abb10**, 1.8.5]. Since the residue field of $\mathcal{O}_K$ is algebraically closed, (II.2.13.1) and (II.2.13.2) coincide mod π, hence (II.2.13.2) is sujective mod π too, hence surjective. Finally, as $\mathcal{O}_{K'}$ is a finite free $\mathcal{O}_K$-module, we deduce that $A\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{K'}=A\otimes_{\mathcal{O}_K}\mathcal{O}_{K'}$ and $B\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{K'}=B\otimes_{\mathcal{O}_K}\mathcal{O}_{K'}$. This proves the lemma. □

II.2.14. We take up again the notation of II.2.8, set $X=\operatorname{Spec}(A)$ and let $\overline{y}\to\mathfrak{X}_s$ be another geometric point. On the one hand, since the diagram

$$\begin{array}{ccc}(\mathfrak{X}_s)_{(\overline{x})} & \longrightarrow & \mathfrak{X}_s\\ \downarrow & \square & \downarrow\\ X_{(\overline{x})} & \longrightarrow & X\end{array} \tag{II.2.14.1}$$

is Cartesian, specialization maps $\overline{y}\rightsquigarrow\overline{x}$ on X and its subscheme $\mathfrak{X}_s$ coincide:

$$\operatorname{Hom}_{\mathfrak{X}_s}(\overline{y},(\mathfrak{X}_s)_{(\overline{x})})\cong\operatorname{Hom}_X(\overline{y},X_{(\overline{x})}). \tag{II.2.14.2}$$

On the other hand, if $\mathcal{F}$ is a sheaf on $\mathfrak{X}_{s,\text{ét}}$, any specialization map $\overline{y}\to(\mathfrak{X}_s)_{(\overline{x})}$ induces a homomorphism $\mathcal{F}_{\overline{x}}\to\mathcal{F}_{\overline{y}}$. Indeed, we have

$$\mathcal{F}_{\overline{x}}=\Gamma\left((\mathfrak{X}_s)_{(\overline{x})},\mathcal{F}|_{(\mathfrak{X}_s)_{(\overline{x})}}\right) \tag{II.2.14.3}$$

and the map $\mathcal{F}_{\overline{x}} \to \mathcal{F}_{\overline{y}}$ is induced by the $\mathfrak{X}$-morphism $\overline{y} \to (\mathfrak{X}_s)_{(\overline{x})}$. In particular, when $\mathcal{F} = \mathcal{O}_{\mathfrak{X}}$, we get a morphism

$$\mathrm{Hom}_{\mathfrak{X}_s}(\overline{y}, (\mathfrak{X}_s)_{(\overline{x})}) \to \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{O}_{\mathfrak{X},\overline{x}}, \mathcal{O}_{\mathfrak{X},\overline{y}}). \tag{II.2.14.4}$$

II.2.15. From here until II.2.19 included, the residue field k is not assumed to be algebraically closed, $\overline{K}$ is an algebraic closure of K, $\mathcal{O}_{\overline{K}}$ is the integral closure of $\mathcal{O}_K$ in $\overline{K}$ and $\overline{k}$ is its residue field. For an affinoid K-algebra A, let us recall that its supremum (semi-) norm $|\cdot|_{\mathrm{sup}}$ is defined as follows. If $f \in A$, then,

$$|f|_{\mathrm{sup}} = \sup_{x \in \mathrm{Sp(A)}} |f(x)|_x, \tag{II.2.15.1}$$

where $\mathrm{Sp}(A)$ is the set of maximal ideals of A and $f(x)$ is the image of f in the finite extension A/x of K and $|\cdot|_x$ is the unique extension of $|\cdot|_K$ to A/x. If A is a standard Tate algebra over K, then the supremum norm coincides with the usual Gauss norm.

II.2.16. Let A_K be affinoid K-algebra. If A is an $\mathcal{O}_K$-algebra which is topologically of finite type over $\mathcal{O}_K$ (hence π-adically complete) such that $A \otimes_{\mathcal{O}_K} K = A_K$, then the formal scheme $\mathfrak{X} = \mathrm{Spf}(A)$ is a model of the affinoid variety $\mathfrak{X}_\eta = \mathrm{Sp}(A_K)$, and $\mathfrak{X}_\eta$ coincide with the *rigid fiber* (in the sense of Raynaud) of $\mathfrak{X}$. To construct such a model, let $\rho : K\{T_1, \dots, T_n\} \to A_K$ be a surjective homomorphism, set $A_\rho = \rho(\mathcal{O}_K\{T_1, \dots, T_n\})$ and take $A = \{f \in A_K \mid |f|_{\mathrm{sup}} \leq 1\}$ to be the unit ball.

Lemma II.2.17 ([**AS02**, Lemma 4.1])**.** *We keep the notation of* II.2.16 *and assume that* A_K *is reduced. Then,*

(*i*) $A_\rho \subseteq A$ *and* A *is the integral closure of* A_ρ *in* A_K*.*
(*ii*) *If* $A_\rho \otimes_{\mathcal{O}_K} k$ *is reduced, then* $A = A_\rho$*.*

As $\mathcal{O}_K$ is a complete discrete valuation ring, the unit ball A of the reduced K-algebra A_K is topologically of finite type over $\mathcal{O}_K$ [**BLR95**, Theorem 1.2]; hence, it defines a formal model $\mathrm{Sp}(A_K)$. Even then, $A \otimes_{\mathcal{O}_K} k$ may not be reduced and thus the formation of A may not commute with base change. However, we have the following generalization of a finiteness result of Grauert and Remmert.

Theorem II.2.18 ([**BLR95**, Theorem 3.1])**.** *Let* A_K *be a geometrically reduced affinoid* K*-algebra. Then, there exists a finite* (*separable*) *extension* K' *of* K *such that the unit ball* $A_{\mathcal{O}_{K'}}$ *of* $A_K \otimes_K K'$ *is topologically of finite type over* $\mathcal{O}_{K'}$ *and has geometrically reduced special fiber* $A_{\mathcal{O}_{K'}} \otimes_{\mathcal{O}_{K'}} k'$*, where* k' *is the residue field of* $\mathcal{O}_{K'}$*. Moreover, the formation of* $A_{\mathcal{O}_{K'}}$ *commutes with any finite extension of* K'*.*

Definition II.2.19. Let A_K be a geometrically reduced affinoid K-algebra. We think of the collection $(\mathrm{Spf}(A_{\mathcal{O}_{K'}}))_{K'}$ of $\mathcal{O}_{K'}$-formal schemes, where K' and $A_{\mathcal{O}_{K'}}$ are as in II.2.18, as a unique model of $\mathrm{Sp}(A_K)$ defined over $\mathcal{O}_{\overline{K}}$ and call it *the normalized integral model of* $\mathrm{Sp}(A_K)$ over $\mathcal{O}_{\overline{K}}$. We say that the normalized integral model is defined over $\mathcal{O}_{K'}$ if the unit ball $A' \subset A_{K'}$ has a geometrically reduced special fiber $A' \otimes_{\mathcal{O}_{K'}} \overline{k}$.

II.2.20. We resume the assumptions of II.2.1. Let $C \to \mathcal{S}$ be a formal relative curve, i.e. an adic morphism of formal schemes which is of finite type, separated and flat, with special fiber of equidimension 1. We assume that $C/\mathcal{S}$ is proper and has smooth rigid fiber C_η. We also assume that C contains an admissible open subset isomorphic to a disjoint union $\coprod_{j=1}^{n} \widehat{\Delta}_j$, where each $\widehat{\Delta}_j$ is isomorphic to the formal torus $\widehat{\mathbb{G}}_{m,\mathcal{O}_K} = \mathrm{Spf}(\mathcal{O}_K\{T_j, T_j^{-1}\})$ with rigid fiber $\widehat{\Delta}_{j,\eta} = \Delta_j$, and also that $C_s - \coprod_j \widehat{\Delta}_{j,s}$ is a finite set. Let f be a rigid analytic function on $\coprod_j \Delta_j$ and define the norm $|f|_j$ to be the sup-norm $|f_{|\Delta_j}|_{\sup}$ on Δ_j. The function f corresponds to a rigid morphism $\coprod_j \Delta_j \to D$, where D is the rigid unit disc over K. If $f \neq 0$ on each Δ_j, we can write $|f|_j = |c_j|$, for some constant $c_j \in L^\times$, where L is some finite extension of K. Then, $c_j^{-1}f$ is defined on $\widehat{\Delta}_j \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_L$, corresponds to a formal morphism $\widehat{\Delta}_j \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_L \to \widehat{\mathbb{A}}^1_{\mathcal{O}_L}$ and thus reduces to a morphism $\widetilde{c_j^{-1}f} : \widehat{\Delta}_{j,s} \to \mathbb{A}^1_k$. As the $\widehat{\Delta}_{j,s}$ are open in C_s and smooth over k, they are dense open subsets of distinct irreducible components $\widetilde{C}_{s,j}$ of the normalization $\widetilde{C}_s$ of C_s. Hence, $\widetilde{c_j^{-1}f}$ defines a rational function on $\widetilde{C}_{s,j}$; the divisor of $\widetilde{c_j^{-1}f}$ on $\widetilde{C}_{s,j}$ is independent of c_j. Therefore, for a point $\widetilde{y} \in \widetilde{C}_s$ which is in the irreducible component $\widetilde{C}_{s,j}$, we define the *order of f at $\widetilde{y}$* by

$$\mathrm{ord}_{\widetilde{y}}(f) = \mathrm{ord}_{\widetilde{y}}(\widetilde{c_j^{-1}f}). \tag{II.2.20.1}$$

Lemma II.2.21. *We keep the notation of II.2.20. Let y be a point in C_s and denote by $\widetilde{y}_1, \ldots, \widetilde{y}_m$ the points in $\widetilde{C}_s$ above y. Let also $\mathrm{sp} : C_\eta \to C_s$ be the specialization map and set $C_+(y) = \mathrm{sp}^{-1}(y)$. Then, we have*

$$\deg(\mathrm{div}(f)_{|C_+(y)}) = \sum_j \mathrm{ord}_{\widetilde{y}_j}(f). \tag{II.2.21.1}$$

Proof. This is [**BL85**, Proposition 3.1 and Remark] since $|\cdot|_j$ is the norm over the irreducible component $C_{s,j}$ of C_s containing $\widehat{\Delta}_{j,s}$, as defined in *loc. cit.* □

II.3. Valuation rings.

II.3.1. Let V be a valuation ring with value group Γ_V and field of fractions K. Then, the group $\Gamma = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_V$ is a totally ordered $\mathbb{Q}$-vector space, whose dimension $r(\Gamma_V)$ is called the *rational rank* of Γ_V. Let L be an algebraic extension of K and W a valuation ring of L extending V. By [**Bou06**, Chap. VI, § 8, n°1, Proposition 1 and Corollaire 1], the value groups $\Gamma_V \subseteq \Gamma_W$ have the same rank (or height). Since the quotient Γ_W/Γ_V is torsion, these value groups have also the same rational rank, thus $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_W \cong \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_V = \Gamma$.

Lemma II.3.2. *Let V be a henselian valuation ring with field of fractions K, L a finite extension of K and W the integral closure of V in L. Then, there is a unique valuation on L extending the valuation of K, and its valuation ring is W.*

Proof. We know from [**Bou06**, Chap. VI, § 8, n°3, Théorème 1 and Remarque] that $W = \cap_{i \in I} V_i$, where the V_i are the valuation rings of all the (inequivalent) valuations of L extending the valuation of V, that W is a semi-local ring whose maximal ideals are the intersections $\mathfrak{m}_i = W \cap \mathfrak{m}(V_i)$, and that $W_{\mathfrak{m}_i} = V_i$. Also the cardinal of I is the number of maximal ideals of W ([**Bou06**, Chap. VI, § 8, n°6, Proposition 6] or simply by going-up). So it is enough to prove that W is a local ring. Since W is integral over V, it is a direct limit of its finite type, hence finite, V-subalgebras W_i. Since V is assumed to be henselian, each domain W_i is a product of local rings, hence local. □

We use the terminology *weakly unramified extension* for an extension of valuation rings $V \to V'$ with ramification index 1.

Lemma II.3.3 ([**Sta19**, Lemma 0ASK])**.** *Let V be a valuation ring, V^h (resp. V^{sh}) the henselization (resp. a strict henselization) of V. Then, the inclusions $V \subseteq V^h \subseteq V^{\mathrm{sh}}$ are weakly unramified extensions of valuation rings.*

Definition II.3.4. A valuation ring V is called a $\mathbb{Z}^2$-*valuation ring* if its value group Γ_V is isomorphic to the lexicographically ordered group $\mathbb{Z} \times \mathbb{Z}$.

Remark II.3.5. If V is a $\mathbb{Z}^2$-valuation ring, then V has height two. Indeed the lexicographically ordered group $\mathbb{Z} \times \mathbb{Z}$ has exactly two isolated subgroups, the trivial subgroup and the second factor. Hence, V has exactly two non zero prime ideals $\mathfrak{p} \subsetneq \mathfrak{m}$ [**Bou06**, Chap. VI, § 4, n°4, Proposition 5].

Lemma II.3.6. *Let K be the field of fractions of a $\mathbb{Z}^2$-valuation ring V whose prime ideals are $(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$ and let $v : K^\times \to \Gamma_V$ be its valuation map. Then, the localization $V_\mathfrak{p}$ is a discrete valuation ring whose valuation map is given by the composition*

$$K^\times \to \Gamma_V \to \Gamma_V/H, \tag{II.3.6.1}$$

where H is the unique non-trivial isolated subgroup of Γ. If we choose an isomorphism of ordered groups $\Gamma_V \cong \mathbb{Z} \times \mathbb{Z}$, then this composition is $K^\times \to \Gamma_V \to \mathbb{Z}$, where the second homomorphism is the first projection.

Proof. The lemma follows from combining [**Bou06**, Chap. VI, § 4, n°1, Proposition 1] and [**Bou06**, Chap. VI, § 4, n°3, Proposition 4], where we also see that the valuation of $V_\mathfrak{p}$ is given (up to equivalence) by the composition $K^\times \to \Gamma_V \to \Gamma_V/H$. □

Definition II.3.7. Let K be the field of fractions of a $\mathbb{Z}^2$-valuation ring V and let $v : K^\times \to \Gamma_V$ be its valuation map. Let ε_K be the minimum element of $\{t \in \Gamma_V \mid t > 0\}$, i.e a generator of the non-trivial isolated subgroup of Γ_V. For a given uniformizer π of the discrete valuation ring $V_\mathfrak{p}$, $(v(\pi), \varepsilon_K)$ is an ordered generating family for the abelian group Γ_V (II.3.6.1). Let $\alpha, \beta : \Gamma_V \to \mathbb{Z}$ be the group homomorphisms characterized respectively by $\alpha(v(\pi)) = 1$, $\alpha(\varepsilon_K) = 0$, and $\beta(v(\pi)) = 0$, $\beta(\varepsilon_K) = 1$. They induce an isomorphism of ordered abelian groups $(\alpha, \beta) : \Gamma_V \xrightarrow{\sim} \mathbb{Z} \times \mathbb{Z}$. The composition

$$K^\times \to \Gamma_V \xrightarrow{(\alpha,\beta)} \mathbb{Z} \times \mathbb{Z} \tag{II.3.7.1}$$

is *the normalized $\mathbb{Z}^2$-valuation map of V*. We set $v^\alpha = \alpha \circ v$ and $v^\beta = \beta \circ v$. Notice that, while v^α does not depend on the chosen π, v^β does.

Lemma II.3.8. *Let V be a henselian $\mathbb{Z}^2$-valuation ring with field of fractions K, L a finite extension of K and W the integral closure of V in L. Then, the valuation ring W* (cf. II.3.2) *is also a $\mathbb{Z}^2$-valuation ring.*

Proof. The group Γ_W is finitely generated with height and rational rank equal 2 (*cf.* II.3.1). So the lemma follows from [**Bou06**, Chap. VI, § 10, n°3, Proposition 4]. □

Lemma II.3.9 ([**Kat87a**, (3.9)])**.** *Let V be a $\mathbb{Z}^2$-valuation ring, K its field of fractions, L a finite extension of K and W the integral closure of V in L. Assume that W is a valuation ring. Let $\mathfrak{m} \supsetneq \mathfrak{p} \supsetneq (0)$ (resp. $\mathfrak{m}' \supsetneq \mathfrak{p}' \supsetneq (0)$) be all distinct prime ideals of V (resp. W). Assume further that $W/\mathfrak{p}'$ is of finite type as a $V/\mathfrak{p}$-module. Then, the following conditions are equivalent.*

(*i*) *W is a V-module of finite type.*
(*ii*) *W is a free V-module.*
(*iii*) *$[L : K] = [\kappa(\mathfrak{p}') : \kappa(\mathfrak{p})]$.*

(iii') *The discrete valuation ring* $W_{\mathfrak{p}'}$ *is weakly unramified over* $V_{\mathfrak{p}}$.

Moreover, if in addition to these conditions the extension $\kappa(\mathfrak{m}')/\kappa(\mathfrak{m})$ *is separable, then* $W = V[a]$ *for any* a *in* W *whose image in* $\kappa(\mathfrak{p}')$ *generates* $W/\mathfrak{p}'$ *over* $V/\mathfrak{p}$ *(such an* a *exists by* [Ser68, Chap. III, Proposition 12]*).*

Proof. The valuation rings V and $V/\mathfrak{p}$ have the same maximal ideal $\mathfrak{m}$, the same residue field $\kappa(\mathfrak{m})$ and $W/\mathfrak{m}W \cong (W/\mathfrak{p}')/\mathfrak{m}(W/\mathfrak{p}')$, so $[(W/\mathfrak{p}')/\mathfrak{m}(W/\mathfrak{p}') : (V/\mathfrak{p})/\mathfrak{m}] = [W/\mathfrak{m} : V/\mathfrak{m}]$. Hence, $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ follows from [Bou06, Chap. VI, § 8, n°5, Théorème 2] applied to the couples $(A, B) = (V/\mathfrak{p}, W/\mathfrak{p}')$ and $(A, B) = (V, W)$. We also see from $[L : K] = e(W_{\mathfrak{p}'}/V_{\mathfrak{p}})[\kappa(\mathfrak{p}') : \kappa(\mathfrak{p})]$ that (iii') is a rewording of (iii). The remaining assertion follows from Nakayama's lemma. □

Remark II.3.10. The finiteness condition on $W/\mathfrak{p}'$ in II.3.9 is satisfied if the field of fractions of the $\mathfrak{m}$-adic completion $\widehat{V/\mathfrak{p}}$ of $V/\mathfrak{p}$ is separable over $\kappa(\mathfrak{p})$ [EGA IV, Chap. 0, 23.1.7 (i)].

Lemma II.3.11. *Let* V *be a henselian* $\mathbb{Z}^2$*-valuation ring,* $0 \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$ *its prime ideals. Assume that the residue field* $\kappa(\mathfrak{p})$ *at* $\mathfrak{p}$ *has characteristic* $p > 0$ *and that* $[\kappa(\mathfrak{p}) : \kappa(\mathfrak{p})^p] = p$*. Then, the field of fractions* $\widehat{\kappa(\mathfrak{p})}$ *of the* $\mathfrak{m}$*-adic completion* $\widehat{V/\mathfrak{p}}$ *is a separable extension of* $\kappa(\mathfrak{p})$*. In particular, by* II.3.10*,* V *satisfies the hypotheses of* Lemma II.3.9.

Proof. By [EGA IV, Chap. 0, 21.4.1], if $\kappa(\mathfrak{p})$ has characteristic p, then every uniformizer of the discrete valuation ring $V/\mathfrak{p}$ (which is also a uniformizer of its completion) is a p-basis of $\widehat{\kappa(\mathfrak{p})}$ over $\widehat{\kappa(\mathfrak{p})}^p$. Hence, $\kappa(\mathfrak{p})(\widehat{\kappa(\mathfrak{p})}^p) = \widehat{\kappa(\mathfrak{p})}$, and $\Omega_{\widehat{\kappa(\mathfrak{p})}/\kappa(\mathfrak{p})} = 0$, and the separability claim follows from [EGA IV, Chap. 0, 20.6.3]. The last statement follows from II.3.10. □

Definition II.3.12. Let V be a valuation ring with field of fractions K. Let L be a finite extension of K and W a valuation ring of L extending V. We say that W/V is a *monogenic integral extension* of valuations rings if W is the integral closure of V in L and $W = V[a]$ for some element a of W.

Proposition II.3.13. *Let* $\mathcal{O}_K$ *be an excellent henselian discrete valuation ring with field of fractions* K*, maximal ideal* $\mathfrak{m}_K$ *and* algebraically closed *residue field* k*. Denote* $S = \operatorname{Spec}(\mathcal{O}_K)$ *and* s *the closed point of* S*. Let* $X \to S$ *be a relative curve, i.e. a separated and flat morphism of finite type with relative dimension* 1*. Let* $\overline{x} \to X$ *be a geometric point with image a closed point* x *of the special fiber* X_s*. Assume that* X *is normal at* x *and* $X - \{x\}$ *is smooth over* S*. Let* $A_{\overline{x}}$ *be the étale local ring* $\mathcal{O}_{X,\overline{x}}$ *and let* $\mathfrak{m}_{\overline{x}}$ *be its maximal ideal. Then,*

(i) $A_{\overline{x}}$ *is a normal and excellent two-dimensional Cohen-Macaulay ring.*

(ii) *The residue field* $A_{\overline{x}}/\mathfrak{m}_{\overline{x}}$ *is isomorphic to* k *and the quotient* $A_{\overline{x}}/\mathfrak{m}_K A_{\overline{x}}$ *is reduced.*

(iii) *Let* K' *be a finite extension of* K*,* $S' = \operatorname{Spec}(\mathcal{O}_{K'})$*,* s' *the closed point of* S' *and* X'/S' *the base change of* X/S*. Since* k *is algebraically closed, the special fibers* X_s *and* $X'_{s'}$ *are canonically isomorphic. Let* $\overline{x}' \to X'$ *be the lift of* $\overline{x}$*. Then,* X' *is normal at* x' *and* $X' - \{x'\}$ *is smooth over* S'*. Moreover* $A'_{\overline{x}'}$ *satisfies*

$$A'_{\overline{x}'} \cong A_{\overline{x}} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \tag{II.3.13.1}$$

Proof. (i) Since S is excellent and X is of finite type over S, X is excellent, hence $\mathcal{O}_{X,x}$ is also excellent. Then, $A_{\overline{x}}$ is excellent [EGA IV, 18.7.6]. By the permanence properties of strict henselization, $A_{\overline{x}}$ is a two dimensional noetherian normal local ring, hence an integrally closed domain. So $A_{\overline{x}}$ is Cohen-Macaulay [EGA IV, Discussion below 16.5.1].

(ii) The residue field $A_{\overline{x}}/\mathfrak{m}_{\overline{x}}$ is also the residue field of $\mathcal{O}_{X,x}$ which is an algebraic extension of k, hence is k. The assumption that $X - \{x\}$ is smooth implies that $A_{\overline{x}}/\mathfrak{m}_K A_{\overline{x}}$ is reduced.

(*iii*) Since $X - \{x\}$ is smooth over S, $X' - \{x'\}$ is smooth over S' by base change. The ring $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ is local because its maximal ideals are the closed points of

$$\operatorname{Spec}\left((\mathcal{O}_{X,x} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'})/\mathfrak{m}_{K'}\right) \xrightarrow{\sim} \operatorname{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}_K). \tag{II.3.13.2}$$

Hence, it follows that

$$\mathcal{O}_{X',x'} \cong \mathcal{O}_{X,x} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}. \tag{II.3.13.3}$$

By the same argument, the ring $A_{\overline{x}} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ is also local; and, being a finite algebra over the henselian ring $A_{\overline{x}}$, it is also henselian, which proves (II.3.13.1). The normality of $\mathcal{O}_{X',x'}$ follows from Serre's criterion [**Ser97**, IV.D.4, Théorème 11] as follows. Being smooth, $X' - \{x'\}$ is regular; if y' is a height 1 prime ideal of $\mathcal{O}_{X',x'}$, it is also a codimension 1 point of $X' - \{x'\}$ and the localization of $\mathcal{O}_{X',x'}$ at y is $\mathcal{O}_{X',y'}$ which is regular. Thus $\mathcal{O}_{X',x'}$ is (R_1). It is also (S_2) because it has the same depth as $\mathcal{O}_{X,x}$, as shown by the following argument. Let$f_1, \dots, f_r$ be generators of the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$ and denote by $\mathfrak{m}_{x'}$ the maximal ideal of $\mathcal{O}_{X',x'}$. Then, we have $V((f_1, \dots, f_r)\mathcal{O}_{X',x'}) = V(m_{x'})$ (II.3.13.3) and thus, by [**SGA 2**, II.5], the local cohomology group $H^q_{\mathfrak{m}_{x'}}(\mathcal{O}_{X',x'})$ is the q-th cohomology of the complex

$$C^\bullet(\mathcal{O}_{X',x'}, f_\bullet) : 0 \to \mathcal{O}_{X',x'} \to \prod_i \mathcal{O}_{X',x'}[\frac{1}{f_i}] \to \prod_{i<j} \mathcal{O}_{X',x'}[\frac{1}{f_i f_j}] \to \cdots \to \mathcal{O}_{X',x'}[\frac{1}{f_1 \cdots f_r}] \to 0.$$

By (II.3.13.3) and faithful flatness of $\mathcal{O}_{K'}$ over $\mathcal{O}_K$, $C^\bullet(\mathcal{O}_{X',x'}, f_\bullet)$ is isomorphic to $C^\bullet(\mathcal{O}_{X,x}, f_\bullet) \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ and this induces the isomophism

$$H^q_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \xrightarrow{\sim} H^q_{\mathfrak{m}_{x'}}(\mathcal{O}_{X',x'}). \tag{II.3.13.4}$$

We conclude by [**SGA 2**, III.3.4] (and faithful flatness of $\mathcal{O}_K \to \mathcal{O}_{K'}$) that $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X',x'}$ have the same depth (which is 2 since $\mathcal{O}_{X,x}$ is Cohen-Macaulay). □

II.3.14. Let $\mathcal{O}_K$ be an excellent henselian discrete valuation ring with field of fractions K and *algebraically closed* residue field k. Set $S = \operatorname{Spec}(\mathcal{O}_K)$, with closed s, and let X/S be a relative curve. Let $\overline{x} = \operatorname{Spec}(k) \to X$ be a geometric point with closed image in X_s. Assume that the couple $(X/S, \overline{x})$ satisfies the following property

(P) X is normal at x and $X - \{x\}$ is smooth over S.

Let $X_{(\overline{x})} = \operatorname{Spec}(A_{\overline{x}})$ be the strict localization of X at $\overline{x}$. Let $\overline{y}$ be a geometric generic point of X_s and let $\overline{y} \rightsquigarrow \overline{x}$ be a specialization map, i.e an X-morphism $\overline{y} \to X_{(\overline{x})}$. Its image corresponds to a height 1 prime ideal $\mathfrak{p}$ of $A_{\overline{x}}$, and it factors through an X-morphism $X_{(\overline{y})} \to X_{(\overline{x})}$. The corresponding homomorphism $A_{\overline{x}} \to \mathcal{O}_{X,\overline{y}}$ factors as $A_{\overline{x}} \to (A_x)_{\mathfrak{p}} \to \mathcal{O}_{X,\overline{y}}$ which induces an isomorphism $(A_{\overline{x}})^{\mathrm{sh}}_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{O}_{X,\overline{y}}$ [**SGA 4**, VIII, 7.6]. It is clear that $(A_{\overline{x}})_{\mathfrak{p}}$ is a discrete valuation ring by II.3.13 (*i*). We define $V_X(\overline{y} \rightsquigarrow \overline{x})$ to be the subring of $(A_{\overline{x}})_{\mathfrak{p}}$ consisting of elements whose images in the residue field $\kappa(\mathfrak{p})$ belong to the normalization of $A_{\overline{x}}/\mathfrak{p}$ in $\kappa(\mathfrak{p})$. Since $A_{\overline{x}}$ is excellent (II.3.13 (*i*)), it is a universally Japanese ring [**EGA IV**, 7.8.3 (vi)]. Thus $A_{\overline{x}}/\mathfrak{p}$ is Japanese. The normalization of $A_{\overline{x}}/\mathfrak{p}$ is therefore a finite algebra over the henselian ring $A_{\overline{x}}/\mathfrak{p}$ and is also an integral domain, hence it is a discrete valuation ring. Therefore $V_X(\overline{y} \rightsquigarrow \overline{x})$ is a normalized $\mathbb{Z}^2$-valuation ring of $\operatorname{Frac}(A_{\overline{x}})$ by II.3.15 below.

Lemma II.3.15. *Let L be a valuation field, with valuation ring R, maximal ideal $\mathfrak{p}$, residue field F and value group Γ_R. We assume that F is also a valuation field with valuation ring $\overline{R}$ and value group $\Gamma_{\overline{R}}$. We put*

$$V = \{x \in R \mid (x \bmod \mathfrak{p}) \in \overline{R}\}. \tag{II.3.15.1}$$

(i) *The subset V of R is a valuation ring of L whose value group Γ_V contains $\Gamma_{\overline{R}}$ as an isolated subgroup, and we have an order-preserving short exact sequence of ordered abelian groups*

$$0 \to \Gamma_{\overline{R}} \to \Gamma_V \to \Gamma_R \to 0. \tag{II.3.15.2}$$

(ii) *If R is a discrete valuation ring, the choice of a uniformizer π of R induces a canonical splitting of* (II.3.15.2)

$$\Gamma_V \cong \mathbb{Z} \oplus \Gamma_{\overline{R}} \tag{II.3.15.3}$$

which is an isomorphism of ordered abelian groups.

(iii) *Assume that R and $\overline{R}$ are discrete valuation rings with normalized valuation maps $v_R : L^\times \to \mathbb{Z}$ and $v_{\overline{R}} : F^\times \to \mathbb{Z}$ respectively. Then, V is a $\mathbb{Z}^2$-valuation ring and $\mathfrak{p}$ is its height 1 prime ideal. For a given uniformizer π of R, the normalization map of V induced by* (II.3.15.3) *is*

$$v : L^\times \to \mathbb{Z}^2, \quad x \mapsto (v_R(x), v_{\overline{R}}(x\pi^{-v_R(x)} \mod \mathfrak{p})) \tag{II.3.15.4}$$

and it coincides with the normalized $\mathbb{Z}^2$-valuation map of V induced by π (II.3.7.1).

Proof. (i) Clearly V is a subring of R. If a and b are elements of R with b non zero, then $a/b = ac/bc$ for any non zero element c of $\mathfrak{p}$, and $\mathfrak{p} \subsetneq V$ (and is clearly a prime ideal of V). Thus, the field of fractions of V is L. Now let $x \in L^\times$. We suppose x is not in R' and show that x^{-1} is in V. If x is not in R, then x^{-1} is a non unit element of R, i.e an element of $\mathfrak{p} \subsetneq V$. If x is in $R - V$, then it a unit in R and $\overline{x} = x \mod \mathfrak{m}$ is not in $\overline{R}$, so $\overline{x}^{-1} = \overline{x^{-1}}$ is in $\overline{R}$, hence x^{-1} is in V. This proves that V is a valuation ring of L. The reduction map $R^\times \to F^\times$ sends $V^\times$ in $\overline{R}^\times$ and the induced quotient map $R^\times/V^\times \to F^\times/\overline{R}^\times$ is clearly an isomorphism of ordered groups. Hence, we have an order-preserving injective homomorphism $F^\times/\overline{R}^\times = \Gamma_{\overline{R}} \hookrightarrow \Gamma_V = L^\times/V^\times$. Then, by [**Bou06**, VI, § 4, n°3, Remarque], the induced short exact sequence (II.3.15.2) is order-preserving.

(ii) If R is a discrete valuation ring with uniformizer π, then the group homomorphism $\mathbb{Z} \to \Gamma_{R'}$ sending 1 to π gives the splitting $\Gamma_V \cong \mathbb{Z} \oplus \Gamma_{\overline{R}}$ of (II.3.15.2).

(ii) By the construction of V, we have $V/\mathfrak{p} = \overline{R}$. Therefore, if, for a totally ordered group G, $\mathrm{ht}(G)$ denotes the height of G, i.e. the number of its isolated proper subgroups, [**Bou06**, VI, §, n°4, Prop. 5] yields $\mathrm{ht}(\Gamma_V) = \mathrm{ht}(\Gamma_R) + \mathrm{ht}(\Gamma_{\overline{R}})$. In particular, if R and $\overline{R}$ are discrete valuation rings, then $\mathrm{ht}(\Gamma_V) = 2 = \mathrm{rk}(\Gamma_V)$ [**Bou06**, V, §10, n° 2, Prop. 3, Cor.] and thus Γ_V is isomorphic to $\mathbb{Z}^2$ with the lexicographic order [**Bou06**, V, §10, n° 2, Prop. 4]. Also, as a prime of V, $\mathfrak{p}$ is clearly of height 1. That the normalized valuation of V is the map (II.3.15.4) follows from the definition of V (II.3.15.1); combined with the splitting (II.3.15.3), it yields the last claim since $v(\pi) = (1, 0)$ and $v(\varepsilon_L) = (0, 1)$ (notation of II.3.7). □

II.3.16. Let $\mathcal{O}_K$ be an excellent henselian discrete valuation ring with field of fractions K and *algebraically closed* residue field k. Following Kato [**Kat87a**, 5.5], we denote by $\mathcal{C}_K$ the category whose objects are the rings A such that A is isomorphic over $\mathcal{O}_K$ to $A_{\overline{x}}$ for some couple $(X/S, \overline{x})$ satisfying property (P) in II.3.14, and whose morphisms are finite $\mathcal{O}_K$-homomorphisms inducing separable extensions of fractions fields. In particular, morphisms in $\mathcal{C}_K$ are injective local homomorphisms.

Proposition II.3.17. *Let $\mathcal{O}_K$ be a complete discrete valuation ring with field of fractions K, maximal ideal $\mathfrak{m}_K$ and* algebraically closed *residue field k. Let $\mathcal{S}$ be* $\mathrm{Spf}(\mathcal{O}_K)$, *$s$ its unique point π a uniformizer of $\mathcal{O}_K$. Let $\mathfrak{X}/\mathcal{S}$ be a formal relative curve* (see II.2.20) *and $\overline{x}$ a geometric point of*

$\mathfrak{X}$ with image a closed point x of the special fiber $\mathfrak{X}_s$. Assume that $\mathfrak{X} - \{x\}$ is smooth over $\mathcal{S}$ and $\mathfrak{X}$ is normal at x. Let $A_{\overline{x}}$ be the formal étale local ring $\mathcal{O}_{\mathfrak{X},\overline{x}}$ (II.2.5.1) and let $\mathfrak{m}_{\overline{x}}$ be its maximal ideal. Then,

(i) *The local ring $A_{\overline{x}}$ is normal and two-dimensional, thus Cohen-Macaulay.*
(ii) *The residue field $A_{\overline{x}}/\mathfrak{m}_{\overline{x}}$ is isomorphic to k and the ring $A_{\overline{x}}/\mathfrak{m}_K A_{\overline{x}}$ is reduced and excellent.*
(iii) *Let K' be a finite extension of K, $\mathcal{S}' = \mathrm{Spf}(\mathcal{O}_{K'})$; s' the unique point of $\mathcal{S}'$ and $\mathfrak{X}'/\mathcal{S}'$ the formal base change of $\mathfrak{X}/\mathcal{S}$. Since k is algebraically closed, the special fibers $\mathfrak{X}_s$ and $\mathfrak{X}'_{s'}$ are canonically isomorphic. Let $x' \in \mathfrak{X}'_{s'}$ be the image of x. Then, the couple $(\mathfrak{X}'/\mathcal{S}', x')$ satisfies the property (P). Moreover, the $\mathfrak{m}_K$-adic completion $\widehat{A'_{\overline{x}'}}$ of $A'_{\overline{x}'}$ satisfies*

$$\widehat{A'_{\overline{x}'}} \cong A_{\overline{x}} \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{K'}, \tag{II.3.17.1}$$

where $\widehat{\otimes}$ denotes the $\mathfrak{m}_K$-completed tensor product.

Proof. (*i*) We can assume that $\mathfrak{X} = \mathrm{Spf}(A)$ is affine, where A is an $\mathfrak{m}_K$-adic $\mathcal{O}_K$-algebra which is topologically of finite type [**EGA I**, 10.13.4]. Then, x corresponds to an open prime ideal $\mathfrak{p}$ of A. On the one hand, since $A/\mathfrak{m}_K$ is of finite type over the excellent ring $\mathcal{O}_K$, it is an excellent ring [**EGA IV**, 7.8.3 (ii)]. Since A is also $\mathfrak{m}_K$-adically complete, it follows from a result of Gabber [**ILO14**, I, Théorème 9.2] that A is itself quasi-excellent. Hence, $A_{\mathfrak{p}}$ is also quasi-excellent [**ILO14**, I, Théorème 5.1] and thus its strict henselization $A_{\mathfrak{p}}^{\mathrm{sh}}$ is quasi-excellent [**ILO14**, I, Théorème 8.1 (iii)], hence excellent [**ILO14**, I, Corollaire 6.3 (ii)]. On the other hand, since $A_{\mathfrak{p}} \to \mathcal{O}_{\mathfrak{X},x}$ is faithfully flat (II.2.4) and $\mathcal{O}_{\mathfrak{X},x}$ is normal, $A_{\mathfrak{p}}$ is also normal by [**EGA IV**, 6.5.4]. Hence, $A_{\mathfrak{p}}^{\mathrm{sh}}$ is a normal and excellent local ring. As the $\mathfrak{m}_K$-adic completion $\widehat{A_{\mathfrak{p}}^{\mathrm{sh}}} \to \widehat{\mathcal{O}_{\mathfrak{X},\overline{x}}}$ is an isomorphism (II.2.8), it thus follows that $\widehat{\mathcal{O}_{\mathfrak{X},\overline{x}}}$ is normal [**EGA IV**, 7.8.3 (v)] and excellent [**ILO14**, I, 9.1 (i)]. Now, as $\mathcal{O}_{\mathfrak{X},\overline{x}} \to \widehat{\mathcal{O}_{\mathfrak{X},\overline{x}}}$ is flat and local, hence faithfully flat, we conclude that $A_{\overline{x}} = \mathcal{O}_{\mathfrak{X},\overline{x}}$ is also normal [**EGA IV**, 6.5.4]. By (II.2.8.2), $\mathcal{O}_{\mathfrak{X},\overline{x}}/\mathfrak{m}_K\mathcal{O}_{\mathfrak{X},\overline{x}} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}_s,\overline{x}}$; hence, $\dim(A_{\overline{x}}) = \dim(\mathcal{O}_{\mathfrak{X}_s,\overline{x}}) + 1 = 2$ [**EGA IV**, Chap. 0, 16.3.4]. It then follows from [**EGA IV**, Chap. 0, discussion below 16.5.1] that $A_{\overline{x}}$ is also a Cohen-Macaulay ring.

(*ii*) We can assume that $\mathfrak{X} = \mathrm{Spf}(A)$ is affine. From II.2.8, we see that the residue field $\kappa(\overline{x})$ of $A_{\overline{x}}$ is algebraic over the residue field of $A_{\mathfrak{p}}$. The latter is k since x is a closed point, hence $\kappa(\overline{x}) \cong k$. From the definition (II.2.5.1), we see that

$$A_{\overline{x}}/\mathfrak{m}_K A_{\overline{x}} \cong A_{\overline{x}} \otimes_{\mathcal{O}_K} k \xrightarrow{\sim} \varinjlim_{\mathrm{Spf}(B)\in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}(\mathfrak{X})} B/\mathfrak{m}_K B. \tag{II.3.17.2}$$

Since $\mathrm{Spf}(B)$ is étale over $\mathfrak{X}$ which is smooth outside x, $\mathrm{Spf}(B)$ is smooth outside the inverse image of x. We deduce that $B/\mathfrak{m}_K B$ is reduced, hence $A_{\overline{x}}/\mathfrak{m}_K A_{\overline{x}}$ is also reduced. From (II.2.8.2), we also have $A_{\overline{x}}/\mathfrak{m}_K A_{\overline{x}} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}_s,\overline{x}}$. Since $\mathfrak{X}_s$ is of finite type over over k [**EGA I**, 10.13.1], its local ring $\mathcal{O}_{\mathfrak{X}_s,x}$ is excellent, hence $A_{\overline{x}}/\mathfrak{m}_K A_{\overline{x}}$ is excellent [**EGA IV**, 18.7.6].

(*iii*) Since $\mathfrak{X} - \{x\}/\mathcal{S}$ is smooth, so is $\mathfrak{X}' - \{x'\}/\mathcal{S}'$. From (II.2.7.1) we have

$$A'_{\overline{x}'} = \varinjlim_{\mathcal{U}'_s \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}'}(\mathfrak{X}'_s)} \mathcal{O}_{\mathfrak{X}'}(\mathcal{U}'_s). \tag{II.3.17.3}$$

Now from the isomorphism of special fibers $\mathfrak{X}'_s \simeq \mathfrak{X}_s$, we deduce

$$A'_{\overline{x}'} \cong \varinjlim_{\mathcal{U} \in \mathrm{Nbh}^{\mathrm{aff}}_{\overline{x}}} \left(\mathcal{O}_{\mathfrak{X}}(\mathcal{U}) \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{K'}\right). \tag{II.3.17.4}$$

Completing $\mathfrak{m}_K$-adically (and using the fact that colimits commute with quotients and tensor products), we get (II.3.17.1). □

II.3.18. Let $\mathcal{O}_K$ be a complete discrete valuation ring with field of fractions K and *algebraically closed* residue field k, let π be a uniformizer. Let $\mathfrak{X}/\mathcal{S}$ be a formal relative curve and $\overline{x}$ a geometric point of $\mathfrak{X}$ with image a closed point x of the special fiber $\mathfrak{X}_s$. Assume that the couple $(\mathfrak{X}/\mathcal{S}, \overline{x})$ satisfies the following property

$$(P) \quad \mathfrak{X} - \{x\} \text{ is smooth over } \mathcal{S} \text{ and } \mathfrak{X} \text{ is normal at } x.$$

Let $\overline{y}$ be a geometric generic point of $\mathfrak{X}_s$. Then, the noetherian local ring $\mathcal{O}_{\mathfrak{X},\overline{y}}$ is normal by the same argument for II.3.17 (i), with maximal ideal $\mathfrak{p}_{\overline{y}} = \pi\mathcal{O}_{\mathfrak{X},\overline{y}}$. Indeed, modulo π, it is the field $\mathcal{O}_{\mathfrak{X}_s,\overline{y}}$ (II.2.8.2). Hence, it is a discrete valuation ring. Let $\overline{y} \to (\mathfrak{X}_s)_{(\overline{x})}$ be a specialization map. It induces a homomorphism $A_{\overline{x}} \to \mathcal{O}_{\mathfrak{X},\overline{y}}$ (II.2.14.2). The inverse image $\mathfrak{p}$ of $\mathfrak{p}_{\overline{y}}$ is a height 1 prime ideal of $A_{\overline{x}}$. Since $A_{\overline{x}}$ is normal (II.3.17 (i)), we hence see that $(A_{\overline{x}})_{\mathfrak{p}}$ is a discrete valuation ring. Moreover, $A_{\overline{x}}/\mathfrak{p}$ is Japanese since it is a quotient of the excellent ring $A_{\overline{x}}/\mathfrak{m}_K A_{\overline{x}}$ (II.3.17 (ii)). Hence, the normalization of $A_{\overline{x}}/\mathfrak{p}$ in $\kappa(\mathfrak{p})$ is a finite algebra over $A_{\overline{x}}/\mathfrak{p}$. We define $V_{\mathfrak{X}}(\overline{y} \rightsquigarrow \overline{x})$ to be the subring of $(A_{\overline{x}})_{\mathfrak{p}}$ consisting of elements whose images in the residue field $\kappa(\mathfrak{p})$ belong to this normalization. By the same argument as in II.3.14, we conclude that $V_{\mathfrak{X}}(\overline{y} \rightsquigarrow \overline{x})$ is a normalized $\mathbb{Z}^2$-valuation ring.

Remark II.3.19. The construction of $V_X(\overline{y} \rightsquigarrow \overline{x})$ in II.3.14 (resp. $V_{\mathfrak{X}}(\overline{y} \rightsquigarrow \overline{x})$ in II.3.18) is functorial in the following sense. Let $(X/S, \overline{x})$ and $(X'/S, \overline{x}')$ (resp. $(\mathfrak{X}/\mathcal{S}, \overline{x})$ and $(\mathfrak{X}'/\mathcal{S}, \overline{x}')$) be couples as in II.3.14 (resp. II.3.18) satisfying property (P) therein, $\overline{y} \to X_s$ and $\overline{y}' \to X'_s$ (resp. $\overline{y} \to \mathfrak{X}_s$ and $\overline{y}' \to \mathfrak{X}'_s$) geometric generic points, $\overline{y} \rightsquigarrow \overline{x}$ and $\overline{y}' \rightsquigarrow \overline{x}'$ specialization maps. Let $X' \to X$ (resp. $\mathfrak{X}' \to \mathfrak{X}$) be an S-morphism (resp. an adic $\mathcal{S}$-morphism) compatible with the geometric points and the specialization maps in the sense that we have the following commutative diagram

(II.3.19.1)
$$\begin{array}{ccccc} \overline{y}' & \longrightarrow & X'_{(\overline{x}')} & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ \overline{y} & \longrightarrow & X_{(\overline{x})} & \longrightarrow & X. \end{array}$$

(resp. a similar diagram with $\mathfrak{X}_s$ and $\mathfrak{X}'_s$ in lieu of X and X' respectively). Then, we get an extension of $\mathbb{Z}^2$-valuation rings $V_X(\overline{y} \rightsquigarrow \overline{x}) \to V_{X'}(\overline{y}' \rightsquigarrow \overline{x}')$ (resp $V_{\mathfrak{X}}(\overline{y} \rightsquigarrow \overline{x}) \to V_{\mathfrak{X}'}(\overline{y}' \rightsquigarrow \overline{x}')$.

II.3.20. Let $\mathcal{O}_K$ be a complete discrete valuation ring with field of fractions K and *algebraically closed* residue field k. We denote by $\widehat{\mathcal{C}}_K$ the category whose objects are the rings A such that A is isomorphic over $\mathcal{O}_K$ to $A_{\overline{x}}$ for some couple $(\mathfrak{X}/S, \overline{x})$ satisfying property (P) in II.3.18, and whose morphisms are finite $\mathcal{O}_K$-homomorphisms inducing separable extensions of fractions fields. In particular, morphisms in $\widehat{\mathcal{C}}_K$ are injective local homomorphisms.

II.3.21. We keep the notation of II.3.16 (resp. II.3.20). Let A be an object of $\mathcal{C}_K$ (resp. $\widehat{\mathcal{C}}_K$) with field of fractions $\mathbb{K}$ and $\mathfrak{p}$ a height 1 prime ideal of A above $\mathfrak{m}_K$. The localization map $A \to A_{\mathfrak{p}}$ induces a homomorphism $A/\mathfrak{m}_K A \to (A_{\mathfrak{p}}/\mathfrak{m}_K A_{\mathfrak{p}})^{\mathrm{sh}}$, where the strict henselization is with respect to a separable closure $\kappa(\mathfrak{p})^{\mathrm{sep}}$ of the residue field $\kappa(\mathfrak{p})$ of $A_{\mathfrak{p}}$, and $\mathfrak{p}$ corresponds to a minimal prime ideal of $A/\mathfrak{m}_K A$. This yields a geometric generic point and a specialization map

(II.3.21.1)
$$\overline{y} = \mathrm{Spec}(\kappa(\mathfrak{p})^{\mathrm{sep}}) \to \mathrm{Spec}\left((A_{\mathfrak{p}}/\mathfrak{m}_K A_{\mathfrak{p}})^{\mathrm{sh}}\right) \to \mathrm{Spec}(A/\mathfrak{m}_K A).$$

Indeed, if A is isomorphic to $A_{\overline{x}}$ for some couple $(X/S, \overline{x})$ (resp. $(\mathfrak{X}/\mathcal{S}, \overline{x})$) satisfying property (P) in II.3.14 (resp. II.3.18), then (II.3.21.1) translates as $\overline{y} \to (X_s)_{(\overline{x})} \to X_s$ (resp. $\overline{y} \to (\mathfrak{X}_s)_{(\overline{x})} \to \mathfrak{X}_s$). In conclusion, the construction of $V_X(\overline{y} \rightsquigarrow \overline{x})$ in II.3.14 (resp. $V_{\mathfrak{X}}(\overline{y} \rightsquigarrow \overline{x})$ in II.3.18) depends only on the choice of a couple $(A, \mathfrak{p})$, where A is an object of $\mathcal{C}_K$ (resp. $\widehat{\mathcal{C}}_K$) corresponding to $(X/S, \overline{x})$ (resp. $(\mathfrak{X}/\mathcal{S}, \overline{x})$) and $\mathfrak{p}$ a height 1 prime ideal of A above $\mathfrak{m}_K$. For such a couple $(A, \mathfrak{p})$, the normalized $\mathbb{Z}^2$-valuation ring obtained is denoted by $V_A(\mathfrak{p})$, its field of fractions by $\mathbb{K}_{\mathfrak{p}} = \mathbb{K}$ and its valuation map by $v_{\mathfrak{p}}$. By Lemma II.3.3, the henselization $V_A^h(\mathfrak{p})$ of $V_A(\mathfrak{p})$ is also a $\mathbb{Z}^2$-valuation ring; let $\mathbb{K}^h$ be its field of fractions. If π is a uniformizer of $\mathcal{O}_K$, then its image by $\mathcal{O}_K \to A_{\mathfrak{p}} = (V_A(\mathfrak{p}))_{\mathfrak{p}}$ is a uniformizer of $(V_A(\mathfrak{p}))_{\mathfrak{p}}$, and, coupled with $\varepsilon_{\mathbb{K}}$ (II.3.7), it gives an isomorphism $\Gamma_{V_A(\mathfrak{p})} \xrightarrow{\sim} \mathbb{Z}^2$ (II.3.7) and thus induces a valuation map $\mathbb{K}^{h\times} \to \mathbb{Z}^2$ (II.3.3) which coincides with the normalized valuation map of $V_A^h(\mathfrak{p})$ (II.3.7.1). We note that, as the chosen uniformizer π comes from K, these normalized valuation maps don't depend on π.

II.3.22. We keep the notation of II.3.16 (resp. II.3.20) and let A be an object of $\mathcal{C}_K$ (resp. $\widehat{\mathcal{C}}_K$). By II.3.13 (i) (resp. II.3.17 (i)), the ring $A_K = A \otimes_{\mathcal{O}_K} K$ is a Dedekind domain. By II.3.13 (ii) (resp II.3.17 (ii)), $A_0 = A/\mathfrak{m}_K A$ is a reduced ring. We denote by $\widetilde{A_0}$ the normalization of A_0 in its total ring of fractions. We denote by $\delta(A)$ the dimension of the quotient $\widetilde{A_0}/A_0$ of k-vector spaces.

For a finite extension K'/K, $A' = A \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ (resp. $A \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{K'}$) is an object of $\mathcal{C}_{K'}$ (II.3.13.1) (resp. $\widehat{\mathcal{C}}_{K'}$ (resp. (II.3.17.1)) and we have

$$A/\mathfrak{m}_K A \xrightarrow{\sim} A'/\mathfrak{m}_{K'} A', \quad \text{hence} \quad \delta(A) = \delta(A'). \tag{II.3.22.1}$$

We denote by $P(A)$ the set of height 1 prime ideals of A, by $P_s(A)$ the subset of $P(A)$ of prime ideals above $\mathfrak{m}_K$ and by $P_\eta(A)$ the complement $P(A) - P_s(A)$ (of primes above 0). Since $P_s(A)$ corresponds to the generic points of $\mathrm{Spec}(A_0)$, it is a finite set. We note also that $P_\eta(A)$ identifies with the set of maximal ideals of the Dedekind domain $A_K = A_{(0)}$. For $\mathfrak{p} \in P_\eta(A)$, we denote by $\mathrm{ord}_{\mathfrak{p}}$ the discrete valuation defined by $A_{\mathfrak{p}} = (A_K)_{\mathfrak{p}}$ and note that the residue field $\kappa(\mathfrak{p})$ is a finite extension of K.

Let $\varphi : A \to B$ be a morphism in $\mathcal{C}_K$ (resp. $\widehat{\mathcal{C}}_K$). For $\mathfrak{p} \in P_s(A)$ and $\mathfrak{q} \in P_s(B)$ above $\mathfrak{p}$, φ induces an extension of $\mathbb{Z}^2$-valuation rings $V_A(\mathfrak{p}) \to V_B(\mathfrak{q})$.

Lemma II.3.23. *The above extension induces a monogenic integral extension of $\mathbb{Z}^2$-valuation rings $V_B^h(\mathfrak{q})/V_A^h(\mathfrak{p})$ (II.3.12). Moreover, the finitely generated $V_A^h(\mathfrak{p})$-module $V_B^h(\mathfrak{q})$ is free.*

Proof. By [**End72**, Thm 17.17], the extension of fields of fractions $\mathrm{Frac}(V_A^h(\mathfrak{p})) \to \mathrm{Frac}(V_B^h(\mathfrak{q}))$ is finite. Since $V_A^h(\mathfrak{p})$ is henselian by II.3.2, we deduce that $V_B^h(\mathfrak{q})$ is its integral closure in $\mathrm{Frac}(V_B^h(\mathfrak{q}))$. By Lemma II.3.3, $V_B^h(\mathfrak{q})/V_A^h(\mathfrak{p})$ is an extension of normalized $\mathbb{Z}^2$-valuation rings. Since B is finite over A, $B/\mathfrak{q}$ is also finite over $A/\mathfrak{p}$. As seen in II.3.14 (resp. II.3.18), $A/\mathfrak{p}$ and $B/\mathfrak{q}$ are Japanese rings, and thus $A/\mathfrak{p} \to \widetilde{A/\mathfrak{p}}$ and $B/\mathfrak{q} \to \widetilde{B/\mathfrak{q}}$ are finite. Hence, $\widetilde{A/\mathfrak{p}} \subseteq \widetilde{B/\mathfrak{q}}$ is a finite extension of henselian rings. Now, since we have

$$V_A^h(\mathfrak{p})/\mathfrak{p} = (V_A(\mathfrak{p})/\mathfrak{p})^h = (\widetilde{A/\mathfrak{p}})^h = \widetilde{A/\mathfrak{p}}, \tag{II.3.23.1}$$

and the same holds with B and $\mathfrak{q}$, this proves that the extension of rings $V_A^h(\mathfrak{p})/\mathfrak{p} \subseteq V_B^h(\mathfrak{q})/\mathfrak{q}$ is finite. The extension $(V_A(\mathfrak{p}))_{\mathfrak{p}} \subseteq (V_B(\mathfrak{q}))_{\mathfrak{q}}$ coincides with $A_{\mathfrak{p}} \subseteq B_{\mathfrak{q}}$ and is thus weakly unramified since the maximal ideals of both rings are generated by the uniformizer π of $\mathcal{O}_K$. Hence, by II.3.3, so is the extension of their henselizations $(V_A(\mathfrak{p}))_{\mathfrak{p}}^h \subseteq (V_B(\mathfrak{q}))_{\mathfrak{q}}^h$. The latter coincides with the extension $\big((V_A^h(\mathfrak{p}))_{\mathfrak{p}}\big)^h \subseteq \big((V_B^h(\mathfrak{q}))_{\mathfrak{q}}\big)^h$ [**SGA 4**, VIII, 7.6], which is thus weakly unramified. Hence,

$(V_A^h(\mathfrak{p}))_{\mathfrak{p}} \subseteq (V_B^h(\mathfrak{q}))_{\mathfrak{q}}$ is also weakly unramified by II.3.3 again. Since the residue field of $V_A^h(\mathfrak{p})$ is the algebraically closed field k, II.3.9 applies and proves our claims. □

II.3.24. Let A be a ring and B an A-algebra which is a projective A-module of finite rank r. Then, the symmetric bilinear trace map $\mathrm{tr} : B \times B \to A$, $(x, y) \mapsto \mathrm{tr}_{B/A}(xy)$ induces a morphism of A-modules

(II.3.24.1) $$T_{B/A} : \det_A(B) \otimes_A \det_A(B) \to A,$$

where $\det_A(B)$ is the invertible A-module $\bigwedge_A^r B$ [Ser68, Chap. III, §2]. If A and B are domains and the induced extension of their fields of fractions is separable, then, the image of $T_{B/A}$ is a non-zero principal ideal of A, the *discriminant ideal of* the extension B/A.

II.3.25. For the rest of this section, we resume with the notation of II.3.16 (resp. II.3.20) and let π be a uniformizer of $\mathcal{O}_K$. Let $A \to B$ be a morphism in $\mathcal{C}_K$ (resp. $\widehat{\mathcal{C}}_K$). On the one hand, since the induced homomorphism $A_K \to B_K$ is an extension of Dedekind domains (II.3.22), the K-linear map T_{B_K/A_K} is well-defined. Following Kato [Kat87a, 5.6], we define the integer

(II.3.25.1) $$d_\eta(B/A) = \dim_K(\mathrm{Coker}(T_{B_K/A_K})).$$

On the other hand, for $\mathfrak{p} \in P_s(A)$ and $\mathfrak{q} \in P_s(B)$ above $\mathfrak{p}$, $T_{V_B^h(\mathfrak{q})/V_A^h(\mathfrak{p})}$ is well defined (II.3.23). Let $c(B/A, \mathfrak{p}, \mathfrak{q}) \in V_A^h(\mathfrak{p})$ be a generator of the image of $T_{V_B^h(\mathfrak{q})/V_A^h(\mathfrak{p})}$. The choice of π induces a $\mathbb{Z}^2$ normalization $(\alpha, \beta) : \Gamma_{V_A^h(\mathfrak{p})} \xrightarrow{\sim} \mathbb{Z}^2$ (II.3.21). Then, as in [Kat87a, 5.6], we define the integer

(II.3.25.2) $$d_s(B/A) = \sum_{(\mathfrak{p},\mathfrak{q})} v_{\mathfrak{p}}^{\beta}(c(B/A, \mathfrak{p}, \mathfrak{q})),$$

where $\mathfrak{p}$ runs over $P_s(A)$ and $\mathfrak{q}$ runs over the elements of $P_s(B)$ above $\mathfrak{p}$. Since any two generators of the image of $T_{V_B^h(\mathfrak{q})/V_A^h(\mathfrak{p})}$ differ by a scalar unit of $V_A^h(\mathfrak{p})$, $d_s(B/A)$ is independent of the choices of such of generators. We note that if, for all $\mathfrak{p}$ and $\mathfrak{q}$ as above, the extension of discrete valuation rings $B_{\mathfrak{q}}/A_{\mathfrak{p}}$ is unramified (i.e. $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is a separable extension), then $v_{\mathfrak{p}}^{\beta}(c(B/A, \mathfrak{p}, \mathfrak{q}))$ is the valuation in $\widetilde{A/\mathfrak{p}}$ of a generator of the discriminant ideal of $\widetilde{B/\mathfrak{q}}$ over $\widetilde{A/\mathfrak{p}}$, where $\widetilde{A/\mathfrak{p}}$ and $\widetilde{B/\mathfrak{q}}$ are the normalizations of $A/\mathfrak{p}$ and $B/\mathfrak{q}$ in their respective fields of fractions $\kappa(\mathfrak{p})$ and $\kappa(\mathfrak{q})$; and thus $d_s(B/A)$ is the k-dimension of the cokernel of $T_{\widetilde{B_0}/\widetilde{A_0}}$, where $A_0 = A/\mathfrak{m}_K A$ and $B_0 = B/\mathfrak{m}_K B$.

Lemma II.3.26 ([Kat87a, Lemma 5.8])**.** *Let A be an object of $\mathcal{C}_K$ (resp. $\widehat{\mathcal{C}}_K$) and denote by $\mathbb{K}$ its field of fractions. Then, for any $x \in \mathbb{K}^\times$, we have*

(II.3.26.1) $$\sum_{\mathfrak{p} \in P_s(A)} v_{\mathfrak{p}}^{\beta}(x) = \sum_{\mathfrak{p} \in P_\eta(A)} [\kappa(\mathfrak{p}) : K] \mathrm{ord}_{\mathfrak{p}}(x).$$

Proof. We note first that both sums in (II.3.26.1) are finite as $P_s(A)$ is finite and A_K is a Dedekind domain with $P_\eta(A)$ identified with the set of its non-zero prime ideals. The K-theoretic proof given by Kato in *loc. cit.* when A is an object of $\mathcal{C}_K$ applies also when $A \in \mathrm{Obj}(\widehat{\mathcal{C}}_K)$. □

Proposition II.3.27 ([Kat87a, proof of (5.7)])**.** *Let $A \to B$ be a morphism in $\mathcal{C}_K$ (resp. $\widehat{\mathcal{C}}_K$). Assume that A and $A_0 = A/\mathfrak{m}_K A$ are regular. Then, B is a free A-module of finite rank and*

(II.3.27.1) $$d_\eta(B/A) = \sum_{\mathfrak{p}_\eta \in P_\eta(A)} [\kappa(\mathfrak{p}_\eta) : K] \mathrm{ord}_{\mathfrak{p}_\eta}(c),$$

(II.3.27.2) $$d_s(B/A) + 2\delta(B) = v_{\mathfrak{p}}^{\beta}(c),$$

where $\mathfrak{p}$ is the unique element of $P_s(A)$ and c is a generator in A of the image of $T_{B/A}$.

PROOF. Recall that $A \to B$ is a finite injective local homomorphism between local domains. As B is Cohen-Macaulay (II.3.17 (i)), we deduce from [**EGA IV**, Chap. 0, 17.3.5 (ii)] that B is indeed a free A-module. Let c be a generator of the image in A of the well defined homomorphism $T_{B/A}$ (II.3.24.1). Then, c is also a generator of the image of T_{B_K/A_K} in the Dedekind domain A_K whose non-zero prime ideals are the elements of $P_\eta(A)$; hence, (II.3.27.1) follows. Let n be the maximum integer such that $\mathrm{Im}(T_{B/A}) \subseteq \pi^n A$ and put $T' = \pi^{-n} T_{B/A}$. With the notation of II.3.22, let $T'_0 : \det_{A_0}(B_0) \otimes_{A_0} \det_{A_0}(B_0) \to A_0$ be the homomorphism induced by T' and let $\widetilde{T}'_0 : \det_{A_0}(\widetilde{B}_0) \otimes_{A_0} \det_{A_0}(\widetilde{B}_0) \to \mathrm{Frac}(A_0)$ be the homomorphism induced by T'_0. As π remains a uniformizer through the composition map $\mathcal{O}_K \to A_{\mathfrak{p}} = (V_A(\mathfrak{p}))_{\mathfrak{p}} \to (V_A^h(\mathfrak{p}))_{\mathfrak{p}}$ between discrete valuation rings, we see that n is also the maximum integer such that $\mathrm{Im}(T_{V_B^h(\mathfrak{q})/V_A^h(\mathfrak{p})}) \subsetneq \pi^n V_A^h(\mathfrak{p})$. It then follows from II.3.15 that $d_s(B/A)$ is the integer $i_{\mathfrak{p}}$ such that $\mathrm{Im}(\widetilde{T}'_0) = \mathfrak{m}_{A_0}^{i_{\mathfrak{p}}}$. Now from [**Ser68**, III, Prop. 5], we see that

$$2\delta(B) = \dim_k\left(\mathrm{Im}(\widetilde{T}'_0)/\mathrm{Im}(T'_0)\right). \tag{II.3.27.3}$$

We also get, looking at the definition of $v_{\mathfrak{p}}^{\beta}$, that

$$v_{\mathfrak{p}}^{\beta}(c) = \dim_k(A_0/\mathrm{Im}(T'_0)). \tag{II.3.27.4}$$

Putting together $d_s(B/A) = i_{\mathfrak{p}}$, (II.3.27.3) and (II.3.27.4) gives us the formula (II.3.27.2). □

II.4. Variation of the discriminant of a rigid morphism.

For the rest of the article, if S is a finite set, $|S|$ will denote its cardinal and the context will help not to confuse it with the absolute value here and elsewhere.

II.4.1. We let K be a complete discrete valuation field, $\mathcal{O}_K$ its valuation ring, $\mathfrak{m}_K$ its maximal ideal, k its residue field, assumed to be algebraically closed of characteristic $p > 0$, and π a uniformizer of $\mathcal{O}_K$. Let $\overline{K}$ be an algebraic closure of K, $v_K : \overline{K}^{\times} \to \mathbb{Q}$ the valuation of $\overline{K}$ normalized by $v_K(K^{\times}) = \mathbb{Z}$, and let $C = \widehat{\overline{K}}$ be the completion of $\overline{K}$ with respect to v. We denote by $D = D_K = \mathrm{Sp}(K\{\xi\})$ the rigid unit disc over K centered at the point 0 corresponding to the maximal ideal (ξ) of $K\{\xi\}$, and by $\mathfrak{D} = \mathfrak{D}_K = \mathrm{Spf}(\mathcal{O}_K\{\xi\})$ the formal unit disc, an admissible formal model of D over $\mathcal{S} = \mathrm{Spf}(\mathcal{O}_K)$, with special fiber $\mathbb{A}^1_k = \mathrm{Spec}(k[\xi])$. For $r \in \mathbb{Q}$, we denote by $D^{(r)} = D_K^{(r)}$ the 0-centered K-subdisc of D of radius $|\pi|^r$, defined by $v(\xi) \geq r$. For rational numbers $r \geq r'$, we denote by $A(r,r')$ (resp. $A^{\circ}(r,r')$) the closed (resp. open) annulus in D, centered at 0, with inner radius $|\pi|^r$ and outer radius $|\pi|^{r'}$. The affinoid K-algebra of $A(r,r')$ is the set of power series

$$f(\xi) = \sum_{i\in\mathbb{Z}} a_i \xi^i \quad \in K[[\xi, \xi^{-1}]] \tag{II.4.1.1}$$

such that $f(x)$ converges in $\overline{K}$ for any x in $A(r,r')$.

II.4.2. By the Weierstrass preparation theorem [**Hen00**, Corollaire 1.5], if a function f on $A(r,r')$ is invertible, it can be written in the form

$$f(\xi) = c\xi^d(1 + h(\xi)), \quad \text{where} \quad h(\xi) = \sum_{i\in\mathbb{Z}-\{0\}} h_i \xi^i, \tag{II.4.2.1}$$

$c \in K^\times$, $d \in \mathbb{Z}$ and h is a function on $A(r, r')$ such that $|h|_{\sup} < 1$. (Therefore, for any $i \in \mathbb{Z} - \{0\}$ and any $r \geq t \geq r'$, we have $|h_i \pi^{it}| < 1$). The integer d is called *the order of* f. After we renormalize by dividing by c in (II.4.2.1), f defines a rigid morphism from $A(r, r')$ to $A(dr, dr')$. If, moreover, f is étale, then, by the jacobian criterion, its derivative $f'(\xi) = \frac{df(\xi)}{d\xi}$ is also an invertible power series and thus has a well-defined order σ; we put

(II.4.2.2) $$\nu = \sigma - d + 1 \in \mathbb{Z}.$$

II.4.3. The open annulus $A^\circ(r, r')$ is the increasing union of the closed annuli $A(\varepsilon, \varepsilon')$, taken over the rational numbers ε and ε' satisfying $r > \varepsilon \geq \varepsilon' > r'$. Hence, the ring $\mathcal{O}(A^\circ(r, r'))$ of functions on $A^\circ(r, r')$ is the projective limit of the rings $\mathcal{O}(A(\varepsilon, \varepsilon'))$ with transition maps the restrictions to smaller annuli. If $f = (f_{\varepsilon,\varepsilon'})$ is an invertible function on $A^\circ(r, r')$, then each $f_{\varepsilon,\varepsilon'}$ is an invertible function on $A(\varepsilon, \varepsilon')$ and thus has a well defined order $d_{\varepsilon,\varepsilon'}$ (II.4.2). As the formula (II.4.2.1) is invariant under restriction, we see that $d_{\varepsilon,\varepsilon'}$ is independent of ε and ε'. Thus, it is a well-defined order of f, which we denote d. If, moreover, f is étale on $A^\circ(r, r')$, likewise, we get a well-defined order σ for its derivative, and put $\nu = \sigma - d + 1$.

Lemma II.4.4 ([**Lüt93**, Lemma 2.3])**.** *Let X be a smooth rigid space over K and $f : X \to A(r, r')$ be a finite flat morphism, étale over a nonempty open subset of $A(r, r')$. Then, there exist a finite extension K' of K, and a finite sequence of rational numbers $r = r_0 > r_1 > \cdots > r_n > r_{n+1} = r'$ in $v_K(K')$ such that, denoting by $f_{K'} : X_{K'} \to A_{K'}(r, r')$ the base change of f to K', the following holds for each $i = 1, \ldots, n+1$.*

(1) *The inverse image $\Delta_i = f_{K'}^{-1}(A^\circ_{K'}(r_{i-1}, r_i))$ decomposes into a finite disjoint union of rigid open annuli $\Delta_i = \Delta_{i1} \cup \cdots \cup \Delta_{i\delta(i)}$.*
(2) *The restriction of $f_{K'}$ to each annulus Δ_{ij} is an étale morphism*

(II.4.4.1) $$f_{ij} : \Delta_{ij} \to A^\circ(r_{i-1}, r_i), \quad \xi_{ij} \mapsto \xi_{ij}^{d_{ij}}(1 + h_{ij}),$$

where h_{ij} is a function on Δ_{ij} satisfying $|h_{ij}|_{\sup} < 1$ and $d_{ij} \geq 1$ is an integer such that $\Delta_{ij} \xrightarrow{\sim} A^\circ_{K'}(r_{i-1}/d_{ij}, r_i/d_{ij})$,
(3) *the sum $d = d_{i1} + \cdots + d_{i\delta(i)}$ is independent of i.*

Proof. The main ingredient in the proof given in *loc. cit.* is the semi-stable reduction theorem [**BL85**, Theorem 7.1]. There, the lemma is proved, over the complete algebraically closed field C, for the base change $X_C = X \widehat{\otimes}_K C$. This is achieved by using a semi-stable formal model $\mathfrak{X}_C$ of X_C over $\mathcal{O}_C$ to produce the rational numbers $r_1, \ldots, r_n$ and the étale morphisms f_{ij}. (See [**Lüt93**, Lemma 2.3] for more details.) The model $\mathfrak{X}_C$ descends to a formal model $\mathfrak{X}_{K'}$ of $X_{K'}$, for some finite extension K' of K which we can take to be large enough to contain all π^{r_i} and the coefficients defining the morphisms f_{ij}. As the latter morphisms are obtained first by restricting to Δ_i the same morphism $f_{K'}$, the independence statement of (3) follows. □

II.4.5. We keep the notation of Lemma II.4.4. From the well-defined orders $\sigma_{ij} = d_{ij} - 1 + \nu_{ij}$ of the derivatives of the f_{ij} (II.4.3), we can define the total order of the the derivative of f on Δ_i

(II.4.5.1) $$\sigma_i = \sigma_{i1} + \cdots + \sigma_{i\delta(i)} = d - \delta(i) + \nu_i, \quad \nu_i = \nu_{i1} + \cdots + \nu_{i\delta(i)}.$$

II.4.6. Let X be a smooth K-affinoid curve and let $f : X \to D$ be a finite flat morphism of degree d, which is étale over a nonempty open subset of D containing 0. Let $r \in \mathbb{Q}$, denote by $X^{(r)}$ the inverse image of $D^{(r)}$ by f, by $f^{(r)} : X^{(r)} \to D^{(r)}$ the induced morphism and set

(II.4.6.1) $$\mathcal{O}^\circ(D^{(r)}) = \{h \in \mathcal{O}_D(D^{(r)}) \mid |h|_{\sup} \leq 1\}, \quad \mathcal{O}^\circ(X^{(r)}) = \{h \in \mathcal{O}_X(X^{(r)}) \mid |h|_{\sup} \leq 1\},$$

for the unit balls of the affinoid algebras $\mathcal{O}(D^{(r)})$ and $\mathcal{O}(X^{(r)})$. If K' is a finite extension of K, $D^{(r)}_{K'}$ (resp. $X^{(r)}_{K'}$) denotes the base change $D^{(r)}\widehat{\otimes}_K K'$ (resp. $X^{(r)}\widehat{\otimes}_K K'$). We denote by D_C, X_C and $f_C : X_C \to D_C$ the base change to C of D, X and f respectively.

Lemma II.4.7. *The unit balls $\mathcal{O}^\circ(D^{(r)})$ and $\mathcal{O}^\circ(X^{(r)})$ are (admissible) π-adic rings and the homomorphism $\mathcal{O}^\circ(D^{(r)}) \to \mathcal{O}^\circ(X^{(r)})$ induced by $f^{(r)}$ is finite and generically étale.*

Proof. Indeed, writing $r = \frac{a}{b}$ with a, b integers, we have $\mathcal{O}^\circ(D^{(r)}) = \mathcal{O}_K\{\xi, \zeta\}/(\xi^b - \zeta\pi^a)$. Being topologically of finite type over $\mathcal{O}_K$, $\mathcal{O}^\circ(D^{(r)})$ is an adic ring. Since f is finite, so is $f^{(r)}$. As $D^{(r)}$ is reduced, [**BGR84**, 6.4.1/6] implies that $\mathcal{O}^\circ(X^{(r)})$ is finite over $\mathcal{O}^\circ(D^{(r)})$, hence also an adic ring [**Abb10**, 1.8.29]. The generic étaleness follows from the étaleness hypothesis on f. □

Lemma II.4.8. *With the notation of II.4.6, $\mathcal{O}^\circ(X^{(r)}_C)$ is a free $\mathcal{O}^\circ(D^{(r)}_C)$-module of finite rank.*

Proof. As $\mathcal{O}^\circ(D^{(r)}_C)/\mathfrak{m}_C\mathcal{O}^\circ(D^{(r)}_C) \cong k[\xi/\pi^r]$ and $\mathcal{O}^\circ(X^{(r)}_C)/\mathfrak{m}_C\mathcal{O}^\circ(X^{(r)}_C)$ are the rings of the special fibers of the normalized integral models of $D^{(r)}_C$ and $X^{(r)}_C$ respectively (II.2.19), they are reduced (II.2.18). Hence, they coincide with $\mathcal{O}^\circ(D^{(r)}_C)/\mathcal{O}^{\circ\circ}(D^{(r)}_C)$ and $\mathcal{O}^\circ(X^{(r)}_C)/\mathcal{O}^{\circ\circ}(X^{(r)}_C)$ respectively, where $\mathcal{O}^{\circ\circ}(D^{(r)}_C)$ and $\mathcal{O}^{\circ\circ}(X^{(r)}_C)$ are the sets of all topologically nilpotent elements of $\mathcal{O}(D^{(r)}_C)$ and $\mathcal{O}(X^{(r)}_C)$ respectively. It follows that $\mathcal{O}^\circ(X^{(r)}_C)/\mathfrak{m}_C\mathcal{O}^\circ(X^{(r)}_C)$ is a finitely generated module over $k[\xi/\pi^r]$ [**BGR84**, Theorem 6.3.4/2]. It is also flat by the following argument. As $X^{(r)}_C \to D^{(r)}_C$ is finite and flat, it is surjective and thus its image is not sent to the tube of a point of the special fiber of the formal model $\mathrm{Spf}(\mathcal{O}^\circ(D^{(r)}_C))$ of $D^{(r)}_C$. Hence, the homomorphism $k[\xi/\pi^r] \to \mathcal{O}^\circ(X^{(r)}_C)/\mathfrak{m}_C\mathcal{O}^\circ(X^{(r)}_C)$ is injective. To see that it is flat, it is enough to show that every localization of $\mathcal{O}^\circ(X^{(r)}_C)/\mathfrak{m}_C\mathcal{O}^\circ(X^{(r)}_C)$ at a prime ideal is Cohen-Macaulay [**EGA IV**, Chap 0, 17.3.5], hence it is enough to show that every localization of $\mathcal{O}^\circ(X^{(r)}_C)$ at a prime ideal is Cohen-Macaulay [**EGA IV**, Chap 0, 16.5.5]. By II.2.18 and II.2.17, the latter ring is integrally closed in the normal ring $\mathcal{O}(X^{(r)}_C)$, hence it is normal. Thus, its localizations at prime ideals are two-dimensional normal rings, hence Cohen-Macaulay [**EGA IV**, Chap. 0, discussion below 16.5.1]. It follows that $\mathcal{O}^\circ(X^{(r)}_C)/\mathfrak{m}_C\mathcal{O}^\circ(X^{(r)}_C)$ is a torsion-free $k[\xi/\pi^r]$-module. Therefore, $\mathcal{O}^\circ(X^{(r)}_C)/\mathfrak{m}_C\mathcal{O}^\circ(X^{(r)}_C)$ is free of finite rank over $k[\xi/\pi^r]$. Then, by [**BGR84**, 6.4.2/3], $\mathcal{O}^\circ(X^{(r)}_C)$ is also free of finite type over $\mathcal{O}^\circ(D^{(r)}_C)$. □

Proposition II.4.9. *There exists a finite extension K' of K, containing an element π^r of valuation r, such that*

(1) *$\mathcal{O}^\circ(X^{(r)}_{K'})$ is a finite free $\mathcal{O}^\circ(D^{(r)}_{K'})$-module.*
(2) *$\mathfrak{D}^{(r)}_{K'} = \mathrm{Spf}\left(\mathcal{O}^\circ(D^{(r)}_{K'})\right)$ and $\mathfrak{X}^{(r)}_{K'} = \mathrm{Spf}\left(\mathcal{O}^\circ(X^{(r)}_{K'})\right)$ are admissible formal models over $\mathcal{S}' = \mathrm{Spf}(\mathcal{O}_{K'})$ of $D^{(r)}_{K'}$ and $X^{(r)}_{K'}$ respectively, with geometrically reduced special fibers.*

Moreover, if K' is a finite extension of K, containing an element π^r of valuation r, satisfying (2), then we also have

(3) *$\mathfrak{D}^{(r)}_{K'}$ is smooth over $\mathcal{S}'$; $\mathfrak{X}^{(r)}_{K'}$ is smooth over $\mathcal{S}'$ outside a finite number of closed points in its special fiber, and is normal.*
(4) *If K'' is a finite extension of K' and $\mathcal{S}'' = \mathrm{Spf}(\mathcal{O}_{K''})$, then*

$$\mathfrak{X}^{(r)}_{K''} \cong \mathfrak{X}^{(r)}_{K'} \times_{\mathcal{S}'} \mathcal{S}''. \qquad \text{(II.4.9.1)}$$

PROOF. The K-affinoid spaces $D_K^{(r)}$ and $X_K^{(r)}$ are smooth, hence geometrically reduced. Then, by Theorem II.2.18, we see that there exists a finite extension K' of K such that $\mathcal{O}^\circ(D_{K'}^{(r)})$ and $\mathcal{O}^\circ(X_{K'}^{(r)})$ have geometrically reduced special fibers and and their formation commutes with finite extensions K''/K', namely

$$\mathcal{O}^\circ(D_{K''}^{(r)}) \cong \mathcal{O}^\circ(D_{K'}^{(r)}) \otimes_{\mathcal{O}_{K'}} \mathcal{O}_{K''} \quad \text{and} \quad \mathcal{O}^\circ(X_{K''}^{(r)}) \cong \mathcal{O}^\circ(X_{K'}^{(r)}) \otimes_{\mathcal{O}_{K'}} \mathcal{O}_{K''}. \tag{II.4.9.2}$$

It then follows from II.2.17 that $\mathcal{O}^\circ(D_{K'}^{(r)})$ and $\mathcal{O}^\circ(X_{K'}^{(r)})$ are formal models of, and integrally closed in, $\mathcal{O}(D_{K'}^{(r)})$ and $\mathcal{O}(X_{K'}^{(r)})$ respectively (and remain so after finite extension (II.4.9.2)). In particular, they are also normal, which implies, by Lemma II.2.3, that $\mathfrak{D}_{K'}^{(r)}$ and $\mathfrak{X}_{K'}^{(r)}$ are normal. Taking the colimit over K'' in (II.4.9.2) and completing π-adically gives

$$\mathcal{O}^\circ(D_C^{(r)}) = \mathcal{O}^\circ(D_{K'}^{(r)}) \widehat{\otimes}_{\mathcal{O}_{K'}} \mathcal{O}_C \quad \text{and} \quad \mathcal{O}^\circ(X_C^{(r)}) = \mathcal{O}^\circ(X_{K'}^{(r)}) \widehat{\otimes}_{\mathcal{O}_{K'}} \mathcal{O}_C. \tag{II.4.9.3}$$

From Lemma II.4.8, we know that $\mathcal{O}^\circ(X_C)$ is a finite free algebra over $\mathcal{O}^\circ(D_C^{(r)})$. Hence, by (II.4.9.2) and Lemma II.2.13, possibly after a finite extension of K', we can assume that $\mathcal{O}^\circ(X_{K'}^{(r)})$ is a finite free algebra over $\mathcal{O}^\circ(D_{K'}^{(r)})$. Smoothness of $\mathfrak{D}_{K'}^{(r)}$ (resp. $\mathfrak{X}_{K'}^{(r)}$) over $\mathcal{S}'$ is tested on the special fiber [**Abb10**, 2.4.6]. Since the latter is a smooth curve (resp. reduced curve), it is smooth outside a finite set of closed points. □

Definition II.4.10. In the situation of II.4.9, we say that the field K' is *r-admissible for f* and that the adic $\mathcal{O}_{K'}$-morphism $\widehat{f}^{(r)} : \mathfrak{X}_{K'}^{(r)} \to \mathfrak{D}_{K'}^{(r)}$ induced by $f^{(r)}$ is *the normalized integral model of $f^{(r)}$ over K'* (see also Definition II.2.19). We then see from II.4.9 (4) that any further finite extension K'' is also r-admissible for f and that the base change $\widehat{f}^{(r)} \widehat{\otimes}_{\mathcal{O}_{K'}} \mathcal{O}_{K''}$ is the normalized integral model of $f^{(r)}$ over K''. Sometimes we drop K' to lighten the notation, especially when dealing with the local rings of these models at some points.

Remark II.4.11. As any homomorphism $h : A \to B$ of K-affinoid algebras satisfy the inequality $|h(a)|_{\sup} \leq |a|_{\sup}$, for any $a \in A$, the construction of the normalized integral models is functorial: if X' a smooth K-affinoid curve and $g : X' \to X$ is a K-morphism such that the composition $f' = f \circ g$ is finite, flat, and étale over a nonempty open subset of D containing 0, then we have an induced adic morphism $\widehat{g}^{(r)} : \mathfrak{X}_{K'}^{'(r)} \to \mathfrak{X}_{K'}^{(r)}$ such that $\widehat{f}^{'(r)} = \widehat{f}^{(r)} \circ \widehat{g}^{(r)}$, where K' is r-admissible for f and f'.

Remark II.4.12. With the notation of II.4.6, assume that K' is r-admissible. Since, by Lemma II.4.7, $\mathcal{O}^\circ(X_{K'}^{(r)})$ is a generically étale $\mathcal{O}^\circ(D_{K'}^{(r)})$-algebra, which is also finite free by II.4.9 (4), it is a finite product of domains which are free over $\mathcal{O}^\circ(D_{K'}^{(r)})$ and whose fields of fractions are separable over the field of fractions of $\mathcal{O}^\circ(D_{K'}^{(r)})$.

Definition II.4.13. We keep the notation of II.4.6 and assume that K' is r-admissible for f. We let $\mathfrak{d}_f(r, K')$ be the discriminant ideal of $\mathcal{O}^\circ(X_{K'}^{(r)})$ over $\mathcal{O}^\circ(D_{K'}^{(r)})$; it is an invertible (i.e. locally monogenic) ideal of $\mathcal{O}^\circ(D_{K'}^{(r)})$ (cf. II.3.24 and II.4.12). For a finite extension K''/K', we have the inclusion $\mathcal{O}^\circ(D_{K'}^{(r)}) \hookrightarrow \mathcal{O}^\circ(D_{K''}^{(r)})$, and II.4.9 (4) implies that

$$\mathfrak{d}_f(r, K')\mathcal{O}^\circ(D_{K''}^{(r)}) = \mathfrak{d}_f(r, K''). \tag{II.4.13.1}$$

Hence, the discriminant ideal does not depend on the field of definition of the normalized integral model of $f^{(r)}$. So we denote it simply by $\mathfrak{d}_f(r)$.

Remark II.4.14. From II.4.9 (4) and the inclusion $\mathcal{O}^\circ(D^{(r)}_{K'}) \hookrightarrow \mathcal{O}^\circ(D^{(r)}_{C})$, we have

$$\mathfrak{d}_f(r)\mathcal{O}^\circ(D^{(r)}_C) = \mathfrak{d}_{f_C}(r), \tag{II.4.14.1}$$

where $\mathfrak{d}_{f_C}(r)$ is the well-defined discriminant ideal of $\mathcal{O}^\circ(X^{(r)}_C)$ over $\mathcal{O}^\circ(D^{(r)}_C)$ (cf. II.4.8 and II.4.12).

II.4.15. Let $r \geq 0$ be a rational number and $\mathfrak{d}$ an invertible ideal of $\mathcal{O}^\circ(A(r,r))$. Let (U_i) be a formal open cover of $\mathrm{Spf}(\mathcal{O}^\circ(A(r,r))) \cong \widehat{\mathbb{G}}_{m,\mathcal{O}_K}$ that trivializes $\mathfrak{d}$ and (g_i) a tuple of local generators of $\mathfrak{d}$ on (U_i). If $x : \mathrm{Spf}(O_K) \to \widehat{\mathbb{G}}_{m,\mathcal{O}_K}$ is a rig-point that lands in one of the opens U_i of the cover, then $|g_i(x)|_x$ (II.2.15) is independent of the chosen generator g_i of $\mathfrak{d}$ on U_i since another choice of local generator differs from g_i by a factor which is a unit in $\mathcal{O}(U_i)$, hence a unit in the valuation ring of $\mathcal{O}(U_{i,\eta})/x$. We put $|\mathfrak{d}(x)|_x = |g_i(x)|_x$ and define the sup-norm of $\mathfrak{d}$ as

$$|\mathfrak{d}|_{\sup} = \sup_x |\mathfrak{d}(x)|_x, \tag{II.4.15.1}$$

where x runs over the rig-points of $\widehat{\mathbb{G}}_{m,\mathcal{O}_K}$. It is clear that $|\mathfrak{d}|_{\sup}$ is independent of the cover trivializing $\mathfrak{d}$. Therefore, $|\mathfrak{d}|_{\sup}$ is well-defined.

II.4.16. Let $r \geq 0$ be a rational number and $d \geq 1$ an integer. If $g : A(r/d, r/d) \to A(r,r)$ is a finite flat morphism of order d, then, g is given by $\xi \mapsto \xi^d(1 + h(\xi))$, where h is a function on $A(r/d, r/d)$ satisfying $|h|_{\sup} < 1$ (II.4.2); thus, $\mathcal{O}^\circ(A(r/d, r/d))$ is a locally free $\mathcal{O}^\circ(A(r,r))$-module of finite rank d. We define the discriminant ideal $\mathfrak{d}_g[r]$ of g as the discriminant of $\mathcal{O}^\circ(A(r/d, r/d))$ over $\mathcal{O}^\circ(A(r,r))$. It is an invertible ideal of $\mathcal{O}^\circ(A(r,r))$, hence has a well-defined supremum norm (II.4.15). By II.4.2, if $g : A(\varepsilon, \varepsilon') \to A(r, r')$ is a finite flat morphism and $r \geq t \geq r'$ is rational number, we have a well-defined discriminant ideal $\mathfrak{d}_g[t]$ by restricting g over $A(t,t)$.

II.4.17. We keep the notation of II.4.6 and assume that K' is r-admissible for f (II.4.10). Let $D^{[r]} = A(r,r) \subset D^{(r)}$ be the annulus of radius $|\pi|^r$ with 0-thickness and $X^{[r]} = f^{-1}(D^{[r]})$ its inverse image, a smooth K-affinoid space. Over K', we have

$$\mathcal{O}(D^{(r)}_{K'}) = K'\{T\}, \quad \mathcal{O}(D^{[r]}_{K'}) = K'\{T, T^{-1}\} \quad \text{and} \quad \mathcal{O}(X^{[r]}_{K'}) = \mathcal{O}(X^{(r)}_{K'})\{T^{-1}\}. \tag{II.4.17.1}$$

As $\mathcal{O}^\circ(D^{(r)}_{K'})\{T, T^{-1}\}$ (resp. $\mathcal{O}^\circ(X^{(r)}_{K'})\{T^{-1}\}$) is a formal model over $\mathcal{O}_{K'}$ of $\mathcal{O}(D^{(r)}_{K'})\{T^{-1}\}$ (resp. $\mathcal{O}(X^{(r)}_{K'})\{T^{-1}\}$) whose special fiber $\mathbb{A}^1_k$ (resp. $(\mathcal{O}^\circ(X^{(r)}_{K'}) \otimes_{\mathcal{O}_{K'}} k)[T^{-1}]$) is geometrically reduced (II.4.9 (2)), Lemma II.2.17(ii) implies that

$$\mathcal{O}^\circ(D^{[r]}_{K'}) = \mathcal{O}^\circ(D^{(r)}_{K'})\{T^{-1}\} = \mathcal{O}_{K'}\{T, T^{-1}\} \quad \text{and} \quad \mathcal{O}^\circ(X^{[r]}_{K'}) = \mathcal{O}^\circ(X^{(r)}_{K'})\{T^{-1}\}. \tag{II.4.17.2}$$

Hence, $\mathfrak{D}^{[r]}_{K'} = \mathrm{Spf}(\mathcal{O}^\circ(D^{[r]}_{K'})) \simeq \widehat{\mathbb{G}}_{m,\mathcal{O}_{K'}}$ is an open subscheme of the formal affine line $\mathfrak{D}^{(r)}_{K'} \simeq \widehat{\mathbb{A}}^1_{K'}$. Moreover, $\mathfrak{D}^{[r]}_{K'}$ and $\mathfrak{X}^{[r]}_{K'} = \mathrm{Spf}(\mathcal{O}^\circ(X^{[r]}_{K'}))$ satisfy the statements (1) to (4) of Proposition II.4.9, and the induced normalized integral model $\widehat{f}^{[r]} : \mathfrak{X}^{[r]}_{K'} \to \mathfrak{D}^{[r]}_{K'}$ of the restriction $f^{[r]} : X^{[r]} \to D^{[r]}$ of f fits into a Cartesian square

$$\begin{array}{ccc} \mathfrak{X}^{[r]}_{K'} & \longrightarrow & \mathfrak{X}^{(r)}_{K'} \\ {\scriptstyle \widehat{f}^{[r]}}\downarrow & \square & \downarrow{\scriptstyle \widehat{f}^{(r)}} \\ \mathfrak{D}^{[r]}_{K'} & \longrightarrow & \mathfrak{D}^{(r)}_{K'}, \end{array} \tag{II.4.17.3}$$

where the bottom horizontal arrow is a formal open immersion. In particular, $\mathfrak{X}^{[r]}_{K'}$ is a formal open subscheme of $\mathfrak{X}^{(r)}_{K'}$, whose complement lies over $\mathfrak{D}^{(r)}_{K'} - \mathfrak{D}^{[r]}_{K'}$. The discriminant ideal $\mathfrak{d}_f[r]$ of

$\mathcal{O}^\circ(X_{K'}^{[r]})$ over $\mathcal{O}^\circ(D_{K'}^{[r]})$ is also well-defined (II.4.9 (1)), independent of the choice of K' which is r-admissible for f (II.4.9 (4)) and consistent with II.4.16. By [Ser68, III, §4, Prop. 9], we have $\mathfrak{d}_f[r] = \mathfrak{d}_f(r)[\frac{1}{T}]$. In particular, with the notation of II.4.15, we see that

(II.4.17.4) $$|\mathfrak{d}_f[r]|_{\sup} = |\mathfrak{d}_f(r)|_{\sup}.$$

Lemma II.4.18 ([Lüt93, Lemma 1.7]). *With the notation of II.4.1, assume that the morphism f is étale and given by*

(II.4.18.1) $$A(r/d, r'/d) \to A(r, r'), \quad \xi \mapsto \xi^d(1+h(\xi)), \quad \text{with} \quad h(\xi) = \sum_{i\neq 0} h_i\xi^i, \ |h(\xi)|_{\sup} < 1,$$

where $r \geq r'$ are in $\mathbb{Q}$. Denote by $\sigma = d-1+\nu$, $\nu \in \mathbb{Z}$, the order of the derivative $f'(\xi) = \frac{df(\xi)}{d\xi}$. Then, for a rational number t such that $r \geq t \geq r'$, we have

(II.4.18.2) $$|\mathfrak{d}_f[t]|_{\sup} = |d|^d \quad \text{if} \quad \nu = 0 \quad \text{and} \quad |\mathfrak{d}_f[t]|_{\sup} = |\nu h_\nu|^d|\pi|^{\nu t} \quad \text{if} \quad \nu \neq 0.$$

Proof. For the sake of completeness, we reproduce here, with some more details, Lütkebohmert's proof. We can write

(II.4.18.3) $$f(\xi) = \sum_{i\in\mathbb{Z}} a_i\xi^i = \xi^d(1+h(\xi));$$

whence we see, putting $h_0 = 1$, that $a_{d+i} = h_i$ for all $i \in \mathbb{Z}$. By the Weierstrass preparation theorem, recalled in II.4.2, the derivative $f'(\xi)$ can also be written in the form

(II.4.18.4) $$f'(\xi) = \left(\sum ia_i\xi^{i-1}\right) = (d+\nu)a_{d+\nu}\xi^\sigma(1+g(\xi)),$$

where g is a function on $A(r/d, r'/d)$ such that $|g|_{\sup} < 1$. It follows that $|f'(\xi)|_{\sup} = |(d+\nu)a_{d+\nu}|$ and thus $(d+\nu)a_{d+\nu}$ is the dominant coefficient of $f'(\xi)$. Then, by [BGR84, 9.7.1/1], applied to f', and the identity $a_{d+i} = h_i$ (with $a_d = h_0 = 1$)), we have the following. If $\nu = 0$, then $|(d+i)h_i\pi^{it}| < |d|$ for all $i \in \mathbb{Z} - \{0\}$ and all $r \geq t \geq r'$. If $\nu \neq 0$, then $|(d+i)h_i\pi^{(i-\nu)t}| < |(d+\nu)h_\nu|$ for all $i \neq \nu$ and all $r \geq t \geq r'$, and thus, taking $i = 0$, we have $|d| < |(d+\nu)h_\nu\pi^{\nu t}|$ for all $r \geq t \geq r'$; in particular, as $|h(\xi)|_{\sup} < 1$, we get $|d| < |d+\nu|$ and thus $|\nu| = |d+\nu|$.

Now, we make the change of coordinates $T = \xi/\pi^{t/d}$ and put $U(T) = f(\pi^{t/d}T)/\pi^t$, $\widetilde{g}(T) = g(\pi^{t/d}T)$. Then, using the identity $\sigma - 1 + d = \nu$, (II.4.18.4) translates into

(II.4.18.5) $$dU(T)/dT = (d+\nu)a_{d+\nu}\pi^{\nu t/d}T^\sigma(1+g).$$

It follows that the different ideal is generated by $(d+\nu)a_{d+\nu}\pi^{t\nu/d}T^\sigma$. Hence, the discriminant ideal $\mathfrak{d}_f[t]$ is generated by its d-th power $(d+\nu)^d a_{d+\nu}^d\pi^{\nu t}T^{d\sigma}$ and thus $|\mathfrak{d}_f[t]|_{\sup} = |(d+\nu)^d h_\nu^d\pi^{\nu t}|$. If $\nu = 0$, then $|\mathfrak{d}_f[t]|_{\sup} = |d|^d$, since $h_0 = 1$. If $\nu \neq 0$, then $|\mathfrak{d}_f[t]|_{\sup} = |\nu h_\nu|^d|\pi|^{\nu t}$, since $|d+\nu| = |\nu|$. □

II.4.19. Let x be the origin point in $\mathbb{A}_k^1$, corresponding to the open maximal ideal $\mathfrak{m}_x = (\pi, \xi)$ of $\mathcal{O}_K\{\xi\}$, a closed point of $\mathfrak{D}$. Let $\overline{x} = \mathrm{Spec}(k) \to \mathbb{A}_k^1$ be the geometric point associated to x. We let $A = \mathcal{O}_{\mathfrak{D},\overline{x}}$ be the étale local ring of $\mathfrak{D}$ at $\overline{x}$ (II.2.5.1). The ideal $\mathfrak{p} = \pi A$ of A is prime of height 1, and $A/\mathfrak{p} = k[\xi]_{(\xi)}^{\mathrm{sh}}$ (II.2.8.2). These data produce a normalized $\mathbb{Z}^2$-valuation ring $V = V_A(\mathfrak{p})$ (see II.3.21) with field of fractions $\mathbb{K} = \mathbb{K}(\mathfrak{p})$. We denote by $v : \mathbb{K}^\times \to \mathbb{Z}^2$ the corresponding normalized valuation map, and by $v^\alpha : \mathbb{K}^\times \to \mathbb{Z}$ and $v^\beta : \mathbb{K}^\times \to \mathbb{Z}$ the associated projections (II.3.7). We note that these normalized valuation maps, defined using the uniformizer π, don't in fact depend on π (see the end of II.3.21). We also note that the degree of imperfection of the residue field $\kappa(\mathfrak{p})$ of V at $\mathfrak{p}$ is $[\kappa(\mathfrak{p}) : \kappa(\mathfrak{p})^p] = [k((\xi)) : k((\xi))^p] = p$ [GO08, 2.1.4].

Lemma II.4.20. *(i) On $k[\xi]$, the valuation map for the discrete valuation ring $A/\mathfrak{p}$ is given by*

(II.4.20.1)
$$\sum_{n=0}^{d} a_n\xi^n \mapsto \min\{n \mid a_n \neq 0\}.$$

(ii) Consider $\mathcal{O}_K\{\xi\} = \mathcal{O}_{\mathfrak{D}}(\mathfrak{D})$ as a subring of $A \subsetneq A_{\mathfrak{p}}$. Then, the restriction to $K\{\xi\}^\times$ of the valuation map $v_{\mathfrak{p}} : \mathbb{K}^\times \to \mathbb{Z}$, associated to the discrete valuation ring $A_{\mathfrak{p}}$ (II.3.18), is given by

(II.4.20.2)
$$\sum_{n\geq 0} a_n\xi^n \mapsto \inf_n\{v_K(a_n)\}.$$

(iii) If $g(\xi) = c\xi^d(1 + h(\xi))$ is in $\mathcal{O}_K\{\xi\}$ with $c \in K^\times$ and $|h|_{\sup} < 1$, then

(II.4.20.3)
$$v^\alpha(g(\xi)) = v_K(c) \quad \text{and} \quad v^\beta(g(\xi)) = d.$$

Proof. (i) and (ii) are clear since both $A/\mathfrak{p}$ and $A_{\mathfrak{p}}$ are discrete valuation rings with maximal ideals generated by ξ and π respectively. Item (iii) follows from (i), (ii) and II.3.15. □

II.4.21. We keep the notation of II.4.19, let $r \in \mathbb{Q}$ and assume that K' is a finite extension of K containing an element π^r of valuation r. In order to describe integral models of $D^{(r)}_{K'}$, we make the change of variable $T = \xi/\pi^r$, which reduces us to the formal unit disc $\mathrm{Spf}(\mathcal{O}_{K'}\{T\})$. We then apply the construction in II.4.19 to this disc and the origin of its special fiber. This gives a $\mathbb{Z}^2$-valuation ring V_r with field of fractions $\mathbb{K}_r$ containing $K'\{T\}$ and normalized valuation map $v'_r : \mathbb{K}_r^\times \to \mathbb{Z}^2$. We denote by v_r the composition $\mathbb{K}_r^\times \xrightarrow{v'_r} \mathbb{Z}^2 \hookrightarrow \mathbb{Q}\times\mathbb{Z}$, where the second map is given by $(a,b) \mapsto (a/e, b)$ with e the ramification index of the extension K'/K, and we denote by v_r^α and v_r^β the composition of v_r with the projections $\mathbb{Q}\times\mathbb{Z} \to \mathbb{Q}$ and $\mathbb{Q}\times\mathbb{Z}\to\mathbb{Z}$ respectively. As in II.4.19, we note the residue field of V_r at the height 1 prime ideal has degree of imperfection p. For $g(\xi) = c\xi^d(1+h(\xi)) = c\pi^{rd}T^d(1+\widetilde{h}(T))$ in $\mathcal{O}_{K'}\{\xi\} \subset \mathcal{O}_{K'}\{T\}$ with $c \in K'$ non-zero, $h(\xi)$ a function on D satisfying $|h|_{\sup} < 1$ and $\widetilde{h}(T) = h(\pi^r T)$ (hence $|\widetilde{h}(T)|_{\sup} < 1$), (II.4.20.3) yields

(II.4.21.1)
$$v_r^\alpha(g(\xi)) = v_K(c) + rd \quad \text{and} \quad v_r^\beta(g(\xi)) = d.$$

In particular, we see that $v_r^\beta(g)$ is independent of r.

II.4.22. For the rest of this section, we keep the notation of II.4.6, II.4.19 and II.4.21, and assume that K' is r-admissible for the morphism $f : X \to D$ (see II.4.10). The discriminant ideal $\mathfrak{d}_f(r)$ is a non-zero principal ideal of $\mathcal{O}_{K'}\{T\} \subseteq V_r$. Therefore, it has well-defined valuations

(II.4.22.1)
$$\partial_f(r) = v_r(\mathfrak{d}_f(r)) \in \mathbb{Q}\times\mathbb{Z}, \quad \partial_f^\alpha(r) = v_r^\alpha(\mathfrak{d}_f(r)) \in \mathbb{Q} \quad \text{and} \quad \partial_f^\beta(r) = v_r^\beta(\mathfrak{d}_f(r)) \in \mathbb{Z}.$$

From (II.4.14.1) and II.4.20 (ii), we have the following equality in $|\overline{K}^\times|$

(II.4.22.2)
$$|\pi|^{\partial_f^\alpha(r)} = |\mathfrak{d}_{f_C}(r)|_{\sup}.$$

Hence, ∂_f^α is the (additive version of the) discriminant function defined in [Lüt93, 1.3]. The following is a rewriting of [Lüt93, Lemma 2.6].

Proposition II.4.23. *Let X be a smooth K-rigid space and let $f : X \to D$ be a finite flat morphism of degree d, which is étale over a nonempty open subset of D containing 0. Then, there exists a finite sequence of rational numbers $(r_i)_{i=1}^n$ such that $0 = r_{n+1} < r_n < \cdots < r_1 < r_0 = +\infty$ and a decomposition of $\Delta_i = f^{-1}(A^\circ(r_{i-1}, r_i))$ into a disjoint union of open annuli $\Delta_i = \coprod_j \Delta_{ij}$ such* *that the restriction* $f|\Delta_i$ *is étale, the function* ∂_f^α *is affine on* $]r_i, r_{i-1}[\cap\mathbb{Q}$ *and its right slope at* $t \in [r_i, r_{i-1}[\cap\mathbb{Q}$ *is*

$$\frac{d}{dt}\partial_f^\alpha(t^+) = \sigma_i - d + \delta_f(i), \tag{II.4.23.1}$$

where σ_i *is the total order of the derivative of* $f|\Delta_i$ (II.4.5) *and* $\delta_f(i)$ *is the number of connected components of* Δ_i (*i.e. the number of* Δ_{ij}*'s*).

Proof. Lemma II.4.4, applied to $f : X \to D = A(0,1)$, gives a finite extension K' of K, a sequence $0 < |\pi|^{r_1} < \cdots < |\pi|^{r_n}$ contained in $v_K(K')$, a decomposition $\Delta_i = \coprod_j \Delta_{ij}$ and étale morphisms defined by functions $f_{ij}(\xi_{ij}) = \xi_{ij}^{d_{ij}}(1 + h_{ij})$, which are the restrictions of $f_{K'}$ to the open annuli $\Delta_{ij} = A_{K'}(r_{i-1}/d_{ij}, r_i/d_{ij})$, $j = 1, \ldots, \delta_f(r_{i-1}, r_i)$. For $t \in]r_i, r_{i-1}[\cap\mathbb{Q}$, we see from this decomposition that $X_{K'}^{[t]} = \coprod_j \Delta_{ij}^{(t)}$, where $\Delta_{ij}^{(t)} = A_{K'}(t/d_{ij}, t/d_{ij})$. It follows that $\mathcal{O}(X_{K'}^{[t]}) = \prod_j \mathcal{O}(\Delta_{ij}^{(t)})$ and thus $\mathcal{O}^\circ(X_{K'}^{[t]}) = \prod_j \mathcal{O}^\circ(\Delta_{ij}^{(t)})$ (notation of II.4.17). Therefore, we have

$$\mathfrak{d}_f[t] = \prod_j \mathfrak{d}_{f_{ij}|\Delta_{ij}^{(t)}}[t]. \tag{II.4.23.2}$$

In the proof of II.4.18, we saw that the ideal $\mathfrak{d}_{f_{ij}|\Delta_{ij}^{(t)}}(t)$ is generated by $(d_{ij} + \nu_{ij})h_{\nu_{ij}}\pi^{\nu_{ij}t}T_{ij}^{\mathfrak{d}_{ij}\sigma_{ij}}$, where T_{ij} is a coordinate of $\Delta_{ij}^{(t)} \cong \mathrm{Sp}(K'\{T_{ij}\})$, $h_{\nu_{ij}}$ is the coefficient of h_{ij} indexed by ν_{ij} and σ_{ij} is the order of the derivative $f'_{ij}(\xi_{ij})$. Combining this with (II.4.23.2) and (II.4.17.4) yields

$$|\mathfrak{d}_f(t)|_{\sup} = |\mathfrak{d}_f[t]|_{\sup} = |\pi|^{\nu_i t}\prod_j |(d_{ij} + \nu_{ij})h_{\nu_{ij}}|, \tag{II.4.23.3}$$

where $\nu_i = \nu_{i1} + \cdots + \nu_{i\delta_f(i)} = \sigma_i - d + \delta_f(i)$ (II.4.5.1). Then, with (II.4.22.2), we see that $\partial_f^\alpha(t) = \nu_i t + c_i$, where c_i is the constant $v_K\left(\prod_j (d_{ij} + \nu_{ij})h_{\nu_{ij}}\right)$. This finishes the proof. □

II.4.24. Let $r \geq 0$ be a rational number and resume the notation and assumption on $f : X \to D$ and K' from II.4.22. Recall, from II.4.19 (and II.2.5), that we have a geometric point $\overline{x} \to \mathfrak{D}_{K'}$ and a height 1 prime ideal $\mathfrak{p}$ of $\mathcal{O}_{\mathfrak{D}_{K'},\overline{x}}$ above $\mathfrak{m}_K$. Through the renormalization $\mathfrak{D}_{K'}^{(r)} \xrightarrow{\sim} \mathfrak{D}_{K'}$, $\xi \mapsto \xi/\pi^r$, they induce a geometric point $\overline{x} \to \mathfrak{D}_{K'}^{(r)}$ and a height 1 prime ideal of $A_{\overline{x}} = \mathcal{O}_{\mathfrak{D}_{K'}^{(r)},\overline{x}}$ which we again denote by $\mathfrak{p}$. We let $S_{f,K'}^{(r)}$ be the set of couples $\tau = (\overline{x}_\tau, \mathfrak{p}_\tau)$, where $\overline{x}_\tau = \overline{x} \to \mathfrak{X}_{K'}^{(r)}$ is a geometric point (of the special fiber) of the normalized integral model $\mathfrak{X}_{K'}^{(r)}$ of $X_{K'}^{(r)}$, above $\overline{x} \to \mathfrak{D}_{K'}^{(r)}$, and $\mathfrak{p}_\tau$ is a height 1 prime ideal of $B_{\overline{x}_\tau} = \mathcal{O}_{\mathfrak{X}_{K'}^{(r)},\overline{x}_\tau}$ above $\mathfrak{p}$. Note that, as the morphism $\widehat{f}^{(r)} : \mathfrak{X}_{K'}^{(r)} \to \mathfrak{D}_{K'}^{(r)}$ is finite and flat (II.4.9 (1)), so is the induced morphism on special fibers which is then surjective; hence, $S_{f,K'}^{(r)}$ is not empty. If K''/K' is a finite extension, then the special fiber of $\mathfrak{X}_{K'}^{(r)}$ is canonically isomorphic to the special fiber of the base change to K'' (II.3.17 (iii)) and the height 1 prime ideals of $B_{\overline{x}_\tau}$ are the minimal primes of $B_{\overline{x}_\tau}/\pi B_{\overline{x}_\tau} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}_{s'},\overline{x}_\tau}$; it follows that we have a canonical bijection $S_{f,K''}^{(r)} \xrightarrow{\sim} S_{f,K'}^{(r)}$. Thus $S_{f,K'}^{(r)}$ is independent of the chosen r-admissible extension K' and we subsequently drop K' from the notation. For $\tau = (\overline{x}_\tau, \mathfrak{p}_\tau)$ in $S_f^{(r)}$, we get a $\mathbb{Z}^2$-valuation ring $V_r(\tau) = V_{B_{\overline{x}_\tau}}(\mathfrak{p}_\tau)$ (II.3.21). The henselization V_r^h (resp. $V_r^h(\tau)$) of V_r (resp. $V_r(\tau)$) is a henselian $\mathbb{Z}^2$-valuation ring (II.3.21) whose field of fractions is denoted $\mathbb{K}_r^h$ (resp. $\mathbb{K}_{r,\tau}^h$). By II.2.12 and II.4.12, $\widehat{f}^{(r)}$ induces a morphism $A_{\overline{x}} \to B_{\overline{x}_\tau}$ in $\widehat{\mathcal{C}}_{K'}$ (II.3.20). By functoriality

(II.3.19) and II.3.23, the latter morphism gives rise to a monogenic integral extension of henselian $\mathbb{Z}^2$-valuation rings $V_r^h(\tau)/V_r^h$, with $V_r^h(\tau)$ a free V_r^h-module of finite rank. We denote again by $v_r : \mathbb{K}_r^{h\times} \to \mathbb{Q}\times\mathbb{Z}$, $v_r^\alpha : \mathbb{K}_r^{h\times} \to \mathbb{Q}$ and $v_r^\beta : \mathbb{K}_r^{h\times} \to \mathbb{Z}$ the maps respectively induced by v_r, v_r^α and v_r^β (II.4.21) through the canonical isomorphism of value groups $\Gamma_{V_r} \xrightarrow{\sim} \Gamma_{V_r^h}$ (II.3.3). We note that the residue field of V_r^h at its height 1 prime ideal has degree of imperfection p and thus V_r^h satisfy the assumptions of II.3.11.

II.4.25. With the notation of II.4.24 above, and as in II.3.25, we define the integer

$$d_{f,s}(r, K') = \sum_{\tau\in S^{(r)}} v_r^\beta(c(V_r^h(\tau)/V_r^h)), \tag{II.4.25.1}$$

where $c(V_r^h(\tau)/V_r^h)$ is a generator of the discriminant ideal of $V_r^h(\tau)/V_r^h$.

Proposition II.4.26. *For $r \in \mathbb{Q}_{\geq 0}$, we have the following equalities of integers*

$$\sum_{j=1}^N d_{\eta,\overline{x}'_j}^{(r)} = d_{f,s}(r, K') + 2\sum_{j=1}^N \delta_{\overline{x}'_j}^{(r)}, \tag{II.4.26.1}$$

where $\overline{x}'_1, \ldots, \overline{x}'_N$ are the geometric points of $\mathfrak{X}_{K'}^{(r)}$ above $\overline{x}$, $\delta_{\overline{x}'_j}^{(r)} = \delta(\mathcal{O}_{\mathfrak{X}^{(r)},\overline{x}'_j})$ is defined as in II.3.22 *and $d_{\eta,\overline{x}'_j}^{(r)} = d_\eta(\mathcal{O}_{\mathfrak{X}^{(r)},\overline{x}'_j}/\mathcal{O}_{\mathfrak{D}^{(r)},\overline{x}})$ is defined as in* (II.3.25.1).

Proof. We denote $\mathcal{A} = \mathcal{O}^\circ(D_{K'}^{(r)})$, $\mathcal{B} = \mathcal{O}^\circ(X_{K'}^{(r)})$, $A_{\overline{x}} = \mathcal{O}_{\mathfrak{D}^{(r)},\overline{x}}$ and $B_j = \mathcal{O}_{\mathfrak{X}^{(r)},\overline{x}'_j}$. By II.4.9 (2), the $\mathcal{S}'$-adic morphism $\widehat{f}^{(r)} : \mathfrak{X}_{K'}^{(r)} \to \mathfrak{D}_{K'}^{(r)}$ is finite. We can thus apply II.2.12 to $\widehat{f}^{(r)}$, $\overline{x}$ of $\mathfrak{D}_{K'}^{(r)}$ and $\overline{x}'_1, \ldots, \overline{x}'_N$ of $\mathfrak{X}_{K'}^{(r)}$, and obtain

$$A_{\overline{x}} \otimes_{\mathcal{A}} \mathcal{B} = B_1 \times \cdots \times B_N. \tag{II.4.26.2}$$

Moreover, by Proposition II.4.9, for each j, $(\mathfrak{X}_{K'}^{(r)}/\mathcal{S}', \overline{x}'_j)$ satisfies property (P) from II.3.18 in an open neighborhood of $\overline{x}'_j$. Hence, $\widehat{f}^{(r)}$ induces a morphism $A_{\overline{x}} \to B_j$ in $\widehat{\mathcal{C}}_{K'}$. Since $A_{\overline{x}}$ and $A_{\overline{x}}/(\pi)$ are regular, its follows from II.3.27 that the discriminant $\mathfrak{d}_{B_j/A_{\overline{x}}}$ is well-defined. Then, from (II.4.26.2), viewing $\mathfrak{d}_f(r)$ in $A_{\overline{x}} \supset \mathcal{A}$ (II.4.21), we get

$$\mathfrak{d}_f(r) = \mathfrak{d}_{B_1/A_{\overline{x}}} \cdots \mathfrak{d}_{B_N/A_{\overline{x}}}. \tag{II.4.26.3}$$

On the one hand, as $d_{\eta,\overline{x}'_j}^{(r)} = d_\eta(B_j/A_{\overline{x}}) = v_r^\beta(\mathfrak{d}_{B_j/A})$ by II.3.26 and II.3.27, it then follows from (II.4.26.3) that $\partial_f^\beta(r) = \sum_j d_{\eta,\overline{x}'_j}^{(r)}$. On the other hand, (II.4.26.3) and II.3.27 also imply that

$$\partial_f^\beta(r) = \sum_{j=1}^N (d_s(B_j/A_{\overline{x}}) + 2\delta(B_j)) = d_{f,s}(r, K') + 2\sum_{j=1}^N \delta(B_j), \tag{II.4.26.4}$$

where the last equality directly uses the definition of $d_s(B_j/A_{\overline{x}})$ given in (II.3.25.2). This establishes (II.4.26.1). □

Remark II.4.27. The integer $d_{f,s}(r, K')$ is independent of the choice of the r-admissible extension K' of K and thus is simply denoted by $d_{f,s}(r)$. Indeed, in the proof of II.4.26, we have shown that

$$\partial_f^\beta(r) = d_{f,s}(r, K') + 2\sum_{j=1}^N \delta_{\overline{x}'_j}^{(r)}, \tag{II.4.27.1}$$

and the $\delta^{(r)}_{\overline{x}'_j}$'s as well as $\mathfrak{d}_f(r)$ (hence $\partial^\beta_f(r)$ too) have already been seen, in (II.3.22.1) and (II.4.13.1) respectively, to be independent of the chosen r-admissible K'.

Proposition II.4.28. *We keep the notation of II.4.23 and II.4.26. Moreover, we assume that X is connected and has trivial canonical sheaf. For $i = 1, \dots, n$ and $t \in]r_i, r_{i-1}[\cap\mathbb{Q}$, we have the following equality*

$$\sum_{j=1}^{N}\left(d^{(t)}_{\eta,\overline{x}'_j} - 2\delta^{(t)}_{\overline{x}'_j} + |P^{(t)}_{s,\overline{x}'_j}|\right) = \sigma_i + \delta_f(i), \tag{II.4.28.1}$$

where the finite set $P^{(t)}_{s,\overline{x}'_j} = P_s(\mathcal{O}_{\mathfrak{X}^{(t)},\overline{x}'_j})$ is defined as in II.3.22.

Proof. The integer i and the rational number $t \in]r_i, r_{i-1}[\cap\mathbb{Q}$ are *fixed* throughout the proof. We recall from II.4.17 that $D^{[t]}$ is the annulus of radius $|\pi|^t$ with 0-thickness, $X^{[t]}$ is its inverse image by f, and $\mathfrak{D}^{[t]}_{K'}$ and $\mathfrak{X}^{[t]}_{K'}$ are their respective normalized integral models over K'. From II.4.4 and II.4.17, we see that

$$X^{[t]}_{K'} = \coprod_{j=1}^{\delta_f(i)} \Delta^{(t)}_{ij}, \quad \text{with} \quad \Delta^{(t)}_{ij} = A\left(\frac{t}{d_{ij}}, \frac{t}{d_{ij}}\right) = D^{[t/d_{ij}]}_{K'}. \tag{II.4.28.2}$$

We get from this the decomposition $\mathfrak{X}^{[t]}_{K'} = \coprod_j \widehat{\Delta}^{(t)}_{ij}$, where the normalized integral model $\widehat{\Delta}^{(t)}_{ij} = \mathrm{Spf}(\mathcal{O}^\circ(\Delta^{(t)}_{ij}))$ of $\Delta^{(t)}_{ij}$ is the formal annulus of radius $|\pi|^{t/d_{ij}}$ with 0-thickness, defined over K', with K' t-admissible for f (II.4.10), and isomorphic to $\mathrm{Spf}(\mathcal{O}_{K'}\{T_j, T_j^{-1}\})$. To get a formal compactification of $\mathfrak{X}^{(t)}_{K'}$, for each j, we glue $\mathfrak{X}^{(t)}_{K'}$ and a formal closed disc $\mathfrak{D}^{(t)}_{ij} = \widehat{\mathbb{A}}^1_{K'} = \mathrm{Spf}(\mathcal{O}_{K'}\{S_j\})$ along the boundary $\widehat{\Delta}^{(t)}_{ij} = \mathrm{Spf}(\mathcal{O}_{K'}\{S_j, S_j^{-1}\})$ (II.4.17.2), with gluing map $T_j \mapsto S_j^{-1}$. The resulting formal relative curve

$$\mathfrak{Y}^{(t)}_{K'} = (\mathfrak{X}^{(t)}_{K'} \cup (\coprod_j \mathfrak{D}^{(t)}_{ij}))/\sim_{\mathfrak{X}^{[t]}} \to \mathcal{S}' = \mathrm{Spf}(\mathcal{O}_{K'}) \tag{II.4.28.3}$$

has smooth rigid fiber and contains $\mathfrak{X}^{(t)}_{K'}$ as a formal open subscheme. As $\mathfrak{X}^{(t)}_{K'}$ is normal (II.4.9(3)), $\mathfrak{Y}^{(t)}_{K'}$ is also normal. Its special fiber $\mathfrak{Y}^{(t)}_{s'}$ is the gluing of $\mathfrak{X}^{(t)}_{s'}$ with the disjoint union of $\delta_f(i)$ copies of $\mathbb{A}^1_k$ along the overlap $\mathfrak{X}^{[t]}_{s'}$, which is a disjoint union of $\delta_f(i)$ copies of $\mathbb{G}_{m,k}$, with gluing map $T_j \to S_j^{-1}$ on each $\mathbb{G}_{m,k}$. It follows that $\mathfrak{Y}^{(t)}_{s'}$ is a proper, hence projective, k-curve. Moreover, by construction, the singular locus of $\mathfrak{Y}^{(t)}_{s'}$ is contained in the set $\mathfrak{X}^{(t)}_{s'} - \mathfrak{X}^{[t]}_{s'}$. As the Cartesian square (II.4.17.3) reduces to a similar Cartesian square on special fibers, the latter set lies over the origin $\mathfrak{D}^{(t)}_{s'} - \mathfrak{D}^{[t]}_{s'} = \{x\}$. By Grothendieck's algebraization theorem, there exists a relative proper algebraic curve $Y^{(t)}_{S'}$ over $S' = \mathrm{Spec}(\mathcal{O}_{K'})$ whose formal completion along its special fiber is $\mathfrak{Y}^{(t)}_{K'}$ [**EGA III**, 5.4.5]. As the rigid fiber $\mathfrak{Y}^{(t)}_{\eta'}$ of $\mathfrak{Y}^{(t)}_{K'}$ is smooth, so is the generic fiber $Y^{(t)}_{\eta'}$ of $Y^{(t)}_{S'}$. It follows from our assumptions that the canonical sheaf of $X^{(t)}_{K'}$ is trivial; so there exists a global section $\omega \in \Gamma(X^{(t)}_{K'}, \Omega^1_{X/K})$ inducing a trivialization $\mathcal{O}_{X^{(t)}_{K'}} \xrightarrow[\times\omega]{\sim} \Omega^1_{X/K}|X^{(t)}_{K'}$. Thus we can write $df^{(t)} = f^\dagger\omega$, where $f^\dagger \in \Gamma(X^{(t)}_{K'}, \mathcal{O}_X)$. As, for each j, both $\omega|\Delta^{(t)}_{ij}$ and dT_j trivialize $\Omega^1_{X/K}$ on $\Delta^{(t)}_{ij}$, we have $\omega|\Delta^{(t)}_{ij} = u_j(T_j)dT_j$, for some $u_j(T_j) \in \Gamma(\Delta^{(t)}_{ij}, \mathcal{O}_{\Delta^{(t)}_{ij}})^\times$. Hence, we deduce that $(f^{(t)})'(T_j) = u_j(T_j)f^\dagger|\Delta^{(t)}_{ij}$. We choose a point y_j in the generic fiber $D^{(t)}_{ij}$ of $\mathfrak{D}^{(t)}_{ij}$ that is not in

$\Delta_{ij}^{(t)}$. By the rigid Runge theorem [Ray94, 3.5.2], we can then approximate $\widehat{f}^{(t)} : \mathfrak{X}_{K'}^{(t)} \to \mathfrak{D}_{K'}^{(t)}$ by the formal completion $\widehat{g}^{(t)} : \mathfrak{Y}_{K'}^{(t)} \to \widehat{\mathbb{P}}_{S'}^{1}$ of an algebraic morphism $g^{(t)} : Y_{S'}^{(t)} \to \mathbb{P}_{S'}^{1}$ satisfying ${g^{(t)}}^{-1}(\infty) \subset \{y_j, j = 1, \ldots, \delta_f(i)\}$, such that the induced morphism $g_{\eta'}^{(t)} : \mathfrak{Y}_{\eta'}^{(t)} \to \mathbb{P}_{K'}^{1,\mathrm{rig}}$ on rigid fibers is meromorphic with poles at most at the y_j and, on each $\Delta_{ij}^{(t)}$, we have

$$|g_{\eta'}^{(t)} - f^{(t)}|_j < |f^\dagger|_j / |u_j^{-1}(T_j)|_{\sup}, \tag{II.4.28.4}$$

where $|\cdot|_j$ is defined as in II.2.20, namely the sup-norm of the restriction to $\Delta_{ij}^{(t)}$ (i is fixed). As for $f^{(t)}$, we have $dg_{\eta'}^{(t)}|X_{K'}^{(t)} = g^\dagger\omega$, for some $g^\dagger \in \Gamma(X_{K'}^{(t)}, \mathcal{O}_X)$, and $(g_{\eta'}^{(t)})'(T_j) = u_j(T_j) g^\dagger|\Delta_{ij}^{(t)}$. Since $Y_{\eta'}^{(t)}$ is a proper smooth curve, hence projective, and $dg^{(t)}$ is a non-zero meromorphic section of the canonical sheaf $\Omega^1_{Y_{\eta'}^{(t)}/K'}$, we have

$$2g(Y_{\overline{\eta}'}^{(t)}) - 2|\pi_0(Y_{\overline{\eta}'}^{(t)})| = \deg(\mathrm{div}(dg_{\eta'}^{(t)})), \tag{II.4.28.5}$$

where $g(Y_{\overline{\eta}'}^{(t)})$ is the total genus of $Y_{\overline{\eta}'}^{(t)}$, i.e. the sum of the genera of its connected components. Let us compute the right-hand side of (II.4.28.5). Taking the derivative of a power series expansion of $g_{\eta'}^{(t)} - f^{(t)}$ on $\Delta_{ij}^{(t)}$ and using the strong triangle inequality gives

$$|(g_{\eta'}^{(t)})'(T_j) - (f^{(t)})'(T_j)|_{\sup} \leq |g_{\eta'}^{(t)} - f^{(t)}|_j. \tag{II.4.28.6}$$

Since $|g^\dagger - f^\dagger|_j \leq |u_j^{-1}(T_j)|_{\sup} |(g_{\eta'}^{(t)} - f^{(t)})'(T_j)|_{\sup}$ and $|f^\dagger|_j \leq |u_j^{-1}(T_j)|_{\sup} |(f^{(t)})'(T_j)|_{\sup}$, (II.4.28.4) and (II.4.28.6) yield both following inequalities

$$|(g_{\eta'}^{(t)})'(T_j) - (f^{(t)})'(T_j)|_{\sup} < |(f^{(t)})'(T_j)|_{\sup} \quad \text{and} \quad |g^\dagger - f^\dagger|_j < |f^\dagger|_j. \tag{II.4.28.7}$$

Therefore, we also have $|(g_{\eta'}^{(t)})'(T_j)|_{\sup} = |(f^{(t)})'(T_j)|_{\sup}$ and $|g^\dagger|_j = |f^\dagger|_j$. Hence, at each point of the normalization $\widetilde{\mathfrak{Y}}_{s'}^{(t)}$ of $\mathfrak{Y}_{s'}^{(t)}$, $g^\dagger$ and $f^\dagger$ have the same order as defined by (II.2.20.1), and so do $(g_{\eta'}^{(t)})'(T_j)$ and $(f^{(t)})'(T_j)$. It follows from II.2.21, that, for each $x'_j \in \mathfrak{X}_{s'}^{(t)} - \mathfrak{X}_{s'}^{[t]}$, we have $\deg(\mathrm{div}(g^\dagger)|C_+(x'_j)) = \deg(\mathrm{div}(f^\dagger)|C_+(x'_j))$. Hence, as

$$\mathrm{div}(dg_{\eta'}^{(t)})|C_+(x'_j) = \mathrm{div}(g^\dagger)|C_+(x'_j) + \mathrm{div}(\omega)|C_+(x'_j), \tag{II.4.28.8}$$

and similarly for $df^{(t)}$ and $f^\dagger$, we obtain $\deg(\mathrm{div}(dg_{\eta'}^{(t)})|C_+(x'_j)) = \deg(\mathrm{div}(df^{(t)})|C_+(x'_j))$. Moreover, as $f^{(t)}$ is étale over $X_{K'}^{[t]}$, so is $g_{\eta'}^{(t)}$; hence, $\mathrm{div}(dg_{\eta'}^{(t)}|X_{K'}^{(t)})$ is supported in the tube of $\mathfrak{X}_{s'}^{(t)} - \mathfrak{X}_{s'}^{[t]}$. Therefore, we have (see also (II.3.27.1))

$$\deg(\mathrm{div}(dg_{\eta'}^{(t)}|X_{K'}^{(t)})) = \sum_{j=1}^{N} \deg(\mathrm{div}(df^{(t)})|C_+(x'_j)) = \sum_{j=1}^{N} d_{\eta,\overline{x}'_j}^{(t)}. \tag{II.4.28.9}$$

We denote by $\Delta_{ij}^{-(t)}$ the annulus $\Delta_{ij}^{(t)}$ seen as the boundary of the disc $D_{ij}^{(t)}$, with coordinate $S_j = T_j^{-1}$. Since $g_{\eta'}^{(t)}$ is étale over $\Delta_{ij}^{-(t)}$, $\mathrm{div}(dg_{\eta'}^{(t)})|D_{ij}^{(t)} - \Delta_{ij}^{-(t)}$ is supported on $C_+(y_j)$. As $D_{ij}^{(t)} - \Delta_{ij}^{-(t)} = C_+(y_j)$, and $(g^{(t)})'(T_j)$ and $(f^{(t)})'(T_j)$ have the same order σ_{ij} on the annulus $\Delta_{ij}^{(t)}$, Lemma II.2.21 again yields

$$\deg(\mathrm{div}((dg_{\eta'}^{(t)}|D_{ij}^{(t)} - \Delta_{ij}^{-(t)}))) = \mathrm{ord}_{y_j}((g_{\eta'}^{(t)})'(T_j^{-1})) = -2 - \sigma_{ij}. \tag{II.4.28.10}$$

Summing (II.4.28.10) over j and adding (II.4.28.9), we find at last that the total degree is

$$\deg(\mathrm{div}(dg^{(t)}_{\eta'})) = \sum_{j=1}^{N} d^{(t)}_{\eta,\overline{x}'_j} - \sigma_i - 2\delta_f(i). \tag{II.4.28.11}$$

Now, let $R\Psi$ be the nearby cycles functor associated to the proper structure morphism $Y^{(t)}_{S'} \to S'$ and let Λ be a finite field of characteristic different from p. Denoting by Z the closed subset $\mathfrak{X}^{(t)}_{s'} - \mathfrak{X}^{[t]}_{s'}$ of the special fiber $\mathfrak{Y}^{(t)}_{s'} \cong Y^{(t)}_{s'}$, $i : Z \to Y^{(t)}_{s'}$ the closed immersion and $j : U = Y^{(t)}_{s'} - Z \to Y^{(t)}_{s'}$ the inclusion of the complement, the long exact sequence of cohomology induced by the short exact sequence $0 \to j_!(\Lambda_{|U}) \to \Lambda \to i_*(\Lambda_{|Z}) \to 0$ of sheaves on $Y^{(t)}_{s'}$ gives the following equality of Euler-Poincaré characteristics

$$\chi(Y^{(t)}_{s'}, \Lambda) = \chi_c(U, \Lambda_{|U}) + \chi(Y^{(t)}_{s'}, i_*(\Lambda_{|Z})), \tag{II.4.28.12}$$

where $\chi_c(\cdot)$ is the Euler-Poincaré characteristic with compact support. As the residue field of the points in Z is the algebraically closed field k, we get $\chi(Y^{(t)}_{s'}, i_*(\Lambda_{|Z})) = \dim_\Lambda H^0_{\text{ét}}(Z, \Lambda_{|Z}) = |Z| = N$. As U is a disjoint union of $\delta_f(i)$ copies of $\mathbb{A}^1_k$, we see that

$$\chi_c(U, \Lambda_{|U}) = \delta_f(i) \cdot \chi_c(\mathbb{A}^1_k, \Lambda) = \delta_f(i)(\chi(\mathbb{P}^1_k, \Lambda) - 1) = \delta_f(i). \tag{II.4.28.13}$$

Since $Y^{(t)}_{S'}$ is normal, the strict localization$Y^{(t)}_{(\overline{x}')}$ at any geometric point $\overline{x}' \to Y^{(t)}_{s'}$ is also normal; hence, $Y^{(t)}_{(\overline{x}')} \times \eta'$ is reduced. Moreover, as $Y^{(t)}_{s'} \cong \mathfrak{Y}^{(t)}_{s'}$ is reduced, so is $Y^{(t)}_{(\overline{x}')} \times s' = Y^{(t)}_{s'(\overline{x}')}$. Therefore, applying [**EGA IV**, 18.9.8] to the flat local homomorphism $Y^{(t)}_{(\overline{x}')} \to S'$, we see that the Milnor tube $Y^{(t)}_{(\overline{x}')} \times \overline{\eta}'$ is connected. As $R^i\Psi(\Lambda_{|Y^{(t)}_{\overline{\eta}'}})_{\overline{x}'} = H^i_{\text{ét}}(Y^{(t)}_{(\overline{x}')} \times \overline{\eta}', \Lambda)$ [**SGA 7**, XIII, 2.1.4], the sheaf $R^0\Psi(\Lambda_{|Y^{(t)}_{\overline{\eta}'}})$ is thus isomorphic to $\Lambda_{|Y^{(t)}_{s'}}$ and $R^i\Psi(\Lambda_{|Y^{(t)}_{\overline{\eta}'}}) = 0$ for $i > 1$ [**SGA 7**, I, Théorème 4.2]. Moreover, $R^1\Psi(\Lambda_{|Y^{(t)}_{\overline{\eta}'}})$ is concentrated in the singular locus of $Y^{(t)}_{s'}$ [**SGA 7**, XIII, 2.1.5], located in Z. It thus follows from (II.4.28.12) and (II.4.28.13) that

$$N + \delta_f(i) - \sum_{j=1}^{N} \dim_\Lambda H^1_{\text{ét}}(Y^{(t)}_{(\overline{x}'_j)} \times \overline{\eta}', \Lambda) = \chi(Y^{(t)}_{s'}, R\Psi(\Lambda)). \tag{II.4.28.14}$$

By the proper base change theorem, we also have the equality

$$\chi(Y^{(t)}_{s'}, R\Psi(\Lambda)) = \chi(Y^{(t)}_{\overline{\eta}'}, \Lambda) = 2|\pi_0(Y^{(t)}_{\overline{\eta}'})| - 2g(Y^{(t)}_{\overline{\eta}'}). \tag{II.4.28.15}$$

It remains to link the cohomology group in (II.4.28.14) to $\delta^{(t)}_{\overline{x}'_j}$ and $P^{(t)}_{s,\overline{x}'_j}$ in the following way. As $\mathfrak{X}^{(t)}_{K'}$ is a formal open subscheme of $\mathfrak{Y}^{(t)}_{K'}$, we have $\mathcal{O}_{\mathfrak{X}^{(t)},\overline{x}'_j} = \mathcal{O}_{\mathfrak{Y}^{(t)},\overline{x}'_j}$. Then, (II.2.8.2) gives that

$$\mathcal{O}_{\mathfrak{X}^{(t)},\overline{x}'_j}/\mathfrak{m}_{K'} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{Y}^{(t)}_{s'},\overline{x}'_j} = \mathcal{O}_{Y^{(t)}_{s'},\overline{x}'_j}. \tag{II.4.28.16}$$

Since, for $A \in \mathrm{Obj}(\mathcal{C}_{K'})$ (resp. $\mathrm{Obj}(\widehat{\mathcal{C}}_{K'})$), $P_s(A)$ identifies with the set of minimal prime ideals of $A/\mathfrak{m}_{K'}$, it then follows that

$$\delta(\mathcal{O}_{\mathfrak{X}^{(t)},\overline{x}'_j}) = \delta(\mathcal{O}_{Y^{(t)},\overline{x}'_j}) \quad \text{and} \quad |P_s(\mathcal{O}_{\mathfrak{X}^{(t)},\overline{x}'_j})| = |P_s(\mathcal{O}_{Y^{(t)},\overline{x}'_j})|. \tag{II.4.28.17}$$

As, locally around $\overline{x}'_j$, the couple $(Y^{(t)}/S', \overline{x}'_j)$ satisfy property (P) in II.3.14, [Kat87a, Prop. 5.9] in conjunction with (II.4.28.17) implies that

$$2\delta(\mathcal{O}_{\mathfrak{X}^{(t)},\overline{x}'_j}) - |P_s(\mathcal{O}_{\mathfrak{X}^{(t)},\overline{x}'_j})| + 1 = \dim_\Lambda H^1_{\text{ét}}(Y^{(t)}_{(\overline{x}'_j)} \times \overline{\eta}', \Lambda). \tag{II.4.28.18}$$

Finally, combining (II.4.28.5), (II.4.28.11), (II.4.28.14), (II.4.28.15) and (II.4.28.18) yields (II.4.28.1), which concludes the proof. □

II.5. Group filtrations and Swan conductors.

We recall here Kato's formalism for group filtrations and conductors [Kat87a, Sections 1 and 2].

II.5.1. Let Γ be a totally ordered $\mathbb{Q}$-vector space, with an order structure compatible with its $\mathbb{Q}$-vector space structure. It induces on the set $\Gamma \cup \{-\infty, \infty\}$ the structure of a totally ordered monoid with $-\infty$ and ∞ set as its minimum and maximum elements respectively.

II.5.2. A function $g : \Gamma \to \mathbb{Q}$ is a *step function* if there is a finite sequence $(a_i)_{0 \le i \le n}$ of elements of Γ such that $a_0 \le a_1 \le \cdots \le a_n$ and such that g is constant on each open interval $]-\infty, a_0[,]a_{i-1}, a_i[$ and $]a_n, \infty[$. For such a function g that takes the value $c_i \in \mathbb{Q}$ on $]a_{i-1}, a_i[$, setting $a = a_0, b = a_n$, we can define the integral

$$\int_a^b g(t)dt = \sum_{i=1}^n c_i(a_i - a_{i-1}) \quad \in \Gamma. \tag{II.5.2.1}$$

Given $a \le b$ in Γ, the integral (II.5.2.1) is independent of the choice of the sequence $(a_i)_{0 \le i \le n}$, such that $a_0 = a, a_n = b$.

If $a > b$, we set $\int_a^b g(t)dt = -\int_b^a g(t)dt$. If the support of g is bounded from above and $a \in \Gamma$, we set $\int_a^\infty g(t)dt = \int_a^b g(t)dt$, where b a big enough element of Γ.

II.5.3. A function $h : \Gamma \to \Gamma$ is *piecewise linear* if there is a finite sequence $(a_i)_{0 \le i \le n}$ of elements of Γ such that $a_0 \le a_1 \le \cdots \le a_n$ and, on each interval $I =]-\infty, a_0]$, $I_i = [a_{i-1}, a_i]$, $I_{n+1} = [a_n, \infty[$, we have $h(t) = b_i t + c_i$, for all $t \in I_i$, where $b_i \in \mathbb{Q}$ and $c_i \in \Gamma$.

Lemma II.5.4 ([Kat87a, 1.5])**.**
1. *If $h : \Gamma \to \Gamma$ is a bijective piecewise linear function, its inverse h^{-1} is also piecewise linear.*
2. *If $g : \Gamma \to \mathbb{Q}$ is a step function such that $g(t) > 0$ for all $t \in \Gamma$, then the function $h : \Gamma \to \Gamma$ defined by $h(t) = \int_a^t g(s)ds$, for a fixed element $a \in \Gamma$, is a bijective piecewise linear function.*

II.5.5. Let G be a finite group. An *upper* (resp. *lower*) *filtration* on G indexed by Γ is a decreasing family of normal subgroups $(G^t)_{t \in \Gamma}$ (resp. $(G_t)_{t \in \Gamma}$) of G indexed by Γ such that $G = G^0$ (resp. $G = G_0$) and for each $\sigma \in G - \{1\}$, the set $\{t \in \Gamma \mid \sigma \in G^t\}$ (resp. $\{t \in \Gamma \mid \sigma \in G_t\}$) has a maximum element denoted $j_G(\sigma)$.

II.5.6. For a lower filtration $(G_t)_{t \in \Gamma}$ on G, the *associated upper filtration* $(G^t)_{t \in \Gamma}$ is obtained as follows. By Lemma II.5.4 (2), the function $\varphi : \Gamma \to \Gamma$, defined by $\varphi(t) = \int_0^t |G_s| ds$, is a piecewise linear bijection. Let $\psi = \varphi^{-1}$ be its inverse and set $G^t = G_{\psi(t)}$ for any $t \in \Gamma$.

Lemma II.5.7 ([Kat87a, 2.3])**.** *If $(G^t)_{t\in\Gamma}$ is an upper filtration on G indexed by Γ, then there is a unique lower filtration $(G_t)_{t\in\Gamma}$ on G indexed by Γ such that $(G^t)_{t\in\Gamma}$ is the upper filtration associated to $(G_t)_{t\in\Gamma}$ in the sense of II.5.6. Explicitly, the function $\psi : \Gamma \to \Gamma$, defined by $\psi(t) = \int_0^t |G^s|^{-1} ds$, is a piecewise linear bijection. Denoting $\varphi = \psi^{-1}$, we have $G_t = G^{\varphi(t)}$.*

II.5.8. For the rest of this section, let $(G_t)_{t\in\Gamma}$ be a lower filtration on G and $(G^t)_{t\in\Gamma}$ the associated upper filtration (see II.5.6).

If H is a subgroup of G, the *induced lower filtration* on H indexed by Γ is defined by $H_t = G_t \cap H$, and the associated upper filtration formally given in II.5.6 is called the *induced upper filtration*. Any subgroup H is implicitly assumed to be endowed with these filtrations.

If H is a normal subgroup of G, the *induced upper filtration* on G/H indexed by Γ is defined by $(G/H)^t = (G^t H)/H$, and the unique asociated lower filtration on G/H, given by Lemma II.5.7, is called the *induced lower filtration* on G/H. Any quotient G/H is implicitly assumed to be endowed with these filtrations.

II.5.9. Let Λ be a field in which $|G_t|$ is invertible for any $t > 0$, which is equivalent to $|G^t|$ being invertible in Λ for any $t > 0$. For a $\Lambda[G]$-module of finite type M, we define its Swan conductor $\mathrm{sw}_G(M) \in \Gamma$ by

(II.5.9.1)
$$\mathrm{sw}_G(M) = \int_0^\infty \dim_\Lambda(M/M^{G^t}) dt = \int_0^\infty |G_t| \cdot \dim_\Lambda(M/M^{G_t}) dt.$$

Lemma II.5.10 ([Kat87a, 2.5])**.**

(1) *For a short exact sequence of $\Lambda[G]$-modules of finite type*

(II.5.10.1)
$$0 \to M' \to M \to M'' \to 0,$$

we have $\mathrm{sw}_G(M) = \mathrm{sw}_G(M') + \mathrm{sw}_G(M'')$.

(2) *If Λ is of characteristic zero, then, in $\Lambda \otimes_{\mathbb{Q}} \Gamma$, we have*

(II.5.10.2)
$$\mathrm{sw}_G(M) = \sum_{\sigma\in G, \sigma\neq 1} (\dim_\Lambda M - \chi_M(\sigma)) \otimes j_G(\sigma),$$

where χ_M is the character of M.

Lemma II.5.11 ([Kat87a, 2.7])**.** *Let H be a subgroup of G.*

(1) *Denote by $\Lambda[G/H]$ the regular Λ-valued representation $\mathrm{Ind}_H^G 1_H$ of G. Then,*

(II.5.11.1)
$$\mathrm{sw}_G(\Lambda[G/H]) = \sum_{\sigma\in G-H} j_G(\sigma).$$

(2) *If M is a $\Lambda[H]$-module of finite type, then*

(II.5.11.2)
$$\mathrm{sw}_G\left(\Lambda[G] \otimes_{\Lambda[H]} M\right) = [G : H]\, \mathrm{sw}_H(M) + (\dim_\Lambda M)\, \mathrm{sw}_G(\Lambda[G/H]).$$

(3) *If the subgroup H is normal and M is a $\Lambda[G/H]$-module of finite type, then*

(II.5.11.3)
$$\mathrm{sw}_G(M) = \mathrm{sw}_{G/H}(M).$$

II.5.12. By Lemma II.5.10 (1), the Swan conductor extends to the Grothendieck group $R_\Lambda(G)$ of $\Lambda[G]$-modules of finite type. If $\chi \in R_\Lambda(G)$ is of dimension zero, then (II.5.11.2) gives

(II.5.12.1) $$\mathrm{sw}_G(\mathrm{Ind}_H^G \chi) = [G : H]\,\mathrm{sw}_H(\chi).$$

Lemma II.5.13 ([Kat87a, 2.9])**.** *Let H be a normal subgroup of G, and let φ_G and ψ_G (resp. φ_H and ψ_H, resp. $\varphi_{G/H}$ and $\psi_{G/H}$) be the bijective functions $\Gamma \to \Gamma$ associated to the filtrations on G (resp. the induced filtrations on H, resp. the induced filtrations on G/H). Then,*

(1) $\varphi_G = \varphi_{G/H} \circ \varphi_H$ *and* $\psi_G = \psi_H \circ \psi_{G/H}$.
(2) *If $t \in \Gamma$ and $s = \psi_{G/H}(t)$, then the induced upper filtration on H is given by $H^s = G^t \cap H$.*

II.6. Ramification of $\mathbb{Z}^2$-valuation rings.

II.6.1. Let V be a valuation ring with value group Γ_V, field of fractions K and valuation map $v_K : K^\times \to \Gamma_V$. Let L be a finite Galois extension of K of group G and W the integral closure of V in L. We assume that W/V is a monogenic integral extension of valuation rings (II.3.12). We put $\Gamma = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_V = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_W$ (*cf.* II.3.1) and let $v : L^\times \to \Gamma$ be the unique valuation of L such that $v|_{K^\times} = v_K$. Let ε be an element of Γ such that, for any $\sigma \in G$ and $x \in W$, we have $v(\sigma(x) - x) \geq |G|^{-1}\varepsilon$ (e.g. $\varepsilon = 0$). Then, set

(II.6.1.1) $$i_G(\sigma) = \min\{v(\sigma(x) - x) \mid x \in W\} \in \Gamma_W \quad \text{for} \quad \sigma \in G - \{1\}, \quad i_G(1) = \infty.$$

The minimum in (II.6.1.1) exists and is equal to $v(\sigma(a) - a)$ for any element a of W such that $W = V[a]$. Indeed, this follows readily from an induction argument using the *almost derivation formula*

(II.6.1.2) $$\sigma(xy) - xy = (\sigma(x) - x)y + \sigma(x)(\sigma(y) - y).$$

For $\sigma \in G$, set also

(II.6.1.3) $$j_{G,\varepsilon}(\sigma) = i_G(\sigma) - \frac{\varepsilon}{|G|} \in \Gamma.$$

Notice that $i_G(\sigma^{-1}) = i_G(\sigma)$ and $j_{G,\varepsilon}(\sigma^{-1}) = j_{G,\varepsilon}(\sigma)$. We now define the *lower ramification filtration* on G indexed by Γ by setting

(II.6.1.4) $$G_{t,\varepsilon} = \{\sigma \in G \mid j_{G,\varepsilon}(\sigma) \geq t\}.$$

Lemma II.6.2 ([Kat87a, Lemma (3.2)])**.** *Let V, K, W, L and G be as above. Let H be a subgroup of G, $K' = L^H$ the corresponding sub-extension of L/K and V' the integral closure of V in K'. Then, for $\tau \in G/H - \{1\}$, the minimum element $i_{G/H}(\tau)$ of the subset $\{v(\tau(y) - y) \mid y \in V'\}$ of Γ exists and equals $\sum_{\sigma \mapsto \tau} i_G(\sigma)$, where σ runs over the representatives of τ in G.*

Corollary II.6.3 ([Kat87a, Corollary (3.3)])**.** *We keep the assumptions of* II.6.2 *and assume moreover that the subgroup H is normal. Then, the induced lower filtration on G/H* (cf. II.5.8) *coincides with the lower ramification filtration on G/H defined by $j_{G/H,\varepsilon}$.*

II.6.4. For the rest of this section, we let V be a *henselian* $\mathbb{Z}^2$-valuation ring with field of fractions K, L a finite Galois extension of K of group G and W the integral closure of V in L. We denote by $(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$ and $(0) \subsetneq \mathfrak{p}' \subsetneq \mathfrak{m}'$ the prime ideals of V and W respectively. We assume that the residue field $\kappa(\mathfrak{m})$ is *perfect*, that $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}')$, that $[L : K] = [\kappa(\mathfrak{p}') : \kappa(\mathfrak{p})]$ and, if $\kappa(\mathfrak{p})$ has characteristic $p > 0$, that $[\kappa(\mathfrak{p}) : \kappa(\mathfrak{p})^p] = p$. Then, by II.3.9, II.3.10 and II.3.11, W/V is a monogenic integral extension of $\mathbb{Z}^2$-valuation rings. Moreover, as $[L : K] = [\Gamma_W : \Gamma_V] \cdot [\kappa(\mathfrak{m}') : \kappa(\mathfrak{m})]$ and

$[\kappa(\mathfrak{p}'):\kappa(\mathfrak{p})]=e_{\mathfrak{p}'/\mathfrak{p}}f_{\mathfrak{p}'/\mathfrak{p}}$, where $e_{\mathfrak{p}'/\mathfrak{p}}$ and $f_{\mathfrak{p}'/\mathfrak{p}}=[\kappa(\mathfrak{m}'):\kappa(\mathfrak{m})]$ are respectively the ramification index and residue degree of the extension of discrete valuation rings $V/\mathfrak{p}\subset W/\mathfrak{p}'$, we have (II.3.9)

(II.6.4.1) $$[\Gamma_W:\Gamma_V]=|G|=e_{\mathfrak{p}'/\mathfrak{p}}.$$

Recall from II.3.7 that ε_K (resp. ε_L) corresponds to a generator of the unique non-trivial isolated subgroup of Γ_V (resp. Γ_W), hence a uniformizer of $V/\mathfrak{p}$ (resp. $W/\mathfrak{p}'$). From II.3.9 (iii'), we see that the ramification index of $W_{\mathfrak{p}'}/V_{\mathfrak{p}}$ is one and thus W/V has (weak) ramification only on the second factor in (II.3.7.1), which corresponds to the extension $V/\mathfrak{p}\subset W/\mathfrak{p}'$. It follows that

(II.6.4.2) $$\varepsilon_K=e_{\mathfrak{p}'/\mathfrak{p}}\varepsilon_L=|G|\varepsilon_L.$$

Therefore, we can put $\varepsilon=\varepsilon_K$ in II.6.1 and thus get a lower ramification filtration on the group G (II.6.1.4). For the remainder of the section, we write simply i_G and j_G for $i_{G,\varepsilon}$ and $j_{G,\varepsilon}$.

II.6.5. Let U_L be the group of units in W

(II.6.5.1) $$U_L=\{x\in L^\times\mid v(x)=0\}.$$

We define a decreasing filtration $(U_t)_{t\in\Gamma_{W\geq 0}}$ on U_L by $U_0=U_L$, and for $t>0$, $U_t=1+\mathfrak{m}'_t$, where $\mathfrak{m}'_t$ is the ideal of elements x in W such that $v(x)\geq t$. Since $\mathfrak{m}'=\mathfrak{m}'_{\varepsilon_L}$, reduction modulo $\mathfrak{m}'$ yields a canonical group isomorphism

(II.6.5.2) $$U_0/U_{\varepsilon_L}\xrightarrow{\sim}\kappa(\mathfrak{m}')^\times.$$

For $t>0$, the surjection from U_t to $\mathfrak{m}'_t$ given by $1+x\mapsto x$ also induces a group isomorphism

(II.6.5.3) $$U_t/U_{t+\varepsilon_L}\xrightarrow{\sim}\mathfrak{m}'_t/\mathfrak{m}'_{t+\varepsilon_L}.$$

The additive group $\mathfrak{m}'_t/\mathfrak{m}'_{t+\varepsilon_L}$ is non canonically isomorphic to the additive group $\kappa(\mathfrak{m}')$ as follows. Choose $b\in\mathfrak{m}'_t-\mathfrak{m}'_{t+\varepsilon_L}$, and define a surjective map from $\mathfrak{m}'_t$ to W by $x\mapsto x/b$. Since $v(x/b)=v(x)-t$, this map is well defined and induces an isomorphism

(II.6.5.4) $$\mathfrak{m}'_t/\mathfrak{m}'_{t+\varepsilon_L}\xrightarrow{\sim}\kappa(\mathfrak{m}').$$

II.6.6. Let $t\geq 0$ be in Γ_W. For $\sigma\in G_t$ and e in W such that $v(e)=\varepsilon_L$, (II.6.1.4) and (II.6.4.2) give

(II.6.6.1) $$v(\sigma(e)/e-1)=v(\sigma(e)-e)-\varepsilon_L\geq t.$$

So $\sigma(e)/e\in U_t$, and we set $\theta_t(\sigma)=\sigma(e)/e \mod U_{t+\varepsilon_L}$. The element $\theta_t(\sigma)$ is independent of the choice of e in W satisfying $v(e)=\varepsilon_L$. Indeed, any other such e' is of the form ue, where u is a unit in W, and since $\sigma\in G_t$, we have

(II.6.6.2) $$v(\sigma(u)/u-1)=v(\sigma(u)-u)\geq t+\varepsilon_L,\quad\text{hence}\quad \frac{\sigma(e')}{e'}=\frac{\sigma(e)}{e}\mod U_{t+\varepsilon_L}.$$

If $\tau\in G_t$, then,

(II.6.6.3) $$\frac{\tau\sigma(e)}{e}=\frac{\tau(\sigma(e))}{\sigma(e)}\cdot\frac{\sigma(e)}{e},\quad\text{and}\quad v(\sigma(e))=\varepsilon_L;$$

hence, by the aforementioned independence, we have $\theta_t(\tau\sigma)=\theta_t(\tau)\theta_t(\sigma)$. Thus, we have defined a group homomorphism

(II.6.6.4) $$\theta_t:G_t\to U_t/U_{t+\varepsilon_L}$$

Now it is clear that the subgroup $G_{t+\varepsilon_L}$ of G_t is exactly the kernel of θ_t. In conclusion we get an injective group homomorphism

$$\overline{\theta}_t : G_t/G_{t+\varepsilon_L} \hookrightarrow U_t/U_{t+\varepsilon_L}. \tag{II.6.6.5}$$

We thus have proved the following.

Proposition II.6.7. *For any $t \in \Gamma_W$, the map $\overline{\theta}_t$ induces an isomorphism between $G_t/G_{t+\varepsilon_L}$ and a subgroup of $U_t/U_{t+\varepsilon_L}$.*

Corollary II.6.8. *The group G/G_{ε_L} is cyclic of order prime to the characteristic of $\kappa(\mathfrak{m}')$.*

Proof. The group U_0/U_{ε_L} is identified with $\kappa(\mathfrak{m}')^\times$ (II.6.5.2). Setting $t = 0$ in proposition II.6.7, we see that G/G_{ε_L} is thus a finite subgroup of the group of roots of unity in $\kappa(\mathfrak{m}')$. Therefore, it is cyclic of order prime to the characteristic of $\kappa(\mathfrak{m}')$ [Bou07, Chap. V, §11, n°1, Théorème 1]. □

Corollary II.6.9. *If the characteristic of $\kappa(\mathfrak{m}')$ is 0, then $G_{\varepsilon_L} = \{1\}$, and G is cyclic.*

Proof. The proof is the same as in [Ser68, Chap. IV, §2, Proposition 7, corollaire 2]. From (II.6.5.3), (II.6.5.4), Proposition II.6.7 and the fact that $\kappa(\mathfrak{m}')$ has no non trivial finite subgroup in characteristic zero, we deduce that $G_t = G_{t+\varepsilon_L}$ for any $t > 0$; since these subgroups are trivial for t large enough, we see that $G_{\varepsilon_L} = \{1\}$, and thus G is cyclic by Corollary II.6.8. □

Corollary II.6.10. *If the characteristic of $\kappa(\mathfrak{m}')$ is $p > 0$, then the quotients $G_t/G_{t+\varepsilon_L}$, $t > 0$ in Γ_W, are directs products of cyclic groups of order p, and G_{ε_L} is a p-group.*

Proof. The proof proceeds as in [Ser68, Chap. IV, §2, Proposition 7, corollaire 3]. We get the first half of the lemma from (II.6.5.3), (II.6.5.4), Proposition II.6.7 and the fact that, in characteristic p, every subgroup of $\kappa(\mathfrak{m}')$ is an $\mathbb{F}_p$-vector space, hence a direct product of cyclic groups of order p; the second half ensues because the order of G_{ε_L} is the product of the orders of the $G_t/G_{t+\varepsilon_L}$ for $t > 0$. □

Remarks II.6.11.
- (i) For every $\sigma \in G - G_{\varepsilon_L}$, it is clear that $i_G(\sigma) = \varepsilon_L$. For every $\sigma \in G_{\varepsilon_L}$ and every $n \in \mathbb{Z} - \{0\}$ prime to the characteristic of $\kappa(\mathfrak{m}')$, we have $i_G(\sigma^n) = i_G(\sigma)$. Indeed, in positive residue characteristic $p > 0$, if σ is in $G_t - G_{t+\varepsilon_L}$ for some $t \geq \varepsilon_L$, i.e. if $\sigma \neq 0$ in $G_t/G_{t+\varepsilon_L}$, so is σ^n since this quotient is p-group and $p \nmid n$.
- (ii) We deduce from (i) that $i_G(\sigma) = i_G(\tau)$ if σ and τ generate the same subgroup of G.

II.6.12. We keep the notation and assumptions of II.6.4. Recall that ε_L is the minimum of the set of positive elements of Γ_W. Define a_G and sw_G to be the following functions from G to Γ. For $\sigma \neq 1$ in G, we set ((II.6.1.1), (II.6.1.3))

$$a_G(\sigma) = -i_G(\sigma) \quad \text{and} \quad \mathrm{sw}_G(\sigma) = -j_G(\sigma). \tag{II.6.12.1}$$

We set also

$$a_G(1) = \sum_{\sigma \in G-\{1\}} i_G(\sigma) \quad \text{and} \quad \mathrm{sw}_G(1) = \sum_{\sigma \in G-\{1\}} j_G(\sigma). \tag{II.6.12.2}$$

Clearly, a_G and sw_G are class functions on G. They satisfy the relation

$$a_G = \mathrm{sw}_G + \varepsilon_L \cdot u_G \tag{II.6.12.3}$$

where u_G is the augmentation character of G.

II.6.13. Let ℓ be a prime number and let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$. For $\Lambda_{\mathbb{Q}}$ a finite extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ with valuation ring Λ, we denote by $R_{\Lambda_{\mathbb{Q}}}(G)$ (resp. $P_\Lambda(G)$) the Grothendieck group of finitely generated $\Lambda_{\mathbb{Q}}[G]$-modules (resp. of finitely generated projective $\Lambda[G]$-modules). For $\chi \in R_{\Lambda_{\mathbb{Q}}}(G)$ and ψ a class function on G with values in Γ, we define the pairing

$$\langle\langle \psi, \chi \rangle\rangle = \sum_{\sigma \in G} \psi(\sigma^{-1}) \otimes \mathrm{tr}_\chi(\sigma) \quad \in \Gamma \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}. \tag{II.6.13.1}$$

Remark II.6.14. If M is a finitely generated $\Lambda_{\mathbb{Q}}[G]$-module of character χ_M, then

$$\mathrm{sw}_G(M) = \langle\langle \mathrm{sw}_G, \chi_M \rangle\rangle. \tag{II.6.14.1}$$

Indeed, this follows from expanding the sum in the formula (II.5.10.2).

Theorem II.6.15 ([**Kat87a**, Theorem 4.7])**.** *Assume that G is abelian. Then, for any $\Lambda_{\mathbb{Q}}[G]$-module of finite type M, we have*

$$\mathrm{sw}_G(M) \in \Gamma_V \subsetneq \Gamma. \tag{II.6.15.1}$$

This is an analogue of the Hasse-Arf theorem. Kato's proof of this result relies on an interpretation of the ramification filtration (G^t) in terms of a filtration of the Milnor K-group $K_2(K)$ via a higher reciprocity map [**Kat87a**, Theorem 4.4].

II.6.16. For the rest of this section, we let π be a uniformizer of $V_{\mathfrak{p}}$. Recall from II.3.7 that π induces group homomorphisms $\alpha, \beta : \Gamma_V \to \mathbb{Z}$ characterized respectively by $\alpha(v(\pi)) = 1$, $\alpha(\varepsilon_K) = 0$, $\beta(v(\pi)) = 0$ and $\beta(\varepsilon_K) = 1$. We also denote by α and β their extensions to Γ. Notice that, because $\varepsilon_K = |G|{\cdot}\varepsilon_L$, β sends Γ_W in $\frac{1}{|G|} \cdot \mathbb{Z}$. The functions a_G and sw_G induce class functions

$$a_G^\beta = \beta \circ a_G : G \to \mathbb{Z} \quad \text{and} \quad \mathrm{sw}_G^\beta = \beta \circ \mathrm{sw}_G : G \to \mathbb{Z}. \tag{II.6.16.1}$$

The relation (II.6.12.3) is carried into

$$a_G^\beta = \mathrm{sw}_G^\beta + u_G. \tag{II.6.16.2}$$

Composing with α, we see that $\alpha \circ a_G = \alpha \circ \mathrm{sw}_G$ which also defines a class function a_G^α on G. Note that, while a_G^α is independent of the chosen uniformizer π, a_G^β and sw_G^β do depend on π a priori (II.3.7).

Remark II.6.17. A straightforward calculation shows that

$$\beta(\langle\langle \mathrm{sw}_G, \chi \rangle\rangle) = |G|{\cdot}\langle \mathrm{sw}_G^\beta, \chi \rangle, \tag{II.6.17.1}$$

for any $\chi \in R_{\Lambda_{\mathbb{Q}}}(G)$, where $\langle \cdot, \cdot \rangle$ denotes the usual pairing between $\Lambda_{\mathbb{Q}}$-valued class functions on G [**Ser98**, 2.2, Remarques]. The same is true with a_G and a_G^α.

Lemma II.6.18 ([**Kat87a**, Lemme 3.5])**.** *Let H be a subgroup of G, K' the corresponding subextension of L/K and V' the valuation ring of K'. Assume that the extension of valuation rings V'/V is monogenic integral (hence V' is a free V-module of finite rank by* II.3.9*) and denote by $\mathfrak{d}_{V'/V}$ a generator of the discriminant ideal of the extension V'/V. Then, with the notation of* II.3.7*, we have*

$$v(\mathfrak{d}_{V'/V}) = [G : H] \sum_{\sigma \in G - H} i_G(\sigma), \tag{II.6.18.1}$$

$$\langle\langle a_G, \Lambda_{\mathbb{Q}}[G/H] \rangle\rangle = v(\mathfrak{d}_{V'/V}), \tag{II.6.18.2}$$

$$\langle\langle \mathrm{sw}_G, \Lambda_{\mathbb{Q}}[G/H]\rangle\rangle = v(\mathfrak{d}_{V'/V}) - ([G:H]-1)\varepsilon_K, \tag{II.6.18.3}$$

$$\langle a_G^\alpha, \Lambda_{\mathbb{Q}}[G/H]\rangle = \frac{1}{|G|}v^\alpha(\mathfrak{d}_{V'/V}), \tag{II.6.18.4}$$

$$\langle \mathrm{sw}_G^\beta, \Lambda_{\mathbb{Q}}[G/H]\rangle = \frac{1}{|G|}v^\beta(\mathfrak{d}_{V'/V}) - \frac{1}{|H|} + \frac{1}{|G|}. \tag{II.6.18.5}$$

Proof. Let b in V' be such that $V' = V[b]$. We see from [Ser68, Chap. III, §6, Prop. 11, Cor. 2] that the different $\mathcal{D}_{V'/V}$ is generated by

$$x = \prod_{\tau\in G/H-\{1\}} (\tau(b) - b) \tag{II.6.18.6}$$

Hence, $\mathfrak{d}_{V'/V}$ is generated by $N_{K'/K}(x)$, where $N_{K'/K} : K'^\times \to K^\times$ is the norm map. Since V'/V is monogenic integral, the extension $V'_{\mathfrak{q}}/V_{\mathfrak{p}}$ has ramification index 1 and thus residue degree $|G/H|$. So,

$$v(\mathfrak{d}_{V'/V}) = v(N_{K'/K}(x)) = |G/H|v(x) = |G/H| \sum_{\tau\in G/H-\{1\}} i_{G/H}(\tau). \tag{II.6.18.7}$$

Therefore, (II.6.18.1) follows from (II.6.18.4) and Lemma II.6.2. We deduce (II.6.18.3) from (II.5.11.1), (II.6.1.3) and (II.6.18.1). Since $a_G = \mathrm{sw}_G + \varepsilon_L u_G = \mathrm{sw}_G + (\varepsilon_K/|G|)u_G$ and $\langle\langle u_G, \Lambda_{\mathbb{Q}}[G/H]\rangle\rangle = |G|([G:H]-1)$, (II.6.18.2) follows from (II.6.18.3). Now, (II.6.18.4) follows from II.6.17 and (II.6.18.2). As $\beta(\varepsilon_K) = |G|$ (II.6.16), the identity (II.6.18.5) is deduced from (II.6.17.1) and (II.6.18.3). □

Lemma II.6.19. *Let H be a subgroup of G, K' the corresponding sub-extension of L/K and V' the valuation ring of K'.*

(*i*) *Denoting r_H the character of the regular representation of H, we have the following relations between class functions*

$$a_G^\alpha|H = \frac{1}{|G|}v^\alpha(\mathfrak{d}_{V'/V})\cdot r_H + a_H^\alpha, \quad a_G^\beta|H = \frac{1}{|G|}v^\beta(\mathfrak{d}_{V'/V})\cdot r_H + a_H^\beta \tag{II.6.19.1}$$

$$\mathrm{sw}_G^\beta|H = (\frac{1}{|G|}v^\beta(\mathfrak{d}_{V'/V}) + 1 - [G:H])\cdot r_H + \mathrm{sw}_H^\beta. \tag{II.6.19.2}$$

(*ii*) *Assume that H is normal. We have the following relations between class functions*

$$a_{G/H}^\alpha = \mathrm{Ind}_G^{G/H} a_G^\alpha, \quad a_{G/H}^\beta = \mathrm{Ind}_G^{G/H} a_G^\beta, \quad \text{and} \quad \mathrm{sw}_{G/H}^\beta = \mathrm{Ind}_G^{G/H}\mathrm{sw}_G^\beta. \tag{II.6.19.3}$$

Proof. (*i*) Since $\mathrm{sw}_G^\beta|H = a_G^\beta|H - (r_G - 1_G)|H$ (II.6.16.2) and $r_G|H = [K':K]\cdot r_H$, (II.6.19.2) follows from (II.6.19.1). We have $a_G^\alpha|H(\sigma) = a_H^\alpha(\sigma)$ and $a_G^\beta|H(\sigma) = a_H^\beta(\sigma)$, if $\sigma \in H - \{1\}$, since, in that case, $i_H(\sigma) = i_G(\sigma)$. Moreover, from (II.6.18.1), we see that

$$\frac{|H|}{|G|}v(\mathfrak{d}_{V'/V}) = \sum_{\sigma\in G-\{1\}} i_G(\sigma) - \sum_{\sigma\in H-\{1\}} i_G(\sigma). \tag{II.6.19.4}$$

Hence, $a_G(1) = \frac{1}{|G|}v(\mathfrak{d}_{V'/V})\cdot r_H(1) + a_H(1)$, which gives (II.6.19.1) by applying α and β.

(*ii*) From Lemma II.6.2, we get

$$i_{G/H}(\tau) = \sum_{\sigma\to\tau} i_G(\sigma) \quad \text{and} \quad j_{G/H}(\tau) = \sum_{\sigma\to\tau} j_G(\sigma). \tag{II.6.19.5}$$

These two equalities imply (ii). □

Proposition II.6.20. *The class function $|G|\mathrm{sw}_G^\beta$ is an element of $P_{\mathbb{Z}_\ell}(G)$.*

Proof. By [Ser98, 16.2, Théorème 37], claim is equivalent to asking that (1) $|G|\mathrm{sw}_G^\beta \in R_{\Lambda_\mathbb{Q}}(G)$, for some finite extension $\Lambda_\mathbb{Q}$ of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$, and that (2) $\mathrm{sw}_G^\beta(\sigma) = 0$ for every σ of order divisible by ℓ. To prove (1), it is enough to show that $|G|\mathrm{sw}_G^\beta \in R_{\overline{\mathbb{Q}}_\ell}(G)$; hence, by Brauer induction, it is enough to prove that $|G|\langle \mathrm{sw}_G^\beta|H, \chi\rangle \in \mathbb{Z}$ for every elementary subgroup H of G and every homomorphism $\chi : H \to \overline{\mathbb{Q}}_\ell^\times$ [Ser98, 11.1, Théorème 22, Corollaire]. By (II.6.19.2), it is enough to prove that $\langle \mathrm{sw}_G^\beta, \chi\rangle \in \mathbb{Z}$ for every homomorphism $\chi : G \to \overline{\mathbb{Q}}_\ell^\times$. (Notice that, in the formula (II.6.19.2), as $\mathfrak{d}_{V'/V} \in V$, $v^\beta(\mathfrak{d}_{V'/V})$ lies in $\mathbb{Z}$ (II.6.16)). Applying (II.6.19.3) to $H = \ker(\chi)$, we can assume that χ is injective, hence that G is abelian. Now, by the Hasse-Arf theorem II.6.15, $\mathrm{sw}_G(M_\chi)$ is in Γ_V, where M_χ is the $\overline{\mathbb{Q}}_\ell[G]$-module defined by χ; hence, $\beta(\mathrm{sw}_G(M_\chi)) \in \mathbb{Z}$ is in $|G|\cdot\mathbb{Z}$. Since $\mathrm{sw}_G(M_\chi) = \langle\langle \mathrm{sw}_G, \chi_M\rangle\rangle$ (II.6.14), we thus have (II.6.17.1)

$$|G|\langle \mathrm{sw}_G^\beta, \chi\rangle = \beta(\mathrm{sw}_G(M_\chi)) \in \mathbb{Z}. \tag{II.6.20.1}$$

For (2), let σ be an element of G of order divisible by ℓ. Since ℓ is different from the characteristic of $\kappa(\mathfrak{m})$, σ is in $G - G_{\varepsilon_L}$ (II.6.9 and II.6.10). By (II.6.1.4) and (II.6.4.2), this means that $j_G(\sigma) < \varepsilon_L$, i.e $i_G(\sigma) < 2\varepsilon_L$, hence $i_G(\sigma) = \varepsilon_L$ by minimality of ε_L. So $j_G(\sigma) = 0$ and $\mathrm{sw}_G^\beta(\sigma) = 0$. Thus, the proposition is proved. □

Remark II.6.21. If $\sigma \in G - G_{\varepsilon_L}$, then $i_G(\sigma) = \varepsilon_L$, hence $a_G^\alpha(\sigma) = 0$ and $\mathrm{sw}_G^\alpha(\sigma) = 0$.

II.7. Variation of conductors.

II.7.1. Let K be a complete discrete valuation field, $\mathcal{O}_K$ its valuation ring, $\mathfrak{m}_K$ its maximal ideal, k its residue field, assumed to be algebraically closed of characteristic $p > 0$, and π a uniformizer of $\mathcal{O}_K$. Let $D = \mathrm{Sp}(K\{\xi\})$ be the rigid unit disc over K, X a smooth and connected K-affinoid curve endowed with a right action of a finite group G, and let $f : X \to D$ be a finite flat morphism such that $X/G \cong D$. The ring $\mathcal{O}(X)^G$ is a K-affinoid closed sub-algebra of $\mathcal{O}(X)$ and $\mathcal{O}(X)$ is finite over $\mathcal{O}(X)^G$ [BGR84, 6.3.3/3]. As D and X are affinoid, assuming that $X/G \cong D$ is equivalent to requiring $\mathcal{O}(D) \cong \mathcal{O}(X)^G$. Assume that f is étale and Galois of group G over an admissible open subset of D containing 0.

Recall that, in II.4.22, we defined the functions $\partial_f^\alpha : \mathbb{Q}_{\geq 0} \to \mathbb{Q}$ and $\partial_f^\beta : \mathbb{Q}_{\geq 0} \to \mathbb{Z}$ that measure the valuation of the discriminant of f. Recall also that in II.4.24, to a rational number $r \geq 0$, denoting by $X^{(r)}$ the inverse image by f of the subdisc $D^{(r)}$ of D of radius $|\pi|^r$, we associated a henselian $\mathbb{Z}^2$-valuation ring $V_r^h = V_{r,K'}^h$, with degree of imperfection p for its residue field at its height 1 prime ideal, and a set $S_f^{(r)}$ of couples $\tau = (\overline{x}_\tau, \mathfrak{p}_\tau)$, where K' is a finite extension of K which is r-admissible for f (see II.4.10), $\overline{x}_\tau$ is a geometric point of the special fiber of the normalized integral model $\mathfrak{X}_{K'}^{(r)}$ of $X_{K'}^{(r)}$ (defined over $\mathcal{O}_{K'}$), and $\mathfrak{p}_\tau$ is a height one prime ideal of $\mathcal{O}_{\mathfrak{X}_{K'}^{(r)},\overline{x}_\tau}$. To a couple $\tau \in S_f^{(r)}$, we associated a henselian $\mathbb{Z}^2$-valuation ring $V_r^h(\tau)$ which is a monogenic integral extension of V_r^h (II.3.12), (II.3.23).

II.7.2. The group G acts also on $X^{(r)}$ and we have $X^{(r)}/G \cong D^{(r)}$. This induces actions of G on $\mathfrak{X}_{K'}^{(r)}$ and $\mathfrak{X}_{s'}^{(r)}$. We thus obtain an action of G on $S^{(r)}$ as follows. For $g \in G$ and $\tau = (\overline{x}_\tau, \mathfrak{p}_\tau)$, we put $g \cdot \tau = (g \circ \overline{x}_\tau, g_{\#}^{-1}(\mathfrak{p}_\tau))$, where $g_{\#}$ is the induced isomorphism

$$\mathcal{O}_{\mathfrak{X}_{K'}^{(r)}, g\circ\overline{x}_\tau} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}_{K'}^{(r)}, \overline{x}_\tau}. \tag{II.7.2.1}$$

Lemma II.7.3. *The above action of G on the set $S_f^{(r)}$ is transitive.*

Proof. The group G acts transitively on the finite set of geometric connected components of $\mathfrak{X}_{K'}^{(r)}$. Hence, enlarging K', we can assume that the connected components of $\mathfrak{X}_{K'}^{(r)}$ are geometrically connected and that G acts transitively on the set formed by these connected components. Let $\mathfrak{X}_c^{(r)}$ be a connected component of $\mathfrak{X}_{K'}^{(r)}$ and let G_c be its stabilizer. To prove the lemma, its enough to show that G_c acts transitively on the subset of $S_f^{(r)}$ formed by couples $(\overline{x}_\tau, \mathfrak{p}_\tau)$ such that the image of $\overline{x}_\tau$ in $\mathfrak{X}_{s'}^{(r)}$ lies in $\mathfrak{X}_{c,s'}^{(r)}$. Therefore, we may assume that $\mathfrak{X}_{K'}^{(r)}$ is geometrically connected; hence $X_{K'}^{(r)}$ and $\mathfrak{X}_{s'}^{(r)}$ are also geometrically connected [**AS02**, 4.4]. As r and K' are fixed, we write $A, \mathcal{A}, B$ and $\mathcal{B}$ for $\mathcal{O}(D_{K'}^{(r)}), \mathcal{O}^\circ(D_{K'}^{(r)}), \mathcal{O}(X_{K'}^{(r)})$ and $\mathcal{O}^\circ(X_{K'}^{(r)})$ respectively.

First, we have the identity $A = B^G$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & B \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & A \end{array} \tag{II.7.3.1}$$

where the horizontal arrows are inclusions. Since $\mathcal{A}$ is a normal integral domain, hence integrally closed, and $\mathcal{B}$ is finite over $\mathcal{A}$ (II.4.9), we find that we have also $\mathcal{A} = \mathcal{B}^G$. As $\mathrm{Spec}(\mathcal{A})$ and $\mathrm{Spec}(\mathcal{B})$ are both noetherian and connected, it follows from [**Fu11**, 3.2.8] that G acts transitively on the set $\mathrm{Spec}(\mathcal{B})(\overline{x})$ of geometric points of $\mathrm{Spec}(\mathcal{B})$ above $\overline{x}$, where $\overline{x}$ (notation from II.4.19) is regarded as a geometric point of $\mathrm{Spec}(\mathcal{A})$ through

$$\overline{x} \to \mathfrak{D}_{s'}^{(r)} = \mathrm{Spec}(\mathcal{A}/(\pi)) \to \mathrm{Spec}(\mathcal{A}). \tag{II.7.3.2}$$

(Note that, even though $\mathrm{Spec}(\mathcal{B}) \to \mathrm{Spec}(\mathcal{A})$ may not be étale, we can still apply [**Fu11**, 3.2.8] since we only use the implication "(i) $\Rightarrow$ (ii)" therein.) We deduce that G acts transitively too on the set $\mathfrak{X}_{s'}^{(r)}(\overline{x}) = \mathfrak{X}_{K'}^{(r)}(\overline{x})$ of geometric points of $\mathfrak{X}_{K'}^{(r)} = \mathrm{Spf}(\mathcal{B})$ above $\overline{x}$.

Second, let $\overline{x}'$ be one such geometric point. Then, its stabilizer $\mathrm{St}_{\overline{x}'}$ in G acts on the noetherian local ring $\mathcal{O}_{\overline{x}'} = \mathcal{O}_{\mathfrak{X}_{K'}^{(r)}, \overline{x}'}$. As $\mathcal{B}^G = \mathcal{A}$ and (II.2.12.1)

$$\mathcal{O}_{\overline{x}} \otimes_{\mathcal{A}} \mathcal{B} \cong \prod_{\overline{x}' \in \mathrm{Spf}(\mathcal{B})(\overline{x})} \mathcal{O}_{\overline{x}'}, \tag{II.7.3.3}$$

where $\mathcal{O}_{\overline{x}}$ is the noetherian local ring $\mathcal{O}_{\mathfrak{D}_{K'}^{(r)}, \overline{x}}$, we have $(\mathcal{O}_{\overline{x}'})^{\mathrm{St}_{\overline{x}'}} = \mathcal{O}_{\overline{x}}$. Then, by [**Fu11**, 3.1.1 (ii)], $\mathrm{St}_{\overline{x}'}$ acts transitively on the set of prime ideals of $\mathcal{O}_{\overline{x}'}$ above the height 1 prime ideal $\mathfrak{p}$ of $\mathcal{O}_{\overline{x}}$ (notation of II.4.19).

Now, let $\tau = (\overline{x}_\tau, \mathfrak{p}_\tau)$ and $\tau' = (\overline{x}_{\tau'}, \mathfrak{p}_{\tau'})$ be two elements of $S_f^{(r)}$. Then, by the first part of the proof, there exists $g \in G$ such that $g \cdot \overline{x}_\tau = \overline{x}_{\tau'}$. Thus, we have a homomorphism $g_{\#} : \mathcal{O}_{\overline{x}_{\tau'}} \to \mathcal{O}_{\overline{x}_\tau}$ and $\mathfrak{p}_1 = g_{\#}^{-1}(\mathfrak{p}_\tau)$ is a height 1 prime ideal of $\mathcal{O}_{\overline{x}_{\tau'}}$ above $\mathfrak{p}$. Hence, by the second part above, there exists $g' \in \mathrm{St}_{\overline{x}_\tau}$ such that $g' \cdot \mathfrak{p}_1 = \mathfrak{p}_{\tau'}$, i.e. $(g'_{\#})^{-1}(\mathfrak{p}_1) = \mathfrak{p}_{\tau'}$, where $g'_{\#}$ is the homomorphism

$\mathcal{O}_{\overline{x}_{\tau'}} \to \mathcal{O}_{\overline{x}_{\tau'}}$ induced by g'. Then, the composition $g'g$ satisfies $g'g \cdot \overline{x}_\tau = \overline{x}_{\tau'}$ and $g'g \cdot \mathfrak{p}_\tau = \mathfrak{p}_{\tau'}$, i.e. $g'g \cdot \tau = \tau'$. □

Lemma II.7.4. *For every $\tau \in S_f^{(r)}$, the extension of fields of fractions $\mathbb{K}_{r,\tau}^h/\mathbb{K}_r^h$ induced by $V_r^h(\tau)/V_r^h$ is finite and Galois of group $G_{r,\tau}$ isomorphic to the stabilizer $\mathrm{St}_{r,\tau}$ of τ under the action of G on $S_f^{(r)}$.*

Proof. Let U be an open subset of D containing 0 such that $f^{-1}(U) \to U$ is étale and Galois of group G. Let $\{\overline{x}_1, \dots, \overline{x}_N\}$ be the geometric points of $\mathfrak{X}_{K'}^{(r)}$ above $\overline{x}$ and denote by $\mathcal{O}_{\overline{x}}$ and $\mathcal{O}_{\overline{x}_j}$ the local rings $\mathcal{O}_{\mathfrak{D}_{K'}^{(r)},\overline{x}}$ and $\mathcal{O}_{\mathfrak{X}_{K'}^{(r)},\overline{x}_j}$ respectively. Denote also by $A, \mathcal{A}, B$ and $\mathcal{B}$ the rings $\mathcal{O}(D_{K'}^{(r)}), \mathcal{O}^\circ(D_{K'}^{(r)}), \mathcal{O}(X_{K'}^{(r)})$ and $\mathcal{O}^\circ(X_{K'}^{(r)})$ respectively. Then, by (II.2.12.1), we have

$$\mathcal{O}_{\overline{x}} \otimes_{\mathcal{A}} \mathcal{B} \cong \prod_{j=1}^{N} \mathcal{O}_{\overline{x}_j}. \tag{II.7.4.1}$$

Since $\mathbb{K}_r^h$ is the field of fractions of $(\mathcal{O}_{\overline{x}})_{\mathfrak{p}}^{\mathrm{sh}}$, where $\mathfrak{p}$ is a height 1 prime ideal of $\mathcal{O}_{\overline{x}}$, it follows from [End72, 17.17] that, for each $j = 1, \dots, N$, we have

$$\mathbb{K}_r^h \otimes_{\mathcal{O}_{\overline{x}}} \mathcal{O}_{\overline{x}_j} \xrightarrow{\sim} \prod_{\mathfrak{q}} \mathrm{Frac}((\mathcal{O}_{\overline{x}_j})_{\mathfrak{q}}^{\mathrm{sh}}), \tag{II.7.4.2}$$

where $\mathfrak{q}$ runs over the height 1 prime ideals of $\mathcal{O}_{\overline{x}_j}$ above $\mathfrak{p}$. Tensoring (II.7.4.1) with $\mathbb{K}_r^h$ and taking into account (II.3.21) yield

$$\mathbb{K}_r^h \otimes_A B \xrightarrow{\sim} \prod_{\tau \in S^{(r)}} \mathbb{K}_{r,\tau}^h. \tag{II.7.4.3}$$

As $(\mathbb{K}_r^h \otimes_A B)^G = \mathbb{K}_r^h$, we are thus reduced to showing that $\mathbb{K}_r^h \otimes_A B$ is finite étale over $\mathbb{K}_r^h$. As the morphism $f^{-1}(U^{(r)}) \to U^{(r)} = U \cap D_{K'}^{(r)}$ induced by f is finite étale and Galois of group G, it is enough to show that $\mathbb{K}_r^h$ defines a point in $U^{(r)}$, i.e. that $\mathbb{K}_r^h \otimes_A \mathcal{O}(U^{(r)}) \neq 0$. Now the admissible open immersion $U^{(r)} \hookrightarrow D_{K'}^{(r)}$ lifts to a formal morphism $\mathcal{U}^{(r)} \hookrightarrow \mathfrak{D}_{K'}^{(r)'} \to \mathfrak{D}_{K'}^{(r)}$, where $\mathfrak{D}_{K'}^{(r)'} \to \mathfrak{D}_{K'}^{(r)}$ is an admissible formal blow-up and $\mathcal{U}^{(r)} \hookrightarrow \mathfrak{D}_{K'}^{(r)'}$ is a formal open immersion. The origin x of $D_{K'}^{(r)}$ defines a rig-point $x : \mathrm{Spf}(\mathcal{O}_{K'}) \to \mathfrak{D}_{K'}^{(r)}$. As the pull-back along this morphism of the coherent open ideal on $\mathfrak{D}_{K'}^{(r)}$ defining $\mathfrak{D}_{K'}^{(r)'}$ is non-zero, hence invertible in $\mathcal{O}_{K'}$, this rig-point lifts to a rig-point $x' : \mathrm{Spf}(\mathcal{O}_{K'}) \to \mathfrak{D}_{K'}^{(r)'}$. As $x \in U^{(r)}$, this rig-point lies in $\mathcal{U}^{(r)}$ and also defines a geometric point $\overline{x}' \to \mathcal{U}$ above $\overline{x}$. Let $\mathcal{V}^{(r)}$ be a formal affine open subset of $\mathcal{U}^{(r)}$ containing x'. Then, $\mathcal{O}_{\mathcal{V},\overline{x}'} = \mathcal{O}_{\mathfrak{D}_{K'}^{(r)'},\overline{x}'}$ and we have a commutative square

$$\begin{array}{ccccc} A & \longrightarrow & \mathcal{O}(U^{(r)}) & \longrightarrow & \mathcal{O}(\mathcal{V}^{(r)}) \otimes_{\mathcal{O}_{K'}} K' \\ \downarrow & & & & \downarrow \\ \mathbb{K}_r & & \longrightarrow & & \mathbb{K}'_r, \end{array} \tag{II.7.4.4}$$

where $\mathbb{K}'_r$ is the field of fractions of $\mathcal{O}_{\mathcal{V}^{(r)},\overline{x}'}$, the vertical arrows are induced by the inclusions $\mathcal{A} \subset \mathcal{O}_{\overline{x}}$ and $\mathcal{O}(\mathcal{V}^{(r)}) \subset \mathcal{O}_{\mathcal{V}^{(r)},\overline{x}'}$ and the bottom horizontal arrow is induced by $\mathcal{O}_{\overline{x}} \to \mathcal{O}_{\mathcal{V}^{(r)},\overline{x}'}$. Then, $\mathbb{K}_r \otimes_A \mathcal{O}(U^{(r)}) \neq 0$; hence $\mathbb{K}_r^h \otimes_A \mathcal{O}(U^{(r)}) \neq 0$. □

II.7.5. From II.7.4 and the independence of $S_f^{(r)}$ (and G) from the choice of a K' which is r-admissible for f (II.4.24), we see that the subgroups $G_{r,\tau} \subset G$ are independent of the choice of such a K'. Moreover, they are conjugate when τ varies over $S_f^{(r)}$. More precisely, if $\tau, \tau' \in S_f^{(r)}$ and $\tau' = g \cdot \tau$, for some $g \in G$, then $gG_{r,\tau}g^{-1} = G_{r,\tau'}$.

For $\tau \in S_f^{(r)}$, as the extension $V_r^h(\tau)/V_r^h$ satisfies the hypotheses in II.6.4, we have a $\mathbb{Q}$-valued class function $a^{\alpha}_{G_{r,\tau}}$ and a $\mathbb{Z}$-valued character $\mathrm{sw}^{\beta}_{G_{r,\tau}}$ on $G_{r,\tau}$ (II.6.16). We note here that, although they are defined using the fixed uniformizer π of $\mathcal{O}_K$ (II.7.1), these functions don't depend on π (see II.4.19). These functions are also conjugate in the sense that $a^{\alpha}_{G_{r,\sigma^{-1}\tau}}(\sigma^{-1}g\sigma) = a^{\alpha}_{G_{r,\tau}}(g)$ and $\mathrm{sw}^{\beta}_{G_{r,\sigma^{-1}\tau}}(\sigma^{-1}g\sigma) = \mathrm{sw}^{\beta}_{G_{r,\tau}}(g)$, for $g \in G_{r,\tau}$. Thus, from the definition of induction, we see that the following class functions on G

$$a^{\alpha}_{f,K'}(r) = \mathrm{Ind}^G_{G_{r,\tau}} a^{\alpha}_{G_{r,\tau}} \quad \text{and} \quad \mathrm{sw}^{\beta}_{f,K'}(r) = \mathrm{Ind}^G_{G_{r,\tau}} \mathrm{sw}^{\beta}_{G_{r,\tau}} \tag{II.7.5.1}$$

are independent of $\tau \in S_f^{(r)}$ used to make the induction.

Lemma II.7.6. *Let L be a finite extension of K'. Then, we have*

$$a^{\alpha}_{f,L}(r) = a^{\alpha}_{f,K'}(r) \quad \text{and} \quad \mathrm{sw}^{\beta}_{f,L}(r) = \mathrm{sw}^{\beta}_{f,K'}(r). \tag{II.7.6.1}$$

In particular, $a^{\alpha}_{f,K'}(r)$ and $\mathrm{sw}^{\beta}_{f,K'}(r)$ are independent of the choice of the extension K' of K which is r-admissible for f; we denote them by $a^{\alpha}_f(r)$ and $\mathrm{sw}^{\beta}_f(r)$ respectively.

Proof. The extension L/K is also r-admissible for f (II.4.10). Let $\mathfrak{D}_L^{(r)}$ and $\mathfrak{X}_L^{(r)}$ be the normalized integral models, defined over $\mathcal{O}_L$, of $D_K^{(r)}$ and $X_K^{(r)}$ respectively. We have a Cartesian diagram

$$\begin{array}{ccc} \mathfrak{X}_L^{(r)} & \xrightarrow{g^{(r)}} & \mathfrak{X}_{K'}^{(r)} \\ {\scriptstyle \widehat{f}_L^{(r)}}\downarrow & \square & \downarrow{\scriptstyle \widehat{f}_{K'}^{(r)}} \\ \mathfrak{D}_L^{(r)} & \longrightarrow & \mathfrak{D}_{K'}^{(r)}. \end{array} \tag{II.7.6.2}$$

We denote by $\overline{x}'$ the unique geometric point of $\mathfrak{D}_L^{(r)}$ (the origin of the special fiber) above the geometric point $\overline{x}$ of $\mathfrak{D}_{K'}^{(r)}$ and $\mathfrak{p}'$ the height 1 prime ideal of $\mathcal{O}_{D_L^{(r)},\overline{x}}$ above $\mathfrak{p}$ (II.4.24). As $\mathfrak{X}_{K'}^{(t)}$ and $\mathfrak{X}_L^{(t)}$ have isomorphic special fibers, we have a canonical bijection $S_{f,L}^{(r)} \xrightarrow{\sim} S_{f,K'} = S_f^{(r)}$, (II.4.24). With this identification, let $\tau' = (\overline{x}_{\tau'}, \mathfrak{p}_{\tau'})$ be an element of $S_f^{(r)}$ and put $\tau = (\overline{x}_{\tau}, \mathfrak{p}_{\tau}) = (g^{(r)}(\overline{x}_{\tau'}), {g_{\#}^{(r)}}^{-1}(\mathfrak{p}_{\tau'}))$. Then, the canonical homomorphism $\mathrm{St}_{r,\tau'} \to \mathrm{St}_{r,\tau}$ is an isomorphism and thus induces an isomorphism $G_{r,\tau'} \xrightarrow{\sim} G_{r,\tau}$ (II.7.4). By functoriality (II.3.19), $\mathfrak{D}_L^{(r)} \to \mathfrak{D}_{K'}^{(r)}$ and $\mathfrak{X}_L^{(r)} \to \mathfrak{X}_{K'}^{(r)}$ induce extensions of $\mathbb{Z}^2$-valuation rings $(V^h_{r,K'}, v_{r,K'}) \to (V^h_{r,L}, v_{r,L})$ and $(V^h_r(\tau), v_{r,\tau}) \to (V^h_r(\tau'), v_{r,\tau'})$. We have a normalized valuation map $(\mathbb{K}^h_{r,K'})^{\times} \to \Gamma_{V^h_{r,K'}} \xrightarrow{\sim} \mathbb{Z} \times \mathbb{Z} \hookrightarrow \mathbb{Q} \times \mathbb{Z}$, with the last inclusion given by $(a, b) \mapsto (a/e_{K'/K}, b)$, where $e_{K'/K}$ is the ramification index of K'/K, and induced projections $v^{\alpha}_{r,K'} : (\mathbb{K}^h_{r,K'})^{\times} \to \mathbb{Q}$ and $v^{\beta}_{r,K'} : (\mathbb{K}^h_{r,K'})^{\times} \to \mathbb{Z}$ (II.4.21). We have a similar normalized valuation map and projections for L/K too. Then, the composition $(\mathbb{K}^h_{r,K'})^{\times} \to (\mathbb{K}^h_{r,L})^{\times} \xrightarrow{v^{\beta}_{r,L}} \mathbb{Z}$ coincides with $v^{\beta}_{r,K'}$. Indeed, as the special fibers of $\mathfrak{X}_{K'}^{(r)}$ and $\mathfrak{X}_L^{(r)}$ are canonically isomorphic, we have $A_{\overline{x}_\tau}/\mathfrak{p}_\tau \xrightarrow{\sim} A_{\overline{x}_{\tau'}}/\mathfrak{p}_{\tau'}$ and thus $V_r^h(\tau)/\mathfrak{p}_\tau \xrightarrow{\sim} V_r^h(\tau')/\mathfrak{p}_{\tau'}$, whence the claim. As $ae_{L/K'}/e_{L/K} = a/e_{K'/K}$, it follows from the above normalizations that the composition $(\mathbb{K}_{r,K'}^h)^\times \to (\mathbb{K}_{r,L}^h)^\times \xrightarrow{v_{r,L}^\alpha} \mathbb{Q}$ coincides with $v_{r,K'}^\alpha$. We can write $V_r^h(\tau) = V_{r,K}^h[a]$ for some $a \in V_r^h(\tau)$ whose reduction mod $\mathfrak{p}_\tau$ is a uniformizer of the totally ramified (discretely valued) extension $\kappa(\mathfrak{p}_\tau)$ of $\kappa(\mathfrak{p})$ (II.3.12 and II.3.23). Then, viewed as an element of $V_r^h(\tau')$, a also satisfies $V_r^h(\tau') = V_{r,L}^h[a]$, as seen from the isomorphism $V_r^h(\tau)/\mathfrak{p}_\tau \xrightarrow{\sim} V_r^h(\tau')/\mathfrak{p}_{\tau'}$. Then, for $\sigma' \in G_{r,\tau'} - \{1\}$, we have (II.6.1)

(II.7.6.3)
$$i_{G_{r,\tau}}(\varphi(\sigma')) = v_{r,\tau}(\varphi(\sigma')(a) - a) \quad \text{and} \quad j_{G_{r,\tau}}(\varphi(\sigma')) = i_{G_{r,\tau}}(\varphi(\sigma')) - \frac{\varepsilon}{|G_{r,\tau'}|},$$

where $v_{r,\tau}$ denotes the valuation map of $V_r^h(\tau)$ and ε is the minimum positive element of $\Gamma_{V_{r,K}^h}$, which is also the minimum positive element of $\Gamma_{V_{r,L}^h}$. As $v_{r,\tau'}(\sigma'(a) - a) = v_{r,\tau}(\varphi(\sigma')(a) - a)$ and $|G_{r,\tau}| = |G_{r,\tau'}|$, we deduce that

(II.7.6.4)
$$i_{G_{r,\tau}}(\varphi(\sigma')) = i_{G_{r,\tau'}}(\sigma') \quad \text{and} \quad j_{G_{r,\tau}}(\varphi(\sigma')) = j_{G_{r,\tau'}}(\sigma'),$$

which proves the lemma. □

Remark II.7.7. We keep the notation of II.7.6 and of its proof. We have also shown in the above proof that we have canonical isomorphisms

(II.7.7.1)
$$V_{r,K'}^h/\mathfrak{p} \xrightarrow{\sim} V_{r,L}^h/\mathfrak{p}' \quad \text{and} \quad V_r^h(\tau)/\mathfrak{p}_\tau \xrightarrow{\sim} V_r^h(\tau')/\mathfrak{p}_{\tau'}.$$

Therefore, the residue Galois group of the extension $(V_r^h(\tau))_{\mathfrak{p}_\tau}/(V_r^h)_{\mathfrak{p}}$ is independent of the choice of K' which is r-admissible for f. Hence, the corresponding inertia subgroup $I_{r,\tau}$ of $G_{r,\tau}$ and wild inertia subgroup $P_{r,\tau}$ of $I_{r,\tau}$ are also independent of the choice of such a K'.

Lemma II.7.8. *Let X' be a smooth K-rigid space and let $g : X' \to X$ be a morphism such that the composition $X' \xrightarrow{g} X \xrightarrow{f} D$, which we denote by f', is a finite flat morphism which is étale and Galois of group G' over an admissible open subset of D containing 0. Then, G is a quotient of G' and we have*

(II.7.8.1)
$$a_f^\alpha(r) = \mathrm{Ind}_{G'}^G a_{f'}^\alpha(r) \quad \text{and} \quad \mathrm{sw}_f^\beta(r) = \mathrm{Ind}_{G'}^G \mathrm{sw}_{f'}^\beta(r).$$

Proof. By functoriality (II.4.11), we have a map $g^{(r)} : S_{f'}^{(r)} \to S_f^{(r)}$. Let τ' be an element of $S_g^{(r)}$ and $\tau \in S_f^{(r)}$ its image by $g^{(r)}$. Functoriality again (II.3.19) induces extensions of $\mathbb{Z}^2$-valuation rings $V_r^h \to V_r^h(\tau) \to V_r^h(\tau')$ and thus a commutative diagram, with inclusion horizontal arrows,

(II.7.8.2)
$$\begin{array}{ccc} G'_{r,\tau'} & \longrightarrow & G' \\ \big\downarrow & & \big\downarrow \\ G_{r,\tau} & \longrightarrow & G. \end{array}$$

The surjectivity of the left vertical homomorphism in (II.7.8.2) stems from functoriality (II.7.4). Since $a_{f'}^\alpha(r) = \mathrm{Ind}_{G'_{r,\tau'}}^{G'} a_{G_{r,\tau}}^\alpha$, from (II.7.8.2) and the transitivity of induction, we get

(II.7.8.3)
$$\mathrm{Ind}_{G'}^G a_{f'}^\alpha(r) = \mathrm{Ind}_{G_{r,\tau}}^G \left(\mathrm{Ind}_{G'_{r,\tau'}}^{G_{r,\tau}} a_{G'_{r,\tau'}}^\alpha \right) = a_f^\alpha(r),$$

where the last equality stems from (II.6.19.3) and (II.7.5.1). The same reasoning applies to $\mathrm{sw}_f^\beta(r)$ and $\mathrm{sw}_{f'}^\beta(r)$. □

II.7.9. We normalize the functions $a_f^\alpha(r)$ and $\mathrm{sw}_f^\beta(r)$ in the following way. We put

$$\widetilde{a}_f^\alpha(r) = \frac{|G|}{|S_f^{(r)}|} a_f^\alpha(r) \quad \text{and} \quad \widetilde{\mathrm{sw}}_f^\beta(r) = \frac{|G|}{|S_f^{(r)}|} \mathrm{sw}_f^\beta(r). \tag{II.7.9.1}$$

As $|S_f^{(r)}|$ has already been seen to be independent of the choice of the r-admissible extension of K defining the normalized integral models (II.4.24), $\widetilde{a}_f^\alpha(r)$ and $\widetilde{\mathrm{sw}}_f^\beta(r)$ are also independent of that choice (II.7.6).

II.7.10. Let H be a subgroup of G. Then, the quotient $Y = X/H = \mathrm{Sp}(\mathcal{O}(X)^H)$ is a smooth K-rigid space and the morphism $f_H : X/H \to D$ induced by f is finite, flat and étale over an open subset of D containing 0. Moreover, for $t \in \mathbb{Q}_{\geq 0}$, the quotient morphism $g : X \to Y$ induces a map $g_S^{(t)} : S_f^{(t)} \to S_{f_H}^{(t)}$.

Lemma II.7.11. *The map $g_S^{(t)} : S_f^{(t)} \to S_{f_H}^{(t)}$ is surjective and its fibers are the orbits of H.*

Proof. As t is fixed, we fix also a finite extension K' of K which is t-admissible for f and f_H. We denote $\mathcal{O}(Y_{K'}^{(t)})$, $\mathcal{O}^\circ(Y_{K'}^{(t)})$, $\mathcal{O}(X_{K'}^{(t)})$ and $\mathcal{O}^\circ(X_{K'}^{(t)})$ by $A, \mathcal{A}, B$ and $\mathcal{B}$ respectively. Let $\overline{y}$ be a geometric point of $\mathrm{Spf}(\mathcal{A})$ and let $\overline{x}$ be a geometric point of $\mathrm{Spf}(\mathcal{B})$ above $\overline{y}$. We denote by $\mathcal{O}_{\overline{y}}$ and $\mathcal{O}_{\overline{x}}$ the respective formal étale local rings (II.2.5.1). Then, as in the proof of II.7.3, we have $\mathcal{A} = \mathcal{B}^H$. Combined with (II.2.12.1), this yields

$$\mathcal{O}_{\overline{y}} \cong (\prod_{h\in H} \mathcal{O}_{h\cdot\overline{x}})^H. \tag{II.7.11.1}$$

By [Fu11, 3.1.1 (ii)], the canonical morphism $\coprod_{h\in H} \mathrm{Spec}(\mathcal{O}_{h\cdot\overline{x}}) \to \mathrm{Spec}(\mathcal{O}_{\overline{y}})$ is then surjective and its fibers are the orbits of H. Hence, $g_S^{(t)}$ is surjective, and if $\tau = (\overline{x}, \mathfrak{q})$ and $\tau' = (h\cdot\overline{x}, \mathfrak{q}')$, for some $h \in H$, are elements of $S_f^{(t)}$, they have the same image by $g_S^{(t)}$ if and only $h_\#^{-1}(\mathfrak{q}') = \mathfrak{q}$ (notation of II.7.2), i.e. if and only if $h\cdot\tau = \tau'$ □

Proposition II.7.12. *We use the notation of II.4.23 and II.7.10. We assume that X/H has* trivial canonical sheaf. *Then, for any $t \in \mathbb{Q}_{\geq 0}$, we have the identity*

$$\partial_{f_H}^\alpha(t) = \langle \widetilde{a}_f^\alpha(t), \mathbb{Q}[G/H]\rangle, \tag{II.7.12.1}$$

where the representation $\mathbb{Q}[G/H] = \mathrm{Ind}_H^G 1_H$ of G stands in abusively for its character. The right derivative of $\partial_{f_H}^\alpha$ at t is

$$\frac{d}{dt}\partial_{f_H}^\alpha(t+) = \langle \widetilde{\mathrm{sw}}_f^\beta(t), \mathbb{Q}[G/H]\rangle. \tag{II.7.12.2}$$

Proof. Let τ be an element of $S_f^{(t)}$.We first note that, as G acts transitively on $S_f^{(t)}$, we have $|G_{t,\tau}| = |G|/|S_f^{(t)}|$. By Frobenius reciprocity, we have the following identities

$$\langle a_f^\alpha(t), \mathbb{Q}[G/H]\rangle = \langle a_{G_{t,\tau}}^\alpha, \mathbb{Q}[G/H]|G_{t,\tau}\rangle, \tag{II.7.12.3}$$

$$\langle \mathrm{sw}_f^\beta(t), \mathbb{Q}[G/H]\rangle = \langle \mathrm{sw}_{G_{t,\tau}}^\beta, \mathbb{Q}[G/H]|G_{t,\tau}\rangle. \tag{II.7.12.4}$$

Let R be a set of representatives in G of the double cosets $G_{t,\tau}\backslash G/H$. From [Ser98, §7.3, Prop. 22], for $\tau \in S_f^{(t)}$, we have the identity

$$\mathbb{Q}[G/H]|G_{t,\tau} = \bigoplus_{\sigma\in R} \mathrm{Ind}_{H_\sigma}^{G_{t,\tau}} 1_{H_{\sigma,\tau}}, \tag{II.7.12.5}$$

where $H_{\sigma,\tau} = \sigma H\sigma^{-1} \cap G_{t,\tau}$. If $\sigma \in R$ and $g\sigma h$ is another representative of the double coset $G_{t,\tau}\sigma H$, then $H_{g\sigma h,\tau} = H_{\sigma,\tau}$. Hence, $H_{\sigma,\tau}$ depends only on the double coset $G_{t,\tau}\sigma H$ and the sum (II.7.12.5) is taken over $G_{t,\tau}\backslash G/H$. Therefore, we have

$$\langle a_f^\alpha(t), \mathbb{Q}[G/H]\rangle = \sum_{\sigma\in G_{t,\tau}\backslash G/H} \langle a_{G_{t,\tau}}^\alpha, \mathbb{Q}[G_{t,\tau}/H_{\sigma,\tau}]\rangle, \tag{II.7.12.6}$$

$$\langle \mathrm{sw}_f^\beta(t), \mathbb{Q}[G/H]\rangle = \sum_{\sigma\in G_{t,\tau}\backslash G/H} \langle \mathrm{sw}_{G_{t,\tau}}^\beta, \mathbb{Q}[G_{t,\tau}/H_{\sigma,\tau}]\rangle. \tag{II.7.12.7}$$

From (II.6.18.4) and (II.6.18.5), we get

$$|G_{t,\tau}|\langle a_{G_{t,\tau}}^\alpha, \mathbb{Q}[G_{t,\tau}/H_{\sigma,\tau}]\rangle = v_t^\alpha(\mathfrak{d}_{V_t^h(\sigma,\tau)/V_t^h}), \tag{II.7.12.8}$$

$$|G_{t,\tau}|\langle \mathrm{sw}_{G_{t,\tau}}^\beta, \mathbb{Q}[G_{t,\tau}/H_{\sigma,\tau}]\rangle = v_t^\beta(\mathfrak{d}_{V_t^h(\sigma,\tau)/V_t^h}) - \frac{|G_{t,\tau}|}{|H_{\sigma,\tau}|} + 1, \tag{II.7.12.9}$$

where $V_t^h(\sigma,\tau) = V_t^h(\tau)^{H_{\sigma,\tau}}$. The subgroup $\sigma^{-1}H_{\sigma,\tau}\sigma$ of $\sigma^{-1}G_{t,\tau}\sigma = G_{t,\sigma^{-1}\cdot\tau}$ (II.7.5) is $H_{\mathrm{id},\sigma^{-1}\cdot\tau}$. Then, σ^{-1} yields an isomorphism of $\mathbb{Z}^2$-valuation rings $V_t^h(\sigma^{-1}\cdot\tau) \xrightarrow{\sigma} V_t^h(\tau)$, via (II.7.2.1), which induces an isomorphism

$$V_t^h(\mathrm{id},\sigma^{-1}\cdot\tau) = (V_t^h(\sigma^{-1}\cdot\tau))^{\sigma^{-1}H_{\sigma,\tau}\sigma} \xrightarrow{\sim} V_t^h(\sigma,\tau). \tag{II.7.12.10}$$

By II.7.11, the map $C : G_{t,\tau}\backslash G/H \to S_{f_H}^{(t)}$, $G_{t,\tau}\sigma H \mapsto g_S^{(t)}(\sigma^{-1}\cdot\tau)$ is well-defined since, for $g \in G_{t,\tau}$ and $h \in H$, $(g\sigma h)^{-1}\cdot\tau = h^{-1}\cdot\tau$. Moreover, as G acts transitively on $S_f^{(t)}$ (II.7.3) and $g_S^{(t)}$ is surjective, C is also surjective. If $C(G_{t,\tau}\sigma H) = C(G_{t,\tau}\sigma' H)$, then there exists $h \in H$ such that $\sigma^{-1}\cdot\tau = h\sigma'^{-1}\cdot\tau$; so $\sigma h\sigma'^{-1} \in G_{t,\tau}$ and thus $\sigma' \in G_{t,\tau}\sigma H$. Hence, C is also injective, hence a bijection. We also have $V_t^h(\mathrm{id},\sigma^{-1}\cdot\tau) = V_t^h(g_S^{(t)}(\sigma^{-1}\cdot\tau))$. It follows that, if $\sigma,\sigma' \in R$ represent double cosets such that $g_S^{(t)}(\sigma^{-1}\cdot\tau) = g_S^{(t)}(\sigma'^{-1}\cdot\tau)$, then $V_t^h(\mathrm{id},\sigma^{-1}\cdot\tau) = V_t^h(\mathrm{id},\sigma'^{-1}\cdot\tau)$.

Combining all this with (II.7.12.6), (II.7.12.7), (II.7.12.8) and (II.7.12.9) yields

$$\langle \widetilde{a}_f^\alpha(t), \mathbb{Q}[G/H]\rangle = \sum_{\tau'\in S_{f_H}^{(t)}} v_t^\alpha(\mathfrak{d}_{V_t^h(\tau')/V_t^h}) = v_t^\alpha\left(\prod_{\tau'\in S_{f_H}(t)} \mathfrak{d}_{V_t^h(\tau')/V_t^h}\right) = \partial_{f_H}^\alpha(t), \tag{II.7.12.11}$$

$$\langle \widetilde{\mathrm{sw}}_f^\beta(t), \mathbb{Q}[G/H]\rangle = v_t^\beta\left(\prod_{\tau'\in S_{f_H}(t)} \mathfrak{d}_{V_t^h(\tau')/V_t^h}\right) - \deg(f_H) + |S_{f_H}^{(t)}|. \tag{II.7.12.12}$$

From the defining formula (II.4.25.1) (for f_H), we thus see that $\langle \widetilde{\mathrm{sw}}_f^\beta(t), \mathbb{Q}[G/H]\rangle = d_{f_H,s}(t) - \deg(f_H) + |S_{f_H}^{(t)}|$. Now, by Proposition II.4.26, applied to f_H, we deduce that

$$\langle \widetilde{\mathrm{sw}}_f^\beta(t), \mathbb{Q}[G/H]\rangle = \sum_{j=1}^N (d_{\eta,\overline{x}'_j}^{(t)} - 2\delta_{\overline{x}'_j}^{(t)}) - \deg(f_H) + |S_{f_H}^{(t)}|, \tag{II.7.12.13}$$

where $\overline{x}'_1, \dots, \overline{x}'_N$ are the geometric points of the special fiber of the normalized integral model of $Y^{(t)}_{K'}$ which are above the geometric point of special fiber $\mathfrak{D}^{(t)}_{s'}$ defined by the origin of $D^{(t)}_{s'}$ and the algebraically closed residue field k (II.4.24) (see II.3.22, (II.3.25.1) and II.4.26 for the definition of the integers $\delta^{(t)}_{\overline{x}'_j}$ and $d^{(t)}_{\eta,\overline{x}'_j}$). As $|S^{(t)}_{f_H}|$ is also the sum over $j = 1, \dots, N$ of the integers $|P_s(\mathcal{O}_{\mathfrak{Y}^{(t)},\overline{x}'_j})|$, the combination of (II.4.28.1), (II.4.23.1), both applied to f_H, and (II.7.12.13), implies that, for $t \in]r_i, r_{i-1}[$,

$$\text{(II.7.12.14)} \qquad \langle \widetilde{\mathrm{sw}}^{\beta}_f(t), \mathbb{Q}[G/H]\rangle = \sigma_i - \deg(f_H) + \delta_{f_H}(i) = \frac{d}{dt}\partial^{\alpha}_{f_H}(t+).$$

□

Corollary II.7.13. *We keep the assumption of II.7.12. The function*

$$\text{(II.7.13.1)} \qquad \mathbb{Q}_{\geq 0} \to \mathbb{Q}, \ t \mapsto \langle \widetilde{a}^{\alpha}_f(t), \mathbb{Q}[G/H]\rangle$$

is continuous and piecewise linear, with finitely many slopes which are all integers. The quantity $\langle \widetilde{\mathrm{sw}}^{\beta}_f(t), \mathbb{Q}[G/H]\rangle$, its derivative at t, is constant on each $]r_i, r_{i-1}[\cap\mathbb{Q}$ (notation of II.4.23).

Proof. This follows from II.7.12 and II.4.23. □

Remark II.7.14. We note that, when the morphism $f : X \to D$ is étale and Galois of group G and H is a subgroup of G, then $X/H \to D$ is also étale and both X and X/H have trivial canonical sheaves.

II.7.15. For the rest of this section, let ℓ be a prime number different from p and $\overline{\mathbb{Q}}_\ell$ an algebraic closure of $\mathbb{Q}_\ell$.

Theorem II.7.16. *We assume that $f : X \to D$ is étale. Let $\chi \in R_{\overline{\mathbb{Q}}_\ell}(G)$. The map*

$$\text{(II.7.16.1)} \qquad \widetilde{a}^{\alpha}_f(\chi, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}, \quad t \mapsto \langle \widetilde{a}^{\alpha}_f(t), \chi\rangle$$

is continuous and piecewise linear, with finitely many slopes which are all integers. Its right derivative at $t \in \mathbb{Q}_{\geq 0}$ is

$$\text{(II.7.16.2)} \qquad \frac{d}{dt}\widetilde{a}^{\alpha}_f(\chi, t+) = \langle \widetilde{\mathrm{sw}}^{\beta}_f(t), \chi\rangle.$$

Proof. By Artin's theorem [Ser98, §9.2, Corollaire], we may assume that $\chi = \mathrm{Ind}^G_H \rho$, where H is a cyclic subgroup of G and $\rho : H \to \overline{\mathbb{Q}}^{\times}_\ell$ a character. By Frobenius reciprocity, we are reduced to proving the following statement.

$S_H(\rho)$ The function $t \mapsto \langle \widetilde{a}^{\alpha}_f(t)_{|H}, \rho\rangle$ is continuous and piecewise linear, and its right derivative is $t \mapsto \langle \widetilde{\mathrm{sw}}^{\beta}_f(t)|H, \rho\rangle$.

We proceed by induction on the order m of ρ, i.e. the smallest integer $i \geq 1$ such that $\rho^i = 1_H$. For $m = 1$, ρ is trivial and χ is the character of the regular representation $\overline{\mathbb{Q}}_\ell[G/H] = \mathrm{Ind}^G_H 1_H$ of G. By II.7.13, the function $t \mapsto \langle \widetilde{a}^{\alpha}_f(t), \overline{\mathbb{Q}}_\ell[G/H]\rangle$ is continuous and piecewise linear and its right derivative is $t \mapsto \langle \widetilde{\mathrm{sw}}^{\beta}_f(t), \overline{\mathbb{Q}}_\ell[G/H]\rangle$. Hence, the statement $S_H(1_H)$ holds. Now, we assume that $m > 1$ and that $S_H(\rho)$ is true if ρ is of order $< m$. As H is cyclic, it has a unique subgroup I of index m. As

$$\text{(II.7.16.3)} \qquad \langle \widetilde{a}^{\alpha}_f(t)_{|H}, \overline{\mathbb{Q}}_\ell[H/I]\rangle = \langle \widetilde{a}^{\alpha}_f(t)_{|I}, 1_I\rangle \quad \text{and} \quad \langle \widetilde{\mathrm{sw}}^{\beta}_f(t)_{|H}, \overline{\mathbb{Q}}_\ell[H/I]\rangle = \langle \widetilde{\mathrm{sw}}^{\beta}_f(t)_{|I}, 1_I\rangle,$$

we see that $S_H(\overline{\mathbb{Q}}_\ell[H/I]) = S_I(1_I)$. Hence, the foregoing argument, where H is replaced by I, implies that the statement $S_H(\overline{\mathbb{Q}}_\ell[H/I])$ also holds. Now, the representation $\overline{\mathbb{Q}}_\ell[H/I]$ lies inside the regular representation $\overline{\mathbb{Q}}_\ell[H]$. In fact, it identifies with the direct sum of all characters of H trivial on I, i.e. of order dividing m. Indeed, the restriction $\overline{\mathbb{Q}}_\ell[H]|I$ is trivial; if $\widetilde{\chi}$ is a $\overline{\mathbb{Q}}_\ell$-valued irreducible character of H, then $\widetilde{\chi}$ is a direct summand of $\overline{\mathbb{Q}}_\ell[H/I]$ as it is readily seen that

$$\langle \overline{\mathbb{Q}}_\ell[H/I], \widetilde{\chi}\rangle = \frac{|I|}{|H|}\dim_{\overline{\mathbb{Q}}_\ell}(\widetilde{\chi}) > 0. \tag{II.7.16.4}$$

Hence, with our assumption above, we deduce that $S_H(\rho')$ is true, where $\rho' = \rho_1 \oplus \cdots \oplus \rho_k$ is the sum of characters of H of order m and trivial on I (among them is ρ). Denote S_m the set formed by these latter characters; they correspond to the generators of the group of characters of H/I and their number k is the cardinal of $(\mathbb{Z}/m\mathbb{Z})^\times$. Consider the natural action of $\mathbb{Z}$ on $R_{\overline{\mathbb{Q}}_\ell}(H)$ given by the operators

$$\Psi^j : \psi \mapsto (\sigma \mapsto \psi(\sigma^j)) \tag{II.7.16.5}$$

[Ser98, §9.2, Exercice 3]. The subgroup $m\mathbb{Z}$ acts trivially on S_m and any $j \in \mathbb{Z}$ prime to m stabilizes S_m, hence an action of $(\mathbb{Z}/m\mathbb{Z})^\times$ on S_m. Moreover, for any $j \in \mathbb{Z}$ prime to m, we have

$$\Psi^j(\widetilde{a}^\alpha_f(t)|H) = \widetilde{a}^\alpha_f(t)|H \quad \text{and} \quad \Psi^j(\widetilde{\mathrm{sw}}^\beta_f(t)|H) = \widetilde{\mathrm{sw}}^\beta_f(t)|H. \tag{II.7.16.6}$$

Indeed this follows from the relations

$$\Psi^j(a^\alpha_{G_{t,\tau}}|H) = a^\alpha_{G_{t,\tau}}|H \quad \text{and} \quad \Psi^j(\widetilde{a}^\alpha_f(t)|H) = (\mathrm{Ind}^G_{G_{t,\tau}}\Psi^j(a^\alpha_{G_{t,\tau}}))|H, \tag{II.7.16.7}$$

and similarly for $\mathrm{sw}^\beta_{G_{t,\tau}}$. The first equality in (II.7.16.7) is directly implied by Remark II.6.11 (ii), applied to σ and $\tau = \sigma^j$ in the notation of that remark, and the second is a straightforward computation. Now for every $j \in \mathbb{Z}$ prime to m, we have

$$\langle \widetilde{a}^\alpha_f(t)|H, \rho\rangle = \langle \Psi^j(\widetilde{a}^\alpha_f(t)|H), \Psi^j(\rho)\rangle = \langle \widetilde{a}^\alpha_f(t)|H, \Psi^j(\rho)\rangle, \tag{II.7.16.8}$$

where the first equality stems from the definition of the pairing $\langle\cdot,\cdot\rangle$ as a sum over H, and the second is (II.7.16.6). A similar equality holds for $\widetilde{\mathrm{sw}}^\beta_f(t)|H$. Thus, since the action of $(\mathbb{Z}/m\mathbb{Z})^\times$ on S_m is transitive, taking the sum over $j \in (\mathbb{Z}/m\mathbb{Z})^\times$ in (II.7.16.8), we obtain

$$\langle \widetilde{a}^\alpha_f(t)|H, \rho\rangle = \frac{\langle \widetilde{a}^\alpha_f(t)|H, \rho'\rangle}{|(\mathbb{Z}/m\mathbb{Z})^\times|}, \tag{II.7.16.9}$$

and similarly with $\widetilde{\mathrm{sw}}^\beta_f(t)|H$, which shows that $S_H(\rho)$ holds and concludes the proof. □

II.7.17. Let $\Lambda_\mathbb{Q}$ be a finite extension of $\mathbb{Q}_\ell$ inside $\overline{\mathbb{Q}}_\ell$ and $\overline{\Lambda}$ its residue field. We use the notation of II.6.13. By [Ser98, 16.1, Théorème 33], the Cartan homomorphism $d_G : R_{\Lambda_\mathbb{Q}}(G) \to R_{\overline{\Lambda}}(G)$ is surjective.

Lemma II.7.18. *Let $\overline{\chi}$ be an element of $R_{\overline{\Lambda}}(G)$ and χ a pre-image of $\overline{\chi}$ in $R_{\Lambda_\mathbb{Q}}(G)$. Then, for $t \in \mathbb{Q}_{\geq 0}$,*

$$\widetilde{a}^\alpha_f(\overline{\chi}, t) = \langle \widetilde{a}^\alpha_f(t), \chi\rangle \quad \text{and} \quad \widetilde{\mathrm{sw}}^\beta_f(\overline{\chi}, t) = \langle \widetilde{\mathrm{sw}}^\beta_f(t), \chi\rangle \tag{II.7.18.1}$$

are independent of the choice of the pre-image χ.

Proof. We use notation from II.7.1 and II.7.5. From the definition (II.7.5.1) and Frobenius reciprocity, for any $\tau \in S^{(t)}$, we have

$$\langle \widetilde{a}^{\alpha}_{f}(t), \chi\rangle = |G_{t,\tau}|\langle a^{\alpha}_{G_{t,\tau}}, \chi|G_{t,\tau}\rangle \quad \text{and} \quad \langle \widetilde{\mathrm{sw}}^{\beta}_{f}(t), \chi\rangle = |G_{t,\tau}|\langle \mathrm{sw}^{\beta}_{G_{t,\tau}}, \chi|G_{t,\tau}\rangle. \tag{II.7.18.2}$$

By II.3.9, the extension $\mathbb{K}^h_{t,\tau}/\mathbb{K}^h_t$ (II.7.5) factors through a subfield $\mathbb{K}'_t$ which is a Galois extension of $\mathbb{K}^h_t$ such that, if V'_t is the integral closure of V^h_t in K'_t and $\mathfrak{p}$ (resp. $\mathfrak{p}'$, resp. $\mathfrak{p}_\tau$) is the height 1 prime ideal of V^h_t (resp. V'_t, resp. $V^h_t(\tau)$), then $(V'_t)_{\mathfrak{p}'}/(V^h_t)_{\mathfrak{p}}$ is an unramified extension and $(V^h_t(\tau))_{\mathfrak{p}_\tau}/(V'_t)_{\mathfrak{p}'}$ has ramification index one with purely inseparable and monogenic residue extension. The Galois group $G'_{t,\tau}$ of $\mathbb{K}_{t,\tau}/\mathbb{K}'_t$ is then a normal subgroup of $G_{t,\tau}$ of cardinality a power of p. The induction, from $G'_{t,\tau}$ to $G_{t,\tau}$, of the character $r_{G'_{t,\tau}}$ of the regular representation of $G'_{t,\tau}$ is the character $r_{G_{t,\tau}}$ of the regular representation of $G_{t,\tau}$. Moreover, by [Ser98, §7.2, Rem. 3)], for a ($\Lambda_{\mathbb{Q}}$-valued) central function φ on $G_{t,\tau}$, we have $\mathrm{Ind}^{G_{t,\tau}}_{G'_{t,\tau}}(\varphi|G'_{t,\tau}) = [G_{t,\tau} : G'_{t,\tau}]\cdot\varphi$. Therefore, taking $\varphi = a^{\alpha}_{G_{t,\tau}}$ and $\varphi = \mathrm{sw}^{\beta}_{G_{t,\tau}}$ successively, we see from II.6.19 and Frobenius reciprocity that

$$\langle a^{\alpha}_{G'_{t,\tau}}, \chi|G'_{t,\tau}\rangle = [G_{t,\tau} : G'_{t,\tau}]\langle a^{\alpha}_{G_{t,\tau}}, \chi|G_{t,\tau}\rangle - \frac{1}{|G_{t,\tau}|}v^{\alpha}(\mathfrak{d}_{V'_t/V^h_t})\chi(1), \tag{II.7.18.3}$$

(II.7.18.4)

$$\langle \mathrm{sw}^{\beta}_{G'_{t,\tau}}, \chi|G'_{t,\tau}\rangle = [G_{t,\tau} : G'_{t,\tau}]\langle \mathrm{sw}^{\beta}_{G_{t,\tau}}, \chi|G_{t,\tau}\rangle - (\frac{1}{|G_{t,\tau}|}v^{\beta}(\mathfrak{d}_{V'_t/V^h_t}) + 1 - [G_{t,\tau} : G'_{t,\tau}])\chi(1).$$

(In fact, as $(V'_t)_{\mathfrak{p}'}/(V^h_t)_{\mathfrak{p}}$ is unramified, $v^{\alpha}(\mathfrak{d}_{V'_t/V^h_t}) = 0$.) Now, as the $\mathrm{char}(\overline{\Lambda}) \neq p$, [Ser98, 18.2, Thm 42, Corollaire 2] implies that, for any element χ' of the kernel of d_G, we have $\mathrm{tr}_{\chi'}|G'_{r,\tau} = 0$ and thus (II.6.13.1)

$$\langle a^{\alpha}_{G'_{t,\tau}}, \chi'\rangle = 0 \quad \text{and} \quad \langle \mathrm{sw}^{\beta}_{G'_{t,\tau}}, \chi'\rangle = 0. \tag{II.7.18.5}$$

Therefore, $\langle a^{\alpha}_{G'_{t,\tau}}, \chi|G'_{t,\tau}\rangle$ and $\langle \mathrm{sw}^{\beta}_{G'_{t,\tau}}, \chi|G'_{t,\tau}\rangle$ depend only on $\overline{\chi}$. As $\chi(1) = \dim_{\Lambda_{\mathbb{Q}}}(\chi)$ depends only on $\overline{\chi}$, it follows from (II.7.18.2), (II.7.18.3)) and (II.7.18.4) that $\widetilde{a}^{\alpha}_{f}(\overline{\chi}, t)$ and $\widetilde{\mathrm{sw}}^{\beta}_{f}(\overline{\chi}, t)$ depend only on $\overline{\chi}$. □

Remark II.7.19. What we have in fact shown in the above proof is the following. Let W/V be a monogenic integral extension of henselian $\mathbb{Z}^2$-valuation rings whose residue characteristic is $p > 0$ and whose induced extension of fields of fractions $\mathbb{L}/\mathbb{K}$ is Galois of group $\mathbb{G}$. Let $\mathbb{K}'$ be the maximal unramified sub-extension of $\mathbb{L}/\mathbb{K}$ with respect to the discrete valuation ring $V_{\mathfrak{p}}$ at the height 1 prime ideal $\mathfrak{p}$ of V and $\mathbb{G}'$ the Galois group of $\mathbb{L}/\mathbb{K}'$. Then, for $\overline{\chi} \in R_{\overline{\Lambda}}(\mathbb{G})$ and χ a pre-image of $\overline{\chi}$ in $R_{\Lambda_{\mathbb{Q}}}(\mathbb{G})$, we have

$$|\mathbb{G}|\langle a^{\alpha}_{\mathbb{G}}, \chi\rangle = |\mathbb{G}'|\langle a^{\alpha}_{\mathbb{G}'}, \chi|\mathbb{G}'\rangle, \tag{II.7.19.1}$$

$$|\mathbb{G}|\langle \mathrm{sw}^{\beta}_{\mathbb{G}}, \chi\rangle = |\mathbb{G}'|\langle \mathrm{sw}^{\beta}_{\mathbb{G}'}, \chi|\mathbb{G}'\rangle, \tag{II.7.19.2}$$

which follows readily from II.6.19 and, by [Ser98, 18.2, Thm 42, Corollaire 2], implies that the pairings

$$\langle a^{\alpha}_{\mathbb{G}}, \chi\rangle \quad \text{and} \quad \langle \mathrm{sw}^{\beta}_{\mathbb{G}}, \chi\rangle \tag{II.7.19.3}$$

depend only on $\overline{\chi}$.

Corollary II.7.20. *We assume that X has* trivial canonical sheaf *and $\overline{\chi} \in R_{\overline{\Lambda}}(G)$. Then, the function $\widetilde{a}^{\alpha}_f(\overline{\chi}, \cdot)$ (II.7.18) is continuous and piecewise linear, with finitely many slopes which are all integers. Its right slope at t is*

$$\frac{d}{dt}\widetilde{a}^{\alpha}_f(\overline{\chi}, t+) = \widetilde{\mathrm{sw}}^{\beta}_f(\overline{\chi}, t). \tag{II.7.20.1}$$

II.8. Characteristic cycles.

II.8.1. In this section, we recall the constructions of Kato's characteristic cycle $\mathrm{KCC}_{\zeta}(\chi)$ and Abbes-Saito's characteristic cycle $\mathrm{CC}_{\psi}(\chi)$ in [**Hu15**, §3 and §4]. Let K be a complete discrete valuation field, $\mathcal{O}_K$ its valuation ring, $\mathfrak{m}_K$ its maximal ideal, k its residue field of characteristic $p > 0$, and π a uniformizer of $\mathcal{O}_K$. Let also $\overline{K}$ be a separable closure of K, $\mathcal{O}_{\overline{K}}$ the integral closure of $\mathcal{O}_K$ in $\overline{K}$, $\overline{k}$ its residue field, G_K the Galois group of $\overline{K}$ over K, and $v : \overline{K}^{\times} \to \mathbb{Q}$ the valuation of $\overline{K}$ normalized by $v(\pi) = 1$.

II.8.2. For a field F and one-dimensional F-vector spaces $V_1, \ldots, V_m$, we define the F-algebra

$$F\langle V_1, \ldots, V_m\rangle = \bigoplus_{(i_1,\ldots,i_m)\in\mathbb{Z}^m} V_1^{\otimes i_1} \otimes \cdots V_m^{\otimes i_m}, \tag{II.8.2.1}$$

We denote additively the law of the group of units $(F\langle V_1, \ldots, V_m\rangle)^{\times}$ of this F-algebra. More precisely, an element x of $(F\langle V_1, \ldots, V_m\rangle)^{\times}$, which is in some $V_1^{\otimes i_1} \otimes \cdots \otimes V_m^{\otimes i_m}$, is denoted $[x]$ and we set $[x^{-1}] = -[x]$ and $[x \cdot y] = [x] + [y]$. If for each $1 \leq i \leq m$, e_i is a non-zero element of V_i, we have an isomorphism

$$F\langle V_1, \ldots, V_m\rangle \xrightarrow{\sim} F[X_1, \ldots, X_m, X_1^{-1}, \ldots, X_m^{-1}], \quad e_i \mapsto X_i. \tag{II.8.2.2}$$

Whence we deduce an isomorphism

$$(F\langle V_1, \ldots, V_m\rangle)^{\times} \xrightarrow{\sim} F^{\times} \oplus \mathbb{Z}^m. \tag{II.8.2.3}$$

II.8.3. Let L be a finite separable extension of K in $\overline{K}$ with residue field $E = \mathcal{O}_L/\mathfrak{m}_L$. We assume that the ramification index of L/K is one and that residue extension E/k is purely inseparable and monogenic, i.e L/K is of type (II) in the terminology of [**Kat87b**] (as opposed to type (I) extensions which are the totally ramified ones). Then, $\mathcal{O}_L$ is monogenic over $\mathcal{O}_K$. Let $h \in \mathcal{O}_L$ be an element whose reduction $\overline{h} \in E$ generates the extension E/k of degree p^n and let $a \in \mathcal{O}_K$ be a lift of $\overline{a} = h^{p^n} \in k$.

II.8.4. Let Q be the kernel of the canonical homomorphism $\Omega^1_k \to \Omega^1_E$. It is a one-dimensional k-vector space generated by $\mathrm{d}\overline{a}$ (see [**Hu15**, 3.4 (i)], where it is denoted by V). The E-vector space $\Omega^1_{E/k}$ of relative differentials is also one-dimensional and is generated by $\mathrm{d}\overline{h}$ [**Hu15**, 3.4 (ii)]. We set

$$S_{K,L} = (k\langle \mathfrak{m}_K/\mathfrak{m}_K^2, Q\rangle)^{\times} \quad \text{and} \quad S_{L/K} = (E\langle \mathfrak{m}_L/\mathfrak{m}_L^2, \Omega^1_{E/k}\rangle)^{\times} \quad \text{(II.8.2.1)}. \tag{II.8.4.1}$$

As L/K has ramification index 1, we get an injective homomorphism of k-algebras

$$k\langle \mathfrak{m}_K/\mathfrak{m}_K^2\rangle \hookrightarrow E\langle \mathfrak{m}_L/\mathfrak{m}_L^2\rangle. \tag{II.8.4.2}$$

We also have a canonical E-linear isomorphism ([**Kat87b**, (1.6.1)] or [**Hu15**, (3.4.1)]

$$E \otimes_k Q \xrightarrow{\sim} (\Omega^1_{E/k})^{\otimes p^n} \tag{II.8.4.3}$$

which maps $y \otimes \mathrm{d}\bar{a}$ to $y(\mathrm{d}\bar{h})^{\otimes p^n}$. Then, (II.8.4.2) and (II.8.4.3) induce a canonical injective map

$$S_{K,L} \hookrightarrow S_{L/K} \tag{II.8.4.4}$$

which sends $[\mathrm{d}\bar{a}]$ to $p^n[\mathrm{d}\bar{h}]$.

II.8.5. We assume that L/K is Galois of type (II) with Galois group G. Let C be an algebraically closed field of characteristic zero and denote by $\widetilde{\mathbb{Z}}$ the integral closure of $\mathbb{Z}$ in C. Let also $\zeta \in \widetilde{\mathbb{Z}}$ be a primitive p-th root of unity. For an element χ of the Grothendieck group $R_C(G)$ of finitely generated $C[G]$-modules, we put $\langle \chi, 1\rangle = |G|^{-1}\sum_{\sigma\in G} \mathrm{tr}_\chi(\sigma)$. We also set

$$\varepsilon(\zeta) = \sum_{r\in\mathbb{F}_p^\times \subseteq E^\times} [r]\otimes\zeta^r \ \in S_{L/K}\otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}. \tag{II.8.5.1}$$

II.8.6. We keep the assumptions of II.8.5. For $\sigma \in G - \{1\}$, we put

$$s_G(\sigma) = [\mathrm{d}\bar{h}] - [h-\sigma(h)] \quad \in S_{L/K}, \tag{II.8.6.1}$$

where $[h-\sigma(h)]$ abusively denotes the class of $h-\sigma(h)$ viewed in $(\mathfrak{m}_L/\mathfrak{m}_L^2)^{\otimes v(h-\sigma(h))}$. This definition is independent of the choice of the generator h [Kat87b, 1.8]. We also put

$$s_G(1) = -\sum_{\sigma\in G-\{1\}} s_G(\sigma) \quad \in S_{L/K}. \tag{II.8.6.2}$$

For $\chi \in R_C(G)$, Kato defines the *Swan conductor with differential values* as

$$\mathrm{sw}_\zeta(\chi) = \sum_\sigma s_G(\sigma)\otimes \mathrm{tr}_\chi(\sigma) + (\dim_C(\chi) - \langle\chi,1\rangle)\varepsilon(\zeta) \ \in S_{L/K}\otimes_{\mathbb{Z}}\widetilde{\mathbb{Z}}. \tag{II.8.6.3}$$

For any $r \in \mathbb{F}_p^\times$, we have $\mathrm{sw}_{\zeta^r}(\chi) = \mathrm{sw}_\zeta(\chi) + (\dim_C(\chi) - \langle\chi,1\rangle)[r]$. For H a subgroup of G and $\chi \in R_C(H)$, we have ([Kat87b, 3.3] or [Hu15, 3.15])

$$\mathrm{sw}_\zeta(\mathrm{Ind}_H^G \chi) = [G:H]\,\mathrm{sw}_\zeta(\chi) + (\dim_C(\chi) - \langle\chi,1\rangle)\mathfrak{D}(L^H/K), \tag{II.8.6.4}$$

where L^H is the subfield of L fixed by H and $\mathfrak{D}(L^H/K) \in S_{L/K}$ is Kato's *different* of L^H/K (see *loc. cit.* for the definition; we will not need to explicitly use (II.8.6.2) anyway). If H is normal subgroup of G, $\chi' \in R_C(G/H)$ and χ the image of χ' under the canonical map $R_C(G/H) \to R_C(G)$, then $\mathrm{sw}_\zeta(\chi) = \mathrm{sw}_\zeta(\chi')$ [Kat87b, 3.3], [Hu15, 3.14].

Kato proved the following generalization of the Hasse-Arf theorem.

Theorem II.8.7. *We keep the notation of II.8.4, II.8.5 and II.8.6. For any $\chi \in R_C(G)$, we have (II.8.4.4)*

$$\mathrm{sw}_\zeta(\chi) \in S_{K,L} \subseteq S_{L/K}. \tag{II.8.7.1}$$

II.8.8. We keep the assumptions of II.8.5 and follow [Hu15, 3.17]. For $\chi \in R_C(G)$, the conductor $\mathrm{sw}_\zeta(\chi) \in S_{K,L}$ can thus be written as

$$\mathrm{sw}_\zeta(\chi) = [\Delta'] + [\pi^c] - m\mathrm{d}\bar{a}, \tag{II.8.8.1}$$

where Δ' is a nonzero element of k, π is the uniformizer of $\mathcal{O}_K$ fixed in II.8.1, c in an integer and $m = \dim_C(\chi) - \langle\chi,1\rangle$. Then, we define as Kato's *characteristic cycle of* χ the differential [1]

$$\mathrm{KCC}_\zeta(\chi) = \Delta'^{-1}(\mathrm{d}\bar{a})^m \ \in (\Omega_k^1)^{\otimes m}. \tag{II.8.8.2}$$

[1] We here point out that there is a mistake in the definition of $\mathrm{KCC}_\zeta(\chi)$ given in [Hu15, (3.17.1)], where Δ' is used instead of Δ'^{-1}. This however does not affect the results in *loc. cit.*, since it is the correct definition that was subsequently used.

We note that, as the decomposition (II.8.8.1) depends on the chosen uniformizer π of $\mathcal{O}_K$, so does $\mathrm{KCC}_\zeta(\chi)$.

II.8.9. If L is an extension of type (II) not over K but over a larger sub-field K' such that K'/K is unramified, then, we can still define $\mathrm{sw}_\zeta(\chi)$ by setting

(II.8.9.1) $$\mathrm{sw}_\zeta(\chi) = \mathrm{sw}_\zeta(\chi|\mathrm{Gal}(L/K'))$$

and we still have $\mathrm{sw}_\zeta(\chi) \in S_{K,L}$ [**Kat87b**, 3.15], [**Hu15**, 3.18].

II.8.10. Let L be a Galois extension of K of group G which is of type (II) over a sub-extension unramified over K. Let $\overline{\Lambda}^{\mathrm{alg}}$ be an algebraically closed field of characteristic $\ell \notin \{0, p\}$. Let C be an algebraic closure of the field of fractions of the ring of Witt vectors $W(\overline{\Lambda}^{\mathrm{alg}})$. Then, the Cartan homomorphism $R_C(G) \to R_{\overline{\Lambda}^{\mathrm{alg}}}(G)$ is surjective [**Ser98**, 16.1, Théorème 33]. Let χ be an element of $R_{\overline{\Lambda}^{\mathrm{alg}}}(G)$ and $\widehat{\chi} \in R_C(G)$ be a pre-image of χ. Let $\zeta \in \overline{\Lambda}^{\mathrm{alg}}$ be a primitive p-th root of unity and $\widehat{\zeta}$ be the unique (primitive) p-th root of unity in C lifting ζ. Then, through II.8.9, we define [**Kat87b**, 3.16]

(II.8.10.1) $$\mathrm{sw}_\zeta(\chi) = \mathrm{sw}_{\widehat{\zeta}}(\widehat{\chi}).$$

This definition is independent of $\widehat{\chi}$ by [**Ser98**, 18.2, Théorème 42, Corollaire 2] and (II.8.6.3). It follows that we also have a well-defined characteristic cycle $\mathrm{KCC}_\zeta(\chi)$ (II.8.8.2).

II.8.11. Abbes and Saito defined a decreasing filtration $(G^r_{K,\log})_{r\in\mathbb{Q}_{>0}}$ of G_K by closed normal subgroups, called *the logarithmic ramification filtration* [**AS02**, 3.12]. We denote by $G^0_{K,\log}$ the inertia subgroup of G_K. For any $r \in \mathbb{Q}_{r\geq 0}$, we put

(II.8.11.1) $$G^{r+}_{K,\log} = \overline{\cup_{s>r} G^r_{K,\log}} \quad \text{and} \quad \mathrm{Gr}^r_{\log} G_K = G^r_{K,\log}/G^{r+}_{K,\log}.$$

By [**AS02**, 3.15], $P = G^{0+}_{K,\log}$ is the wild inertia subgroup of G_K, i.e. the p-Sylow subgroup of $G^0_{K,\log}$.

Now, let $L \subseteq \overline{K}$ be a finite separable extension of K. For a rational number $r \geq 0$, we say that the (logarithmic) ramification of L/K is bounded by r (resp. $r+$) if $G^r_{K,\log}$ (resp. $G^{r+}_{K,\log}$) acts trivially on $\mathrm{Hom}_K(L, \overline{K})$ through its action on $\overline{K}$. The *logarithmic conductor* $c = c(L/K)$ of L/K is defined to be the infimum of rational numbers $r > 0$ such that the ramification of L/K is bounded by r. Then, c is a rational number and the ramification of L/K is bounded by $c+$ [**AS02**, 9.5]. However, if $c > 0$, then the ramification L/K is not bounded by c.

Theorem II.8.12 ([**AS03**, Theorem 1])**.** *For every rational number $r > 0$, the group $\mathrm{Gr}^r_{\log}$ is abelian and lies in the center of the pro-p group $P/G^{r+}_{K,\log}$.*

Lemma II.8.13 ([**Ka88**, 1.1], [**AS11**, 6.4])**.** *Let M be a $\mathbb{Z}[\frac{1}{p}]$-module on which P acts through a finite quotient, say by $\rho : P \to \mathrm{Aut}_{\mathbb{Z}}(M)$. Then,*

(*i*) *The module M has a unique direct sum decomposition*

(II.8.13.1) $$M = \bigoplus_{r\in\mathbb{Q}_{\geq 0}} M^{(r)}$$

into P-stables submodules $M^{(r)}$ such that $M^{(0)} = M^P$ and for every $r > 0$,

(II.8.13.2) $$(M^{(r)})^{G^r_{K,\log}} = 0 \quad \text{and} \quad (M^{(r)})^{G^{r+}_{K,\log}} = M^{(r)}.$$

(ii) *If* $r > 0$, *then* $M^{(r)} = 0$ *for all but the finitely many values of* r *for which* $\rho(G^{r+}_{K,\log}) \subsetneq \rho(G^{r}_{K,\log})$.

(iii) *For a fixed* $r \geq 0$ *and variable* M, *the functor* $M \mapsto M^{(r)}$ *is exact.*

(iv) *For* M, N *as above, we have* $\mathrm{Hom}_{P-\mathrm{mod}}(M^{(r)}, N^{(r')}) = 0$ *if* $r \neq r'$.

Definition II.8.14. The decomposition $M = \bigoplus_r M^{(r)}$(II.8.13.1) is called *the slope decomposition* of M. The values $r \geq 0$ for which $M^{(r)} \neq 0$ are the *slopes* of M. We say that M is *isoclinic* if it has only one slope.

II.8.15. For the rest of this section, we fix a prime number ℓ different from p. From now on until II.8.22 included, Λ is a local $\mathbb{Z}_\ell$-algebra which is of finite type as a $\mathbb{Z}_\ell$-module and $\psi : \mathbb{F}_p \to \Lambda^\times$ is a nontrivial character.

Lemma II.8.16 ([AS11, 6.7])**.** *Let* M *be a* Λ*-module on which* P *acts* Λ*-linearly through a finite discrete quotient, which is isoclinic of slope* $r > 0$. *So the action of* P *on* M *factors through the quotient group* $P/G^{r+}_{K,\log}$.

(i) *Let* $X(r)$ *be the set of isomorphism classes of characters* $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda_\chi^\times$ *such that* Λ_χ *is a finite étale* Λ*-algebra, generated by the image of* χ *and with connected spectrum. Then,* M *has a unique direct sum decomposition*

$$M = \bigoplus_{\chi \in X(r)} M_\chi, \tag{II.8.16.1}$$

where each M_χ *is a* P*-stable* Λ*-submodule on which* $\Lambda[G^r_{K,\log}]$ *acts through* Λ_χ.

(ii) *There are finitely many characters* $\chi \in X(r)$ *such that* $M_\chi \neq 0$.

(iii) *For a fixed* χ *and variable* M, *the functor* $M \mapsto M_\chi$ *is exact.*

(iv) *For* M, N *as above, we have* $\mathrm{Hom}_{\Lambda[P]}(M_\chi, N_\chi) = 0$ *if* $\chi \neq \chi'$.

Definition II.8.17. The decomposition $M = \bigoplus_\chi M_\chi$ (II.8.16.1) is called the *central character decomposition* of M. The characters $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda^\times_\chi$ for which $M_\chi \neq 0$ are the *central characters* of M.

Remark II.8.18. Let P_0 be a finite discrete quotient of $P/G^{r+}_{K,\log}$ through which P acts on M and let G^r_0 be the image of $\mathrm{Gr}^r_{\log} G_K$ in P_0. Then, G^r_0 is abelian by II.8.12 and thus $\Lambda[G^r_0]$ is a commutative ring. The connected components of $\mathrm{Spec}(\Lambda[G^r_0])$ correspond to the isomorphism classes of characters $\chi : G^r_0 \to \Lambda_\chi$, where Λ_χ is a finite étale Λ-algebra generated by the image of χ and with connected spectrum. If $p^n G^r_0 = 0$ and Λ contains a p^n-th primitive root of unity, then $\Lambda_\chi = \Lambda$ for every χ satisfying $M_\chi \neq 0$.

II.8.19. For the rest of this section, recalling the notation fixed in II.8.1, we assume that k is of finite type over a perfect subfield k_0. We define the k-vector space $\Omega^1_k(\log)$ by

$$\Omega^1_k(\log) = (\Omega^1_{k/k_0} \oplus (k \otimes_{\mathbb{Z}} K^\times))/(\mathrm{d}\bar{b} - \bar{b} \otimes b; b \in \mathcal{O}_K^\times). \tag{II.8.19.1}$$

For a rational number r, we also define $\mathfrak{m}^r_{\overline{K}}$ and $\mathfrak{m}^{r+}_{\overline{K}}$ by

$$\mathfrak{m}^r_{\overline{K}} = \{x \in \overline{K} \mid v(x) \geq r\} \quad \text{and} \quad \mathfrak{m}^{r+}_{\overline{K}} = \{x \in \overline{K} \mid v(x) > r\}. \tag{II.8.19.2}$$

Theorem II.8.20 ([Sai09, 1.24], [Sai12, Theorem 2])**.** *For every rational number* $r > 0$, *we have*

(i) *The abelian group* $\mathrm{Gr}^r_{\log} G_K$ *is killed by* p.

(*ii*) *There is an injective group homomorphism*

$$\mathrm{rsw} : \mathrm{Hom}(\mathrm{Gr}^r_{\log} G_K, \mathbb{F}_p) \to \mathrm{Hom}_{\overline{k}}(\mathfrak{m}^r_{\overline{K}}/\mathfrak{m}^{r+}_{\overline{K}}, \Omega^1_k(\log) \otimes_k \overline{k}). \tag{II.8.20.1}$$

The morphism (II.8.20.1) is called the *Refined Swan conductor.*

II.8.21. Let $L \subset \overline{K}$ be a finite Galois extension of K of group G. Recall that Λ is a local $\mathbb{Z}_\ell$-algebra which is of finite type as a $\mathbb{Z}_\ell$-module. Let M be a free Λ-module of finite rank on which G acts linearly. Following (II.8.13.1), let

$$M = \bigoplus_{r \in \mathbb{Q}_{\geq 0}} M^{(r)} \tag{II.8.21.1}$$

be the slope decomposition of M. We define the *Abbes-Saito logarithmic Swan conductor* to be the following rational number

$$\mathrm{sw}^{\mathrm{AS}}_G(M) = \sum_{r \in \mathbb{Q}_{\geq 0}} r \cdot \dim_\Lambda M^{(r)}, \tag{II.8.21.2}$$

where $\dim_\Lambda(M^{(r)}$ abusively denotes the rank of the Λ-module $M^{(r)}$. Clearly, $\mathrm{sw}^{\mathrm{AS}}_G(M) = 0$ if and only if the wild inertia P acts trivially on M (II.8.13 (i)).

For each rational number $r > 0$, following (II.8.16.1), let

$$M^{(r)} = \bigoplus_{\chi \in X(r)} M^{(r)}_\chi \tag{II.8.21.3}$$

be the central character decomposition of $M^{(r)}$. Then, each $M^{(r)}_\chi$ is a free Λ-module. As $\mathrm{Gr}^r_{\log}$ is killed by p, the existence of ψ ensures that χ factors as

$$\mathrm{Gr}^r_{\log} G_K \xrightarrow{\overline{\chi}} \mathbb{F}_p \xrightarrow{\psi} \Lambda^\times. \tag{II.8.21.4}$$

Following [Hu15, (4.12.1)], we define the Abbes-Saito *characteristic cycle* $\mathrm{CC}_\psi(M)$ of M by

$$\mathrm{CC}_\psi(M) = \bigotimes_{r \in \mathbb{Q} > 0} \bigotimes_{\chi \in X(r)} (\mathrm{rsw}(\overline{\chi})(\pi^r))^{\otimes(\dim_\Lambda M^{(r)}_\chi)} \in (\Omega^1_k(\log) \otimes_k \overline{k})^{\otimes \dim_\Lambda M/M^{(0)}}. \tag{II.8.21.5}$$

Denoting by b a common denominator of all the slopes of M, for any such slope r, the element π^r is defined up to a choice of a b-th root of 1. Another choice changes the right-hand side of (II.8.21.5) by a factor $\xi^{b\,\mathrm{sw}^{\mathrm{AS}}_G(M)}$, where ξ is a b-th root of unity, which disappears as $\mathrm{sw}^{\mathrm{AS}}_G(M)$ is an integer ([Xia10, 4.4.3], [Xia12, 4.5.14] and [Sai20, 4.3.1]). Thus, $\mathrm{CC}_\psi(M)$ is unambiguously defined. We note that, just like $\mathrm{KCC}_\zeta(\chi_M)$, where χ_M is the image of M in $R_\Lambda(G)$ (II.8.8.2), $\mathrm{CC}_\psi(M)$ depends on the chosen uniformizer π of $\mathcal{O}_K$.

The main result of [Hu15] is the following comparison theorem for characteristic cycles.

Theorem II.8.22 ([Hu15, Theorem 10.4])**.** *We use the notation of* II.8.5*,* II.8.8 *and* II.8.21*. We assume that p is not a uniformizer of K, that $\Lambda \subset C$ and that the extension L/K is of type* (II)*. Let M be a finite free Λ-module with a Λ-linear action of G. Then, for the same uniformizer π, we have*

$$\mathrm{KCC}_{\psi(1)}(\chi_M) = \mathrm{CC}_\psi(M) \quad \text{in} \quad (\Omega^1_k)^{\otimes m}, \tag{II.8.22.1}$$

where χ_M is the image in $R_C(G)$ (of the base change to C) of M and the exponent m is the integer $\dim_C \chi_M - \langle \chi_M, 1\rangle = \dim_\Lambda(M/M^{(0)})$.

II.8.23. *For the rest of this section*, let $\overline{\Lambda}$ be a finite field of characteristic $\ell \neq p$ which contains a primitive p-th root of unity, $\Lambda = W(\overline{\Lambda})$ its ring of Witt vectors, $\Lambda_{\mathbb{Q}}$ the field of fractions of Λ, C an algebraic closure of $\Lambda_{\mathbb{Q}}$ and $\psi : \mathbb{F}_p \to \overline{\Lambda}^{\times}$ be a non-trivial character.

Here, we follow [**Hu15**, 10.7]. Assume that p is not a uniformizer of K. Let L be a finite Galois extension of K of group G which of type (II) over a larger subfield K' such that K'/K is unramified. The group $\mathrm{Gal}(L/K')$ has cardinality a power of p. Let M be a $\overline{\Lambda}$-vector space of finite dimension with a linear action of G. Then, using the same uniformizer π, the Abbes-Saito characteristic cycle $\mathrm{CC}_{\psi}(M)$ (II.8.21.5) and Kato's characteristic cycle $\mathrm{KCC}_{\psi(1)}(\chi_M)$ (II.8.9, II.8.10), where χ_M is (any pre-image by the Cartan homomorphism $R_{\Lambda_{\mathbb{Q}}}(G) \to R_{\overline{\Lambda}}(G)$ of) the image of M in $R_{\overline{\Lambda}}(G)$, are both well-defined and lie in $(\Omega^1_k)^{\otimes m}$, where the exponent m is $\dim_{\overline{\Lambda}}(M/M^{(0)})$. Then, H. Hu shows that we still have

$$\mathrm{KCC}_{\psi(1)}(\chi_M) = \mathrm{CC}_{\psi}(M) \quad \text{and} \quad (\Omega^1_k)^{\otimes m}. \tag{II.8.23.1}$$

II.8.24. Let V be a henselian $\mathbb{Z}^2$-valuation ring and W/V a monogenic integral extension of (henselian) $\mathbb{Z}^2$-valuation rings (II.3.12) with residue characteristic $p > 0$ and trivial residue extension. Let $\mathfrak{p}$ (resp. $\mathfrak{q}$) be the height 1 prime ideal of V (resp. W). Let $\mathbb{K}$ (resp. $\mathbb{L}$) be the field of fractions of V (resp W) and assume that $\mathbb{L}/\mathbb{K}$ is finite and Galois of group $\mathbb{G}$. The valuation ring V is equipped with a normalized $\mathbb{Z}^2$-valuation map $v : \mathbb{K}^{\times} \to \mathbb{Z}^2$, and first and second projection maps $v^{\alpha}, v^{\beta} : \mathbb{K}^{\times} \to \mathbb{Z}$ (II.3.7). Let $\widehat{\mathbb{K}}_{\mathfrak{p}}$ (resp. $\widehat{\mathbb{L}}_{\mathfrak{q}}$) be the field of fractions of the $\mathfrak{p}$-adic (resp. $\mathfrak{q}$-adic) completion $\widehat{V_{\mathfrak{p}}}$ of $V_{\mathfrak{p}}$ (resp. $\widehat{W_{\mathfrak{q}}}$ of $W_{\mathfrak{q}}$). Then, $\widehat{V_{\mathfrak{p}}} \subset \widehat{W_{\mathfrak{q}}}$ is a monogenic integral extension of complete discrete valuation rings and $\widehat{\mathbb{L}}_{\mathfrak{q}}/\widehat{\mathbb{K}}_{\mathfrak{q}}$ is Galois of group denoted $\widehat{\mathbb{G}}$. Let $\pi_{\mathfrak{p}}$ be a uniformizer of $V_{\mathfrak{p}}$. Then, we have well-defined class functions on G, $a^{\alpha}_{\widehat{\mathbb{G}}}$ and $\mathrm{sw}^{\beta}_{\widehat{\mathbb{G}}}$ with values in $\mathbb{Q}$ and $\mathbb{Z}$ respectively (II.6.16).

The quotient $V/\mathfrak{p}$ is a discrete valuation ring, whose field of fraction is the residue field $\kappa(\mathfrak{p})$ of $V_{\mathfrak{p}}$ (and $\widehat{V_{\mathfrak{p}}}$) with valuation map $\mathrm{ord}_{\mathfrak{p}} : \kappa(\mathfrak{p})^{\times} \to \mathbb{Z}$ which satisfies $v^{\beta}(x) = \mathrm{ord}_{\mathfrak{p}}(x\pi_{\mathfrak{p}}^{-v^{\alpha}(x)} \bmod \mathfrak{p})$ for any $x \in V$. We denote again by $\mathrm{ord}_{\mathfrak{p}} : \Omega^1_{\kappa(\mathfrak{p})} - \{0\} \to \mathbb{Z}$ the valuation defined by $\mathrm{ord}_{\mathfrak{p}}(b\mathrm{d}a) = \mathrm{ord}_{\mathfrak{p}}(b)$, if $a, b \in \kappa(\mathfrak{p})^{\times}$ and $\mathrm{ord}_{\mathfrak{p}}(a) = 1$. It can be canonically extended multiplicatively to $\mathrm{ord}_{\mathfrak{p}} : (\Omega^1_{\kappa(\mathfrak{p})})^{\otimes m} - \{0\} \to \mathbb{Z}$, for any integer $m > 0$.

Moreover, with respect to $W_{\mathfrak{q}}/V_{\mathfrak{p}}$, the extension $\mathbb{L}/\mathbb{K}$ is of type (II) over a subfield of $\mathbb{L}$ which is unramified over $\mathbb{K}$. Hence, so is the extension $\widehat{\mathbb{L}}_{\mathfrak{q}}/\widehat{\mathbb{K}}_{\mathfrak{p}}$. Hence, by II.8.9, we can use the notation of II.8.5 and put $K = \widehat{\mathbb{K}}_{\mathfrak{p}}$, $L = \widehat{\mathbb{L}}_{\mathfrak{q}}$, $G = \widehat{\mathbb{G}}$ and $\pi = \pi_{\mathfrak{p}}$.

We use the notation of II.8.23. For a finite dimensional $\overline{\Lambda}$-representation M of the quotient G of G_K, we have well-defined characteristic cycles $\mathrm{KCC}_{\psi(1)}(\chi_M)$ and $\mathrm{CC}_{\psi}(M)$ (II.8.23) attached to the same π. Moreover, if p is not a uniformizer of $V_{\mathfrak{p}}$, then the following identity holds (II.8.23.1)

$$\mathrm{KCC}_{\psi(1)}(\chi_M) = \mathrm{CC}_{\psi}(M) \quad \text{in} \quad (\Omega^1_{\kappa(\mathfrak{p})})^{\otimes(\dim_{\overline{\Lambda}}(M/M^{(0)}))} \tag{II.8.24.1}$$

where $M^{(0)}$ is the tame part of M, i.e. the sub-module of M fixed by wild inertia subgroup of G (II.8.13 (i)).

Independent of the chosen pre-image of χ_M in $R_{\Lambda_{\mathbb{Q}}}(G)$, we also have well-defined pairings (II.7.19.3)

$$\langle a^{\alpha}_G, \chi_M \rangle \quad \text{and} \quad \langle \mathrm{sw}^{\beta}_G, \chi_M \rangle. \tag{II.8.24.2}$$

Proposition II.8.25. *We use the notation of* (II.8.24). *Assume that p is not a uniformizer of $V_{\mathfrak{p}}$. Let M be a finite dimensional $\overline{\Lambda}$-representation of G. Then, we have the identities*

$$|G|\langle a_G^{\alpha}, \chi_M\rangle = \mathrm{sw}_G^{\mathrm{AS}}(M), \tag{II.8.25.1}$$

$$|G|\langle \mathrm{sw}_G^{\beta}, \chi_M\rangle = -\operatorname{ord}_{\mathfrak{p}}(\mathrm{CC}_\psi(M)) - \dim_{\overline{\Lambda}}(M/M^{(0)}). \tag{II.8.25.2}$$

Proof. We denote by ζ the p-th root of unity $\psi(1)$. We will prove the identities (II.8.25.1) and (II.8.25.2) by comparing the Swan conductor $\mathrm{sw}_G(M) \in \mathbb{Z}^2$ (*cf.* Theorem II.6.15) with Kato's Swan conductor with differential values $\mathrm{sw}_\zeta(\chi_M)$ (II.8.6.3), where $\chi_M \in R_{\Lambda_{\mathbb{Q}}}(G)$ now denotes a characteristic zero lift of the image of M in $R_{\overline{\Lambda}}(G)$. We may assume that L/K is an extension of type (II). Indeed, this follows from (II.7.19) and the stability of the logarithmic ramification filtration under tame base change [AS02, 3.15 (3)]. We use (some of) the notation of II.8.4. We denote by Q the kernel of the canonical map $\Omega^1_{\kappa(\mathfrak{p})} \to \Omega^1_{\kappa(\mathfrak{q})}$, which is a 1-dimensional $\kappa(\mathfrak{p})$-vector space generated by $\mathrm{d}\overline{a}$. We recall also that $\Omega^1_{\kappa(\mathfrak{q})/\kappa(\mathfrak{p})}$ is also a one-dimensional $\kappa(\mathfrak{q})$-vector space generated by $\mathrm{d}\overline{h}$. As $V/\mathfrak{p} \to W/\mathfrak{q}$ is an extension of discrete valuation rings with induced extension of fields of fractions $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ and trivial residue extension (II.8.24), we can assume that $W/\mathfrak{q} = V/\mathfrak{p}[\overline{h}]$, with $\overline{h}$ is a uniformizer of $W/\mathfrak{q}$ [Ser68, III, §6, Prop. 12 & Lemme 4]. Then, since $\overline{h}^{p^n} = \overline{a}$, with $p^n = |G| = [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})]$, $\overline{a}$ is a uniformizer of $V/\mathfrak{p}$. Recall that C is an algebraic closure of $\Lambda_{\mathbb{Q}}$ (II.8.24) and let

$$\varphi : S_{L/K} = (\kappa(\mathfrak{q})\langle \mathfrak{q}/\mathfrak{q}^2, \Omega^1_{\kappa(\mathfrak{q})/\kappa(\mathfrak{p})}\rangle)^\times \to \kappa(\mathfrak{q})^\times \oplus \mathbb{Q}^2 \hookrightarrow \kappa(\mathfrak{q})^\times \oplus C^2 \tag{II.8.25.3}$$

be the composition of the canonical inclusion $\kappa(\mathfrak{q})^\times \oplus \mathbb{Q}^2 \hookrightarrow \kappa(\mathfrak{q})^\times \oplus C^2$ with the non-canonical homomorphism which sends $\pi \bmod \mathfrak{q}^2$ and $\mathrm{d}\overline{h}$ to $(1,1,0)$ and $(1,0,1)/|G|$ respectively. As the injection $S_{K,L} \hookrightarrow S_{L/K}$ identifies $\mathrm{d}\overline{a}$ with $|G|\mathrm{d}\overline{h}$ (II.8.4.4), the composition $S_{K,L} \hookrightarrow S_{L/K} \xrightarrow{\varphi} \kappa(\mathfrak{q})^\times \oplus C^2$ sends $\mathrm{d}\overline{a}$ to $(1,0,1)$, inducing an isomorphism $S_{K,L} \xrightarrow{\sim} \kappa(\mathfrak{p})^\times \oplus \mathbb{Z}^2$. We consider also the map

$$\varpi : \kappa(\mathfrak{q})^\times \oplus C^2 \to C^2, \quad (x,y,z) \mapsto (y, \operatorname{ord}_{\mathfrak{p}}(x) + z\operatorname{ord}_{\mathfrak{p}}(\mathrm{d}\overline{a})). \tag{II.8.25.4}$$

We can write $\mathrm{sw}_\zeta(\chi_M) = [\Delta'] + [\pi^c] - m[\mathrm{d}\overline{a}]$ (II.8.8.1), where $\Delta' \in \kappa(\mathfrak{p})^\times$, c is an integer and $m = \dim_{\overline{\Lambda}}(M/M^{(0)}) = \dim_{\overline{\Lambda}}\chi_M - \langle \chi_M, 1\rangle$. It follows that

$$\varpi \circ \varphi(\mathrm{sw}_\zeta(\chi_M)) = (c, \operatorname{ord}_{\mathfrak{p}}(\Delta') - m\operatorname{ord}_{\mathfrak{p}}(\mathrm{d}\overline{a})). \tag{II.8.25.5}$$

Therefore, from the definition of $\mathrm{KCC}_\zeta(\chi_M)$, we deduce that

$$\beta \circ \varpi \circ \varphi(\mathrm{sw}_\zeta(\chi_M)) = -\operatorname{ord}_{\mathfrak{p}}(\mathrm{KCC}_\zeta(\chi_M)). \tag{II.8.25.6}$$

We also have, just by definition,

$$\mathrm{sw}_\zeta(\chi_M) = \sum_{\sigma \in G} s_G(\sigma) \otimes \mathrm{tr}_{\chi_M}(\sigma) + m \sum_{r \in \mathbb{F}_p^\times \subseteq \kappa(\mathfrak{p})^\times} [r] \otimes \zeta^r. \tag{II.8.25.7}$$

Now, we know from that $|G|[\mathrm{d}\overline{h}] = [\mathrm{d}\overline{a}]$ in $S_{L/K}$; so $\varpi \circ \varphi([\mathrm{d}\overline{h}]) = \operatorname{ord}_{\mathfrak{p}}(\mathrm{d}\overline{a})\varepsilon/|G|$, where $\varepsilon = (0,1)$ (*cf.* II.6.4). For $\sigma \in G - \{1\}$, we also clearly have $\varpi \circ \varphi([h - \sigma(h)]) = i_G(\sigma)$ (II.6.1.1). It thus follows from (II.6.12.1) and (II.8.6.1) that

$$\varpi \circ \varphi(s_G(\sigma)) = \frac{(\operatorname{ord}_{\mathfrak{p}}(\mathrm{d}\overline{a}) - 1)\varepsilon}{|G|} - j_{G,\varepsilon}(\sigma) = \frac{(\operatorname{ord}_{\mathfrak{p}}(\mathrm{d}\overline{a}) - 1)\varepsilon}{|G|} + \mathrm{sw}_G(\sigma), \tag{II.8.25.8}$$

and we deduce from (II.6.12.2), (II.8.6.2)) and (II.8.25.8) that

$$\varpi\circ\varphi(s_G(1)) = \frac{(\mathrm{ord}_{\mathfrak{p}}(\mathrm{d}\overline{a})-1)\varepsilon}{|G|} + \mathrm{sw}_G(1) - (\mathrm{ord}_{\mathfrak{p}}(\mathrm{d}\overline{a})-1)\varepsilon. \tag{II.8.25.9}$$

We can also compute

$$\varpi\circ\varphi\otimes\mathrm{Id}_C\left(\sum_{r\in\mathbb{F}_p^\times\subsetneq\kappa(\mathfrak{p})^\times}[r]\otimes\zeta^r\right) = (\sum_r \mathrm{ord}_{\mathfrak{p}}(r), 0) = (0,0), \tag{II.8.25.10}$$

where $\varpi\circ\varphi\otimes\mathrm{Id}_C(x\otimes y) = y\varpi\circ\varphi(x)$ for $x\in S_{L/K}$ and $y\in C$. Hence, combining (II.8.25.7), (II.8.25.8), (II.8.25.9) and (II.8.25.10) with (II.6.13.1) and (II.6.14.1), we obtain

$$\begin{aligned}\varpi\circ\varphi(\mathrm{sw}_\zeta(\chi_M)) &= \mathrm{sw}_G(M) + (\mathrm{ord}_{\mathfrak{p}}(\mathrm{d}\overline{a})-1)\langle\chi,1\rangle\varepsilon - (\mathrm{ord}_{\mathfrak{p}}(\mathrm{d}\overline{a})-1)\chi_M(1)\varepsilon\\ &= \mathrm{sw}_G(M) - m(\mathrm{ord}_{\mathfrak{p}}(\mathrm{d}\overline{a})-1)\varepsilon.\end{aligned} \tag{II.8.25.11}$$

Now we use the fact that $\overline{a}$ is a uniformizer of the valuation ring $V/\mathfrak{p}$ of $\kappa(\mathfrak{p})$ and thus $\mathrm{ord}_{\mathfrak{p}}(\mathrm{d}\overline{a}) = 0$ to get the simplified identity

$$\varpi\circ\varphi(\mathrm{sw}_\zeta(\chi_M)) = \mathrm{sw}_G(M) + m\varepsilon. \tag{II.8.25.12}$$

Applying the projection β to this identity, we deduce from (II.6.17.1), (II.8.24.1) and (II.8.25.6) that indeed $|G|\mathrm{sw}_G^\beta(M) = -\mathrm{ord}_{\mathfrak{p}}(\mathrm{CC}_\psi(M)) - m$. Applying the projection α to (II.8.25.12) and using (II.8.25.5) (and that $a_G^\alpha = \mathrm{sw}_G^\alpha$ (II.6.16)) yields

$$|G|a_G^\alpha(M) = |G|\mathrm{sw}_G^\alpha(M) = c. \tag{II.8.25.13}$$

Finally, as $W_{\mathfrak{q}}/V_{\mathfrak{p}}$ is a monogenic extension, we see from [**AS02**, 6.7] that its logarithmic ramification is bounded by a rational number $r\geq 0$ if and only if $\alpha(i_G(\sigma))\geq r$. (Note that, in *loc. cit.*, Proposition 6.7 holds for all monogenic separable extensions and thus is validly applied here). Therefore, the logarithmic filtration of G (II.8.11), relative to $W_{\mathfrak{q}}/V_{\mathfrak{p}}$, coincide with the filtration defined by $\alpha\circ i_G$. Hence, we have

$$\mathrm{sw}_G^{\mathrm{AS}}(M) = \alpha(\mathrm{sw}_G(M)) = |G|a_G^\alpha(M) = c, \tag{II.8.25.14}$$

which finishes the proof. □

II.9. Proof of Theorem II.1.9.

II.9.1. Let K be a complete discrete valuation field, $\mathcal{O}_K$ its valuation ring, $\mathfrak{m}_K$ its maximal ideal, k its residue field, assumed to be algebraically closed of characteristic $p>0$, and π a uniformizer of $\mathcal{O}_K$. Let also $\overline{\Lambda}$ be a finite field of characteristic $\ell\neq p$ and fix a nontrivial character $\psi:\mathbb{F}_p\to\overline{\Lambda}^\times$.

II.9.2. Let D be the rigid unit disc over K and $\mathcal{F}$ a *lisse* étale sheaf of $\overline{\Lambda}$-modules on D. Let $\overline{0}\to D$ be a geometric point above the origin 0 of D. By [**deJ95**, 2.10], the datum $\mathcal{F}$ is equivalent to the data of a finite Galois étale connected cover $f:X\to D$ and a finite dimensional continuous $\overline{\Lambda}$-representation $\rho_{\mathcal{F}}$ of $\pi_1^{\text{ét}}(D,\overline{0})$ which factors through the quotient $G=\mathrm{Aut}(X/D)$ of $\pi_1^{\text{ét}}(D,\overline{0})$. Let $\chi_{\mathcal{F}}$ be the image of $\rho_{\mathcal{F}}$ in the Grothendieck group $R_{\overline{\Lambda}}(G)$. Let $t\in\mathbb{Q}_{\geq 0}$, $\mathfrak{p}^{(t)}$ the generic point of the special fiber of the normalized integral model the sub-disc $D^{(t)}$ of D and $\tau=(\overline{x}_\tau,\mathfrak{p}_\tau)$ an element of the set $S_f^{(t)}$ (notation of II.7.1) associated to the normalized integral model of $f^{(t)}:X^{(t)}\to D^{(t)}$ defined over a finite extension K' of K which is t-admissible for f (II.4.10). (Recall that $S_f^{(t)}$ is independent of the choice of such a K' (II.4.24).) The group G acts transitively on $S_f^{(t)}$, and any element $\tau \in S_f^{(t)}$ defines an monogenic integral extension of henselian $\mathbb{Z}^2$-valuation rings $V_t^h(\tau)/V_t^h$ whose induced extension of fields of fractions $\mathbb{K}_{t,\tau}^h/\mathbb{K}_t^h$ is Galois of group $G_{t,\tau}$, the stabilizer of τ under the action of G (II.7.3). We complete this extension, which puts as in the situation of II.8.24. As $|G_{t,\tau}| = |G|/|S_f^{(t)}|$ (II.7.3), we deduce from II.8.25, II.7.9 and Frobenius reciprocity that (notation of II.1.8)

$$\mathrm{sw}^{\mathrm{AS}}_{G_{t,\tau}}(\rho_{\mathcal{F}}|G_{t,\tau}) = \langle \widetilde{a}_f^{\alpha}(t), \chi_{\mathcal{F}} \rangle, \tag{II.9.2.1}$$

$$-\operatorname{ord}_{\overline{\mathfrak{p}}^{(t)}}(\mathrm{CC}_\psi(\rho_{\mathcal{F}}|G_{t,\tau})) - \dim_{\overline{\Lambda}}\left(\rho_{\mathcal{F}}|G_{t,\tau}/(\rho_{\mathcal{F}}|G_{t,\tau})^{(0)}\right) = \langle \widetilde{\mathrm{sw}}_f^{\beta}(t), \chi_{\mathcal{F}} \rangle. \tag{II.9.2.2}$$

It follows that $\mathrm{sw}^{\mathrm{AS}}_{G_{t,\tau}}(\rho_{\mathcal{F}}|G_{t,\tau})$ is independent of the choice of both the t-admissible extension K' and $\tau \in S_f^{(t)}$. Since $G_{t,\tau}$ and its wild inertia subgroup $P_{t,\tau}$, with respect to the extension of discrete valuation rings induced by $V_t^h(\tau)/V_t^h$ at the height 1 prime ideals, are independent of the choice of K' (see II.7.5 and II.7.7), so are $\rho_{\mathcal{F}}|G_{t,\tau}$ and its tame part $(\rho_{\mathcal{F}}|G_{t,\tau})^{(0)} = (\rho_{\mathcal{F}}|G_{t,\tau})^{P_{t,\tau}}$. As the $G_{t,\tau}$ (resp. $P_{t,\tau}$), for all $\tau \in S_f^{(t)}$, are conjugate, $\rho_{\mathcal{F}}|G_{t,\tau}$ and $(\rho_{\mathcal{F}}|G_{t,\tau})^{(0)}$ are also independent of the choice of $\tau \in S_f^{(t)}$. Hence, by (II.9.2.2), $-\operatorname{ord}_{\overline{\mathfrak{p}}^{(t)}}(\mathrm{CC}_\psi(\rho_{\mathcal{F}}|G_{t,\tau}))$ is also independent of the choice of both K' and τ. Finally, we remark also that $\operatorname{ord}_{\overline{\mathfrak{p}}^{(t)}}(\mathrm{CC}_\psi(\rho_{\mathcal{F}}|G_{t,\tau}))$ is independent of both the chosen uniformizer π (by (II.7.5) and (II.9.2.2)) and the nontrivial character ψ.

Definition II.9.3. We keep the notation and assumptions of II.9.2 above. We define the normalized logarithmic Swan conductor of $\mathcal{F}$ at t by

$$\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, t) = \mathrm{sw}^{\mathrm{AS}}_{G_{t,\tau}}(\rho_{\mathcal{F}}|G_{t,\tau}). \tag{II.9.3.1}$$

We define the normalized order of the characteristic cycle of $\mathcal{F}$ at t by

$$\varphi_s(\mathcal{F}, t) = -\operatorname{ord}_{\overline{\mathfrak{p}}^{(t)}}(\mathrm{CC}_\psi(\rho_{\mathcal{F}}|G_{t,\tau})) - \dim_{\overline{\Lambda}}\left(\rho_{\mathcal{F}}|G_{t,\tau}/(\rho_{\mathcal{F}}|G_{t,\tau})^{(0)}\right). \tag{II.9.3.2}$$

Theorem II.9.4 (Theorem II.1.9)**.** *We keep the notation and assumptions of II.9.2 and II.9.3. Then, the function $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}$ is continuous and piecewise linear, with finitely many slopes which are all integers. Its right derivative is the function $\varphi_s(\mathcal{F}, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}$ which is locally constant.*

Proof. This follows from (II.9.2.1), (II.9.2.2) and Corollary II.7.20. □

Chapitre III

Variation of the Swan conductor of an $\mathbb{F}_\ell$-sheaf on a rigid annulus

Sommaire

III.1. Introduction

III.1.1. Let K be a complete discrete valuation field, $\mathcal{O}_K$ its valuation ring and π a uniformizer of $\mathcal{O}_K$. Let also $\overline{K}$ be a separable closure of K and $v_K : \overline{K}^{\times} \to \mathbb{Q}$ the valuation of $\overline{K}$ normalized by $v(\pi) = 1$. Assume that the residue field k of $\mathcal{O}_K$ is *algebraically closed* of characteristic $p > 0$. Denote by D the closed rigid unit disc over K. Let $0 \leq r < r'$ be rational numbers and denote by $C = A(r', r)$ the closed sub-annulus of D of radii $|\pi|^{r'} < |\pi|^r$. For a rational number $r \leq t \leq r'$, we denote by $C^{[t]}$ the sub-annulus of C of radius t with 0-thickness. Let Λ be a finite field of characteristic $\ell \neq p$. An étale sheaf of Λ-modules $\mathcal{F}$ on C is said to be *meromorphic* if it is lisse, i.e. locally constant and constructible, on the complement of a finite set of rigid points of C. In this paper, we associate to such an $\mathcal{F}$ a Swan conductor function $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot) : [r, r'] \cap \mathbb{Q}_{\geq 0} \to \mathbb{Q}$, defined by the (logarithmic) ramification theory of Abbes and Saito, and which, for the variable t, measures the ramification of $\mathcal{F}|C^{[t]}$ along the special fiber of the normalized integral model of $C^{[t]}$. We show that this function is continuous and piecewise linear outside the radii of the ramification points of $\mathcal{F}$, with finitely many slopes which are all integers. For two distinct radii t and t' lying between consecutive radii of ramification points of $\mathcal{F}$, we compute the difference of the slopes of $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$ at t and t' as the difference of the orders of the characteristic cycles of $\mathcal{F}$ at t and t'.

III.1.2. Let $\psi : \mathbb{F}_p \to \Lambda^{\times}$ be a nontrivial character. Let $r \leq t \leq r'$ be a rational number distinct from the radii of the ramification points of $\mathcal{F}$. Then, the restriction $\mathcal{F}|C^{[t]}$ corresponds to a connected Galois étale cover $f^{[t]} : X^{[t]} \to C^{[t]}$ and a continuous finite dimensional Λ-representation $\rho_{\mathcal{F}}[t]$ of $G^{[t]} = \mathrm{Aut}(X^{[t]}/C^{[t]})$ [deJ95, 2.10]. Let $\mathfrak{X}^{[t]}_{K'} \to \mathscr{C}^{[t]}_{K'}$ be the normalized integral model of $f^{[t]}$ over $\mathcal{O}_{K'}$, for some large finite extension K' of K (III.4.3), $\overline{\mathfrak{p}}^{(t)}$ a geometric generic point of the special fiber $\mathscr{C}^{[t]}_{s'}$ of $\mathscr{C}^{[t]}_{K'}$ and $\mathfrak{X}^{[t]}_{K'}(\overline{\mathfrak{p}}^{(t)})$ the set of geometric generic points of the special fiber $\mathfrak{X}^{[t]}_{s'}$ above $\overline{\mathfrak{p}}^{(t)}$. The canonical right action of $G^{[t]}$ on $X^{[t]}$ induces a transitive action of $G^{[t]}$ on $\mathfrak{X}^{[t]}_{K'}(\overline{\mathfrak{p}}^{(t)})$. The stabilizer of any $\overline{\mathfrak{q}}^{(t)} \in \mathfrak{X}^{[t]}_{K'}(\overline{\mathfrak{p}}^{(t)})$ is isomorphic to the Galois group $G_{\overline{\mathfrak{q}}^{(t)}}$ of a finite

Galois extension of henselian discrete valuation fields (III.5.11). Therefore, the ramification theory of Abbes and Saito [**AS02**, **AS11**] applies to the restriction $M_{\overline{\mathfrak{q}}^{(t)}} = \rho_{\mathcal{F}}[t]|G_{\overline{\mathfrak{q}}^{(t)}}$; it yields the Swan conductor

(III.1.2.1) $$\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, t) = \mathrm{sw}^{\mathrm{AS}}_{G_{\overline{\mathfrak{q}}^{(t)}}}(M_{\overline{\mathfrak{q}}^{(t)}}) \quad \in \mathbb{Q}$$

and the characteristic cycle $\mathrm{CC}_\psi(M_{\overline{\mathfrak{q}}^{(t)}})$ of $M_{\overline{\mathfrak{q}}^{(t)}}$, which are independent of the choice of both K' and $\overline{\mathfrak{q}}^{(t)} \in \mathfrak{X}^{[t]}_{K'}(\overline{\mathfrak{p}}^{(t)})$ (III.6.2). In our setting, the characteristic cycle lies in $(\Omega^1_{\kappa(\overline{\mathfrak{p}}^{(t)})})^{\otimes m_t}$, where $m_t = \dim_\Lambda(M_{\overline{\mathfrak{q}}^{(t)}}/(M_{\overline{\mathfrak{q}}^{(t)}})^{(0)})$, with $(M_{\overline{\mathfrak{q}}^{(t)}})^{(0)}$ being the part of $M_{\overline{\mathfrak{q}}^{(t)}}$ fixed by the tame inertia subgroup of $G_{\overline{\mathfrak{q}}^{(t)}}$, and $\kappa(\overline{\mathfrak{p}}^{(t)})$ coincides with the field $\mathcal{O}_{\mathscr{C}^{[t]}_{s'}, \overline{\mathfrak{p}}^{(t)}}$. The latter has a normalized discrete valuation map $\mathrm{ord}_{\overline{\mathfrak{p}}^{(t)}} : \kappa(\overline{\mathfrak{p}}^{(t)})^\times \to \mathbb{Z}$ defined by $\mathrm{ord}_{\overline{\mathfrak{p}}^{(t)}}(\xi) = 1$; it extends uniquely to $(\Omega^1_{\kappa(\overline{\mathfrak{p}}^{(t)})})^{\otimes m_t}$. We put

(III.1.2.2) $$\varphi_s(\mathcal{F}, t) = -\,\mathrm{ord}_{\overline{\mathfrak{p}}^{(t)}}(\mathrm{CC}_\psi(M_{\overline{\mathfrak{q}}^{(t)}})) - m_t.$$

This integer is independent not only of $\overline{\mathfrak{q}}^{(t)} \in \mathfrak{X}^{[t]}_{K'}(\overline{\mathfrak{p}}^{(t)})$, but also of the choice of both ψ and a uniformizer of $\mathcal{O}_{K'}$ [**Hu15**, 11.6].

We note that definitions (III.1.2.1) and (III.1.2.2) extend to the radii of the ramification points of $\mathcal{F}$: one just replaces $C^{[t]}$ by the complement of the ramification points of radius t and make the same constructions again, with the same arguments. Our main result takes the following general form.

Theorem III.1.3. *Let $\mathcal{F}$ be a meromorphic étale sheaf of Λ-modules on the annulus C as above and $\{r_1 > \ldots > r_n\}$ the ordered set of the radii of its ramification points. Then, the function*

(III.1.3.1) $$\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot) : [r, r'] \cap \mathbb{Q}_{\geq 0} \to \mathbb{Q},$$

is continuous and piecewise linear on $\mathbb{Q} - \{r_1, \ldots, r_n\}$, with finitely many slopes which are all integers. Moreover, for rational numbers $r_i < t < t' < r_{i-1}$, the difference of the right and left derivatives of $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$ at t and t' respectively is

(III.1.3.2) $$\frac{d}{dt}\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, t+) - \frac{d}{dt}\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, t'-) = \varphi_s(\mathcal{F}, t) - \varphi_s(\mathcal{F}, t').$$

We note first that if $\mathcal{F}$ is a lisse sheaf on the entire rigid unit disc D, we proved a more precise statement in [**Bah20**, Theorem 1.9], namely that the function $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$, defined by the formula (III.1.2.1) on the whole $\mathbb{Q}_{\geq 0}$, is continuous and piecewise linear, with finitely many slopes which are all integers, and its right derivative *is exactly* the function $\varphi_s(\mathcal{F}, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}$, defined by the formula (III.1.2.2). This may still be true in the setting of III.1.2, but our techniques are unlikely to yield it. Instead, what we get is the variation of the slope between two distinct radii.

In *loc. cit.*, we also shared the expectation that the main theorem there [**Bah20**, Theorem 1.9] should hold for an étale sheaf on D (or C) with a finite number of ramification points and expressed the hope of tackling the question soon. The present paper fulfills this expectation, in the form of Theorem III.1.2, by extending the techniques of *loc. cit.* to handle morphisms to annuli. In fact, by excluding the radii of the ramification points of $\mathcal{F}$, we see that we are reduced to treating the case of a lisse étale sheaf of Λ-modules on the annulus C, hence to studying étale morphisms to annuli.

III.1.4. The overall strategy of proof is the same as the one employed for in [**Bah20**]. Therefore, rather than repeating it here, we refer the reader to the introduction of *loc. cit.* for an overview and briefly indicate the most salient changes.

We more generally consider a smooth K-affinoid space X and a finite flat morphism $f : X \to C = A(r', r)$ which is generically étale. We extend the constructions and results in [**Bah20**, §4, 7] to this setting. Most notably, we generalize the nearby cycles formula [**Bah20**, 4.28] linking the derivative of the discriminant function ∂_f studied by W. Lütkebohmert [**Lüt93**, §1, 2] with cohomological invariants associated to the local homomorphisms $\mathcal{O}_{\mathscr{C}_{K'},o} \to \mathcal{O}_{\mathfrak{X}_{K'},\overline{x}}$, for the formal étale topology, induced by the normalized integral model $\widehat{f} : \mathfrak{X}_{K'} \to \mathscr{C}_{K'}$ of f over $\mathcal{O}_{K'}$, where the $\overline{x}$'s are the geometric points of the special fiber $\mathfrak{X}_{s'}$ of $\mathfrak{X}_{K'}$ above the origin o of $\mathscr{C}_{s'}$ (III.4.7). We do this under the assumption that f is étale over $A(r, r)$ and $A(r', r')$ and that the inverse images by f of $A(r, r)$ and $A(r', r')$ decomposes into finite disjoint unions of annuli

$$f^{-1}(A(r, r)) = \coprod_i A(r/d_i, r/d_i) \quad \text{and} \quad f^{-1}(A(r', r')) = \coprod_j A(r'/d'_j, r'/d'_j) \tag{III.1.4.1}$$

respectively, which is later satisfied for all but a finite number of the radii of interest, thanks to the semi-stable reduction theorem (III.4.8). We have to compactify both sets of annuli, unlike in [**Bah20**, 4.28] where we have only one side. Taking into account orientation issues, the final nearby cycles formula exhibits the difference $\frac{d}{dt}\partial_f(r+) - \frac{d}{dt}\partial_f(r'-)$. This difference between the inner and outer radii is a feature that carries over throughout all the text.

From this formula and the non-vanishing, at non-smooth points, of the nearby cycles sheaf $R^1\Psi(\Lambda)$, associated to (an algebraization of the compactification of) $\widehat{f}$ (III.4.12), we deduce the following.

Proposition III.1.5 (cf. III.5.9)**.** *Assume that $f : X \to C$ is étale. Then, the discriminant function ∂_f is convex.*

III.1.6. Let G be a finite group with a right action on X such that f is G-equivariant. Just like in [**Bah20**, §7], for a rational number $r \leq t \leq r'$, Kato's ramification theory for $\mathbb{Z}^2$-valuation rings [**Bah20**, §6] yields an Artin class function $\widetilde{a}_f(t)$ on G (III.5.11). But, unlike in *loc. cit.*, for the aforementioned reason, the appropriate Swan class function to consider is $\widetilde{\mathrm{sw}}^{\beta}_f([t, t'])$ for an interval $[t, t']$ such that $t \neq t'$, accounting for the two branches $A(t, t)$ and $A(t', t')$. (We don't use this notation in the body of the text, writing instead $\widetilde{\mathrm{sw}}^{\beta}_{f[t,t']} + \widetilde{\mathrm{sw}}'^{\beta}_{f[t,t']}$ (III.5.12).)

Let r_G be the character of the regular representation of G and $\langle \cdot, \cdot \rangle$ the usual pairing of class functions on G. Then, we have (III.5.13)

$$\partial_f(t) = \langle \widetilde{a}_f(t), r_G \rangle, \tag{III.1.6.1}$$

$$\frac{d}{dt}\partial_f(t+) - \frac{d}{dt}\partial_f(t'-) = \langle \widetilde{\mathrm{sw}}^{\beta}_f([t, t']), r_G \rangle. \tag{III.1.6.2}$$

The proof of III.1.6.2 requires not only the nearby cycles formula (III.4.7.2) but also a Hurwitz formula, due to Kato for local homomorphisms like $\mathcal{O}_{\mathscr{C}_{K'},o} \to \mathcal{O}_{\mathfrak{X}_{K'},\overline{x}}$ above. For the analogous statement [**Bah20**, 7.12], we could get by with a special case of this Kato-Hurwitz formula (by the regularity of the affine line !), as we considered only sub-discs of D. For annuli with thickness however, the more general formula (III.3.4.1) is needed.

From the above identities and III.1.5, we deduce that the function $t \mapsto \langle \widetilde{a}_f(t), r_G \rangle$ is continuous, convex and piecewise linear with (finitely many) integer slopes. Theorem III.1.3 is ultimately deduced from this result. In light of this (and the further identities in III.6.3), we expect the function $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F},\cdot)$ also to be convex.

III.1.7. The article is organized as follows. In §III.3, we recall a few properties of the category of formal étale local rings studied in [**Bah20**, §3], which is a formal analogue of an algebraic category studied by Kato in [**Kat87a**, §5]. We give a detailed proof of the Kato-Hurwitz formula for morphisms of this category, following closely Kato's algebraic proof. Section III.4 is devoted to the nearby cycles formula, after the requisite preparation. In section III.5, the identities (III.1.4.1) and (III.1.6.2) are proved. For any character χ of G, the variation with t of $\langle \widetilde{a}_f(t), \chi\rangle$, is deduced and its change of slope is shown to be $\langle \widetilde{\mathrm{sw}}_f^{\beta}([t,t'],\chi\rangle$ (III.5.14 - III.5.16). The last section hearkens back to III.1.2. Theorem III.6.5 is deduced by identifying $\langle \widetilde{a}_f(t), \chi\rangle$ with $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F},t)$ and $\langle \widetilde{\mathrm{sw}}_f^{\beta}([t,t'],\chi\rangle$ with $\varphi(\mathcal{F},t)-\varphi_s(\mathcal{F},t')$ (III.6.3), which implies Theorem III.1.2.

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III.2. Notations and Conventions

Let K be a complete discrete valuation field, $\mathcal{O}_K$ its valuation ring, $\mathfrak{m}_K$ its maximal ideal, k its residue field, assumed to be algebraically closed of characteristic $p>0$, and π a uniformizer of $\mathcal{O}_K$. Let also $\overline{K}$ be a separable closure of K, $\mathcal{O}_{\overline{K}}$ the integral closure of $\mathcal{O}_K$ in $\overline{K}$, $\overline{k}$ its residue field, G_K the Galois group of $\overline{K}$ over K, and $v_K : \overline{K}^{\times} \to \mathbb{Q}$ the valuation of $\overline{K}$ normalized by $v_K(\pi)=1$. Let $D=\mathrm{Sp}(K\{\xi\})$ be the closed rigid unit disc over K. For rational numbers $r'\geq r\geq 0$, we denote by $A(r',r)=A_K(r',r)$ the closed sub-annulus of D defined by $r'\geq v_K(\xi)\geq r$ and by $A^{\circ}(r',r)$ the open sub-annulus given by $r'>v_K(\xi)>r$. We put $\mathscr{S}=\mathrm{Spf}(\mathcal{O}_K)$ and denote by s its unique point. All formal schemes are assumed to be locally noetherian.

III.3. Kato-Hurwitz formula

III.3.1. For a formal relative curve $\mathfrak{X}/\mathscr{S}$ and a geometric point $\overline{x}$ of $\mathfrak{X}$ [**Bah20**, 2.5] with image a closed point x of $\mathfrak{X}$, we consider the following property :

$$\text{(P)}\qquad \mathfrak{X}-\{x\} \text{ is smooth over } \mathscr{S} \text{ and } \mathfrak{X} \text{ is normal at } x.$$

We denote by $\widehat{\mathcal{C}}_K$ the category whose objects are the rings that are isomorphic over $\mathcal{O}_K$ to the formal étale local rings $\mathcal{O}_{\mathfrak{X},\overline{x}}$ [**Bah20**, 2.5] for some couple $(\mathfrak{X}/\mathscr{S},\overline{x})$ satisfying property (P), where $\mathfrak{X}$ is a formal relative curve over $\mathscr{S}$ and $\overline{x}$ is a geometric point of $\mathfrak{X}$ over a closed point x of $\mathfrak{X}$. The morphisms of $\widehat{\mathcal{C}}_K$ are the finite $\mathcal{O}_K$-homomorphisms inducing separable extensions of fields of fractions. By [**Bah20**, 3.17], if A is an object of $\widehat{\mathcal{C}}_K$, then A is a two-dimensional, henselian, normal, local ring with residue field k and special fiber $A/\mathfrak{m}_KA$ reduced and excellent. We refer the reader to [**Bah20**, 2.5-2.12 and 3.17-3.23] for other relevant properties of the category $\widehat{\mathcal{C}}_K$ and its objects, when needed, as well as for the definition of its motivating algebraic analog $\mathcal{C}_K$ introduced by Kato in [**Kat87a**, §5].

We briefly recall a few definitions associated to an object A and a morphism $A\to B$ of $\widehat{\mathcal{C}}_K$ (valid verbatim for $\mathcal{C}_K$). We denote by $P(A)$ the set of height one prime ideals of A, by $P_s(A)$

its (finite) subset of prime ideals above $\mathfrak{m}_K$ and by $P_\eta(A)$ the complement of $P_s(A)$ in $P(A)$. The quotient $A_0 = A/\mathfrak{m}_K A$ is a reduced ring [Bah20, 3.17 (ii)]; let $\widetilde{A_0}$ be its integral closure in its total ring of fractions. We denote by $\delta(A)$ the k-dimension of the quotient vector space $\widetilde{A_0}/A_0$. The ring $A_K = A \otimes_{\mathcal{O}_K} K$ is a Dedekind domain [Bah20, 3.17 (i)], and the homomorphism $A_K \to B_K$ induced by $A \to B$ is an extension of Dedekind domains, hence projective. The corresponding bilinear trace map induces a well-defined K-linear determinant homomorphism

$$T_{B_K/A_K} : \det_{A_K}(B_K) \otimes_{A_K} \det_{A_K}(B_K) \to A_K, \tag{III.3.1.1}$$

where $\det_{A_K}(B_K)$ is the invertible A_K-module $\bigwedge^r_{A_K} B_K$, with r the rank of B_K as an A_K-module [Bah20, 3.24-3.25]. Following Kato [Kat87a, §5], we define the integer $d_\eta(B/A)$ to be the dimension of the cokernel of T_{B_K/A_K}.

For $\mathfrak{p} \in P(A)$, $A_\mathfrak{p}$ is a discrete valuation ring with residue field $\kappa(\mathfrak{p})$; we denote by $w_\mathfrak{p}$: $\mathrm{Frac}(A)^\times \to \mathbb{Z}$ the associated normalized valuation map. Moreover, $\kappa(\mathfrak{p})$ is a discrete valuation field. Indeed, if $\mathfrak{p} \in P_s(A)$, the integral closure of the henselian Japanese ring $A/\mathfrak{p}$ in $\kappa(\mathfrak{p})$ is a discrete valuation ring; if $\mathfrak{p} \in P_\eta(A)$, $\kappa(\mathfrak{p})$ is a finite extension of K. In both cases, we denote by $\mathrm{ord}_\mathfrak{p} : \kappa(\mathfrak{p})^\times \to \mathbb{Z}$ the associated normalized valuation map.

For any $\mathfrak{p} \in P_s(A)$, the couple $(A, \mathfrak{p})$ gives rise to a $\mathbb{Z}^2$-valuation ring $V_A(\mathfrak{p})$ and to a henselian $\mathbb{Z}^2$-valuation ring $V^h_A(\mathfrak{p})$, the henselization of $V_A(\mathfrak{p})$ [Bah20, 3.18], whose field of fractions we denote by $\mathbb{K}^h_A(\mathfrak{p})$. We let $v_\mathfrak{p} : (\mathbb{K}^h_A(\mathfrak{p}))^\times \to \mathbb{Z}^2$ be the associated normalized valuation map and $v^\alpha_\mathfrak{p}$ (resp. $v^\beta_\mathfrak{p}$) the composition of $v_\mathfrak{p}$ with the first (resp. second) projection $\mathbb{Z}^2 \to \mathbb{Z}$ [Bah20, 3.7].

For any $\mathfrak{q} \in P_s(B)$ above $\mathfrak{p}$, the valuation ring $V^h_B(\mathfrak{q})$ is the integral closure of $V^h_A(\mathfrak{p})$ in $\mathbb{K}^h_B(\mathfrak{q})$. It is a finite free $V^h_A(\mathfrak{p})$-module and $V^h_B(\mathfrak{q}) = V^h_A(\mathfrak{p})[b]$ for some $b \in V^h_B(\mathfrak{q}))$ [Bah20, 3.23]. Therefore, we have again a well-defined K-linear determinant homomorphism $T_{V^h_B(\mathfrak{q})/V^h_A(\mathfrak{p})}$, whose image is a nonzero principal ideal of $V^h_A(\mathfrak{p})$. Let $c(B/A, \mathfrak{p}, \mathfrak{q})$ be a generator of this image. Following Kato [Kat87a, §5], we have a well-defined integer

$$d_s(B/A) = \sum_{(\mathfrak{p},\mathfrak{q})} v^\beta_\mathfrak{p}(c(B/A, \mathfrak{p}, \mathfrak{q})), \tag{III.3.1.2}$$

where $\mathfrak{p}$ runs over $P_s(A)$ and $\mathfrak{q}$ runs over the elements of $P_s(B)$ above $\mathfrak{p}$.

Lemma III.3.2 ([Kat87a, Lemma 5.8]). *Let A be an object of $\mathcal{C}_K$ (resp. $\widehat{\mathcal{C}}_K$) and denote by $\mathbb{K}$ its field of fractions. Then, for any $x \in \mathbb{K}^\times$, we have*

$$\sum_{\mathfrak{p} \in P_s(A)} v^\beta_\mathfrak{p}(x) = \sum_{\mathfrak{p} \in P_\eta(A)} [\kappa(\mathfrak{p}) : K] w_\mathfrak{p}(x). \tag{III.3.2.1}$$

Proof. We first note that the valuation map $w_\mathfrak{p}$ coincide with $v^\alpha_\mathfrak{p}$ for $\mathfrak{p} \in P_s(A)$ [Bah20, 3.7, 3.15]. Denoting by $K_2(\mathbb{K})$ the second Milnor K-group of $\mathbb{K}$, for any $p \in P(A)$, we have the Steinberg tame symbol $\partial_\mathfrak{p} : K_2(\mathbb{K}) \to \kappa(\mathfrak{p})^\times$ of $w_\mathfrak{p}$. By definition, for any $x \in \mathbb{K}^\times$, it satisfies

$$\partial_\mathfrak{p}(\{x, \pi\}) = (-1)^{v^\alpha_\mathfrak{p}(x)} x^{-1} \pi^{v^\alpha_\mathfrak{p}(x)} \mod \mathfrak{p} \quad \text{if } \mathfrak{p} \in P_s(A), \tag{III.3.2.2}$$

$$\partial_\mathfrak{p}(\{x, \pi\}) = \pi^{w_\mathfrak{p}(x)} \mod \mathfrak{p} \quad \text{if } \mathfrak{p} \in P_\eta(A). \tag{III.3.2.3}$$

Note that, for any $\mathfrak{p} \in P(A)$, as k is algebraically closed, the residue field of the discrete valuation ring associated to $\mathrm{ord}_\mathfrak{p} : \kappa(\mathfrak{p}) \to \mathbb{Z}$ is of degree $d_\mathfrak{p} = 1$ over k. If $\mathfrak{p} \in P_s(A)$, then $\mathrm{ord}_\mathfrak{p}(x\pi^{-v^\alpha_\mathfrak{p}(x)} \mod \mathfrak{p}) = v^\beta_\mathfrak{p}(x)$ for any $x \in \mathbb{K}^\times$ [Bah20, 3.15].

The K-theory for $\mathrm{Spec}(A)$ yields a localization complex

$$K_2(\mathbb{K}) \xrightarrow{(\partial_\mathfrak{p})_\mathfrak{p}} \bigoplus_{\mathfrak{p}\in P(A)} \kappa(\mathfrak{p})^\times \xrightarrow{(\partial'_\mathfrak{p})_\mathfrak{p}} \mathbb{Z} = K_0(\kappa(\mathfrak{m}_A)), \tag{III.3.2.4}$$

where $\mathfrak{m}_A$ is the maximal ideal of A and $\partial'_\mathfrak{p} : \kappa(\mathfrak{p})^\times \to \mathbb{Z}$ is the composition of $\mathrm{ord}_\mathfrak{p}$ with the multiplication map $\mathbb{Z} \xrightarrow{\times d_\mathfrak{p}} \mathbb{Z}$. Then,

$$\partial'_\mathfrak{p} \circ \partial_\mathfrak{p}(\{x,\pi\}) = \mathrm{ord}_\mathfrak{p}(x^{-1}\pi^{v^\alpha_\mathfrak{p}(x)} \mod \mathfrak{p}) = -v^\beta_\mathfrak{p}(x) \quad \text{if } \mathfrak{p} \in P_s(A), \tag{III.3.2.5}$$

$$\partial'_\mathfrak{p} \circ \partial_\mathfrak{p}(\{x,\pi\}) = [\kappa(\mathfrak{p}) : K] w_\mathfrak{p}(x) \quad \text{if } \mathfrak{p} \in P_\eta(A). \tag{III.3.2.6}$$

Hence, equation (III.3.2.1) follows from the relation $\sum_{\mathfrak{p}\in P(A)} \partial'_\mathfrak{p} \circ \partial_\mathfrak{p} = 0$ (III.3.2.4). □

Lemma III.3.3. *Let $\mathscr{D} = \mathrm{Spf}(\mathcal{O}_K\{T\})$ be the formal closed unit disc over $\mathcal{O}_K$ and o the origin of its special fiber $\mathrm{Spec}(k[T])$. Then, for any object A of $\widehat{\mathcal{C}}_K$, there exists a morphism $\mathcal{O}_{\mathscr{D},o} \to A$ of the category $\widehat{\mathcal{C}}_K$.*

Proof. Let $\mathfrak{X}/\mathscr{S}$ be an affine formal relative curve and $\overline{x} \to \mathfrak{X}$ a geometric point at a closed point of $\mathfrak{X}$ such that the couple $(\mathfrak{X}/\mathscr{S}, \overline{x})$ satisfies property (P) and $A \cong \mathcal{O}_{\mathfrak{X},\overline{x}}$. As $\mathfrak{X}_s$ is a geometrically reduced k-curve, *Noether normalization with separating transcendence basis* [?, 16.18] implies that we have a finite morphism $g_s : \mathfrak{X}_s \to \mathrm{Spec}(k[T])$ which is generically étale. As $\overline{x}$ is over a closed point of $\mathfrak{X}$ and k is algebraically closed, after replacing T by a translation $T - z$, for some $z \in k$, we can assume that $g_s(\overline{x}) = o$. Let $a \in \mathcal{O}_\mathfrak{X}(\mathfrak{X})$ be a lift of the image of T in $\mathcal{O}_{\mathfrak{X}_s}(\mathfrak{X}_s)$ by g_s. Then, the assignment $T \mapsto a$ defines an $\mathscr{S}$-morphism $g : \mathfrak{X} \to \mathscr{D}$ that lifts g_s and is thus finite. As the $\mathscr{S}$-curve $\mathfrak{X}$ is normal, it is also Cohen-Macaulay [**EGA IV**, Discussion below 0.16.5.1]. It follows that g is flat [**EGA IV**, 0.17.3.5 (i)]. Then, as g_s is étale over a nonempty open admissible subset of $\mathfrak{X}_s$, so is g [**Abb10**, 2.4.7, 2.4.8]. Hence, the homomorphism of formal étale stalks $\mathcal{O}_{\mathscr{D},o} \to \mathcal{O}_{\mathfrak{X},\overline{x}}$ induced by g gives the desired morphism of $\widehat{\mathcal{C}}_K$. □

Proposition III.3.4 ([**Kat87a**, 5.7])**.** *Let $A \to B$ be a morphism of $\widehat{\mathcal{C}}_K$. Then, with the notation of III.3.1, we have the following formula*

$$d_\eta(B/A) - d_s(B/A) = 2\delta(B) - 2\deg(B/A)\delta(A), \tag{III.3.4.1}$$

where $\deg(B/A)$ is the degree of the extension of fields of fractions induced by $A \to B$.

Proof. Let $g : A \to B$ and $h : B \to C$ be morphisms of $\widehat{\mathcal{C}}_K$. By [**Ser68**, III, Prop. 8], we have

$$d_\eta(C/A) = d_\eta(C/B) + \deg(C/B) d_\eta(B/A). \tag{III.3.4.2}$$

As the formation of the two-dimensional henselian $\mathbb{Z}^2$-valuation ring in III.3.1 is functorial [**Bah20**, 3.22], [**Ser68**, III, Prop. 8] implies also that, for $\mathfrak{p} \in P_s(A)$, $\mathfrak{q} \in P_s(B)$ above $\mathfrak{p}$ and $\mathfrak{r} \in P_s(C)$ above $\mathfrak{q}$, with the notation at the end of III.3.1, we have

$$v^\beta_\mathfrak{p}(c(C/A,\mathfrak{p},\mathfrak{r})) = v^\beta_\mathfrak{q}(c(C/B,\mathfrak{q},\mathfrak{r})) + \deg(B/A) v^\beta_\mathfrak{p}(c(B/A,\mathfrak{p},\mathfrak{q})). \tag{III.3.4.3}$$

Since, for each $\mathfrak{p} \in P_s(A)$, the set of $\mathfrak{r} \in P_s(C)$ above $\mathfrak{p}$ is the disjoint union of the sets of $\mathfrak{r} \in P_s(C)$ above $\mathfrak{q}$, for $\mathfrak{q}$ ranging over the subset of elements of $P_s(B)$ above $\mathfrak{p}$, it follows that

$$d_s(C/A) = d_s(C/B) + \deg(C/B) d_s(B/A). \tag{III.3.4.4}$$

Therefore, by (III.3.4.2) and (III.3.4.4), if (III.3.4.1) holds for two morphisms among g, h and $h \circ g$, it holds also for the third. Hence, by III.3.3, we can assume that $A = \mathcal{O}_{\mathscr{D},o}$ in the statement of the proposition. In particular, A and $A_0 = A/\mathfrak{m}_K A$ are regular rings, $\delta(A) = 0$ and $P_s(A)$ is a singleton. Then, the proposition follows from III.3.2 and [Bah20, 3.27]. □

Remark III.3.5. The above proof follows verbatim Kato's proof for the morphisms of $\mathcal{C}_K$ [Kat87a, 5.7-5.8].

III.4. The nearby cycles formula

III.4.1. For an integer $n \geq 0$, we let $C_n = C_{K,n}$ be the affinoid curve $\mathrm{Sp}(K\{\xi,\zeta\}/(\xi\zeta - \pi^n))$. It is the annulus in the closed unit disc D defined by $n \geq v(\xi) \geq 0$. The admissible adic ring $\mathcal{O}_K\{\xi,\zeta\}/(\xi\zeta - \pi^n)$ defines a formal model $\mathscr{C}_n = \mathscr{C}_{K,n} = \mathrm{Spf}(\mathcal{O}_K\{\xi,\zeta\}/(\xi\zeta - \pi^n))$ of C_n over $\mathcal{O}_K$. The special fiber $\mathscr{C}_{n,s} = \mathrm{Spec}(k[\xi,\zeta]/(\xi\zeta))$ is a union of two copies of $\mathbb{A}^1_k$ that intersect at the unique singular point $o = (0,0)$ of $\mathscr{C}_{n,s}$, an ordinary double point, and is geometrically reduced. It follows that $\mathcal{O}_K\{\xi,\zeta\}/(\xi\zeta - \pi^n)$ is the unit ball of $K\{\xi,\zeta\}/(\xi\zeta - \pi^n)$ for the sup-norm [AS02, 4.1] and that $\mathscr{C}_n$ is the *normalized integral model of* C_n *defined over* $\mathcal{O}_K$ [Bah20, 2.18-2.19]. In particular, $\mathscr{C}_n$ is normal. Moreover, $\mathscr{C}_n - \{o\}$ is smooth over $\mathscr{S}$. Hence, the couple $(\mathscr{C}_n, o)$ satisfies property (P) in III.3.1. Therefore, the formal étale local ring $\mathcal{O}_{\mathscr{C}_n,o}$ is an object of the category $\widehat{\mathcal{C}}_K$. If $\overline{\mathfrak{p}}$ is a geometric generic point of $\mathscr{C}_{n,s}$, then its image $\mathfrak{p}$ in $\mathscr{C}_{n,s}$ corresponds to a minimal prime ideal of $\mathcal{O}_{\mathscr{C}_{n,s},o}$; thus, the inverse image of $\mathfrak{p}$ through the natural reduction map $\mathcal{O}_{\mathscr{C}_n,o} \to \mathcal{O}_{\mathscr{C}_{n,s},o}$ of geometric stalks [Bah20, 2.7.2] is a height one prime ideal denoted $\mathfrak{p}$ again.

III.4.2. For the remainder of this section, we fix rational numbers $r' \geq r \geq 0$ and put $C = A(r',r)$, $C^{[r]} = A(r,r)$, $C^{[r']} = A(r',r')$, $C^\circ = A^\circ(r',r)$. There exist a finite extension K' of K and integers $m \geq n \geq 0$ such that $r, r' \in v(K')$ and

$$C_{K'} = C \otimes_K K' \simeq A_{K'}(m,n) = \{x \in \overline{K} \mid m \geq v_{K'}(x) \geq n\}, \tag{III.4.2.1}$$

where $v_{K'}$ is the valuation of K' normalized by $v_{K'}(\pi') = 1$, for π' a uniformizer of K'. By the change of coordinate $\xi \mapsto \frac{\xi}{\pi^n_{K'}}$, we get an isomorphism $C_{m-n} \xrightarrow{\sim} C_{K'}$ over K' and deduce the following formal isomorphism over $\mathscr{S}' = \mathrm{Spf}(\mathcal{O}_{K'}) = \{s'\}$

$$\mathscr{C}_{K',m-n} \xrightarrow{\sim} \mathscr{C}_{K'}. \tag{III.4.2.2}$$

With III.4.1, the isomorphism (III.4.2.2) shows that we have a distinguished geometric point o of $\mathscr{C}_{K'}$, that $\mathscr{C}_{K'}$ is normal, that $\mathscr{C}_{K'} - \{o\}$ is smooth over $\mathscr{S}'$ and that the special fiber $\mathscr{C}_{s'}$ has exactly two irreducible components corresponding, via specialization, to the annuli with 0-thickness $C^{[r]}_{K'}$ and $C^{[r']}_{K'}$, and intersecting at the unique singular point o of $\mathscr{C}_{s'}$. Let $\mathfrak{p}$ and $\mathfrak{p}'$ be the corresponding generic points. Let $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{p}}'$ be geometric generic points at $\mathfrak{p}$ and $\mathfrak{p}'$. Let $V^h = V^h_{\mathcal{O}_{\mathscr{C},o}}(\mathfrak{p})$ (resp. $V'^h = V^h_{\mathcal{O}_{\mathscr{C},o}}(\mathfrak{p}')$) be the henselian $\mathbb{Z}^2$-valuation ring induced by the couple $(\mathcal{O}_{\mathscr{C},o}, \mathfrak{p})$ (resp. $(\mathcal{O}_{\mathscr{C},o}, \mathfrak{p}')$) (III.3.1) and denote by $\mathbb{K}^h$ (resp. $\mathbb{K}'^h$) its field of fractions. Let $v : (\mathbb{K}^h)^\times \to \mathbb{Q} \times \mathbb{Z}$ (resp. $v' : (\mathbb{K}'^h)^\times \to \mathbb{Q} \times \mathbb{Z}$) be the composition of the normalized $\mathbb{Z}^2$-valuation map $v_\mathfrak{p}$ (resp. $v_{\mathfrak{p}'}$) (III.3.1) with the injection $\mathbb{Z}^2 \to \mathbb{Q} \times \mathbb{Z}$, $(a,b) \mapsto (a/e, b)$, where e is the ramification index of K'/K. Let v^α and v^β (resp. v'^α and v'^β) be the composition of v (resp. v') with the first and second projections $\mathbb{Q} \times \mathbb{Z} \to \mathbb{Q}$ and $\mathbb{Q} \times \mathbb{Z} \to \mathbb{Z}$.

III.4.3. We keep the notation of III.4.2. Let X be a smooth K-affinoid space and $f : X \to C$ a finite flat morphism. As in [Bah20, 4.9], with the same justifications (we don't need the étaleness assumption made in *loc. cit.*), there exists a finite extension K' of K (taken to be larger than in III.4.2) such that $r, r' \in v_K(K')$ and, as for $C_{K'}$, the formal spectrum $\mathfrak{X}_{K'} = \mathrm{Spf}(\mathcal{O}^\circ(X_{K'}))$ of the unit ball $\mathcal{O}^\circ(X_{K'})$ of $\mathcal{O}(X_{K'})$ for its sup-norm is an admissible formal model of $X_{K'}$ over $\mathcal{O}_{K'}$, with a geometrically reduced special fiber. In other words, $\mathfrak{X}_{K'}$ is the normalized integral model of X defined over $\mathcal{O}_{K'}$. In particular, $\mathfrak{X}_{K'}$ is normal. Moreover, $\mathfrak{X}_{K'}$ is smooth over $\mathscr{S}'$ outside a finite set of its closed points. Let $\widehat{f}_{K'} : \mathfrak{X}_{K'} \to \mathscr{C}_{K'}$ be the adic morphism induced by f, called *the normalized integral model of f over $\mathcal{O}_{K'}$*. Then, the formation of $\mathscr{C}_{K'}$, $\mathfrak{X}_{K'}$ and $\widehat{f}_{K'}$ commute with further finite extensions of K'. An extension K' of K as above is said to be *admissible for* f; from the aforementioned commutation, we see that any further extension of K' is also admissible for f.

Lemma III.4.4. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be formal relative curves over $\mathscr{S}$ and $\overline{x}$ and $\overline{y}$ respective geometric points of $\mathfrak{X}$ and $\mathfrak{Y}$ over closed points such that the couples $(\mathfrak{X}/\mathscr{S}, \overline{x})$ and $(\mathfrak{Y}/\mathscr{S}, \overline{y})$ satisfy property (P). Let $g : \mathfrak{X} \to \mathfrak{Y}$ be a finite flat $\mathscr{S}$-morphism such that $g(\overline{x}) = \overline{y}$.*

(i) *If the generic fiber $g_\eta : \mathfrak{X}_\eta \to \mathfrak{Y}_\eta$ is étale over a nonempty open admissible subset of $\mathfrak{Y}_\eta$ contained in the tube of $\overline{y}$, then g induces a morphism $\mathcal{O}_{\mathfrak{Y},\overline{y}} \to \mathcal{O}_{\mathfrak{X},\overline{x}}$ of the category $\widehat{\mathcal{C}}_K$.*

(ii) *Let G be a finite group with a right action on $\mathfrak{X}$. If $g : \mathfrak{X} \to \mathfrak{Y}$ is G-equivariant and $\mathcal{O}(\mathfrak{X})^G \cong \mathcal{O}(\mathfrak{Y})$, then g induces a morphism $\mathcal{O}_{\mathfrak{Y},\overline{y}} \to \mathcal{O}_{\mathfrak{X},\overline{x}}$ of $\widehat{\mathcal{C}}_K$ whose extension of field of fractions is Galois of group a subgroup of G.*

Proof. We can assume that $\mathfrak{X} = \mathrm{Spf}(B)$ and $\mathfrak{Y} = \mathrm{Spf}(A)$ are affine. Denote by $\mathbb{K}_{\overline{y}}$ (resp. $\mathbb{K}_{\overline{x}}$) the field of fractions of $\mathcal{O}_{\mathfrak{Y},\overline{y}}$ (resp. $\mathcal{O}_{\mathfrak{X},\overline{x}}$). By [Bah20, 2.12], we have

$$\mathcal{O}_{\mathfrak{Y},\overline{y}} \otimes_A B \cong \prod_{\overline{x} \in \mathfrak{X}(\overline{y})} \mathcal{O}_{\mathfrak{X},\overline{x}}, \tag{III.4.4.1}$$

where $\mathfrak{X}(\overline{y})$ is the set of geometric points of $\mathfrak{X}$ above $\overline{y}$. This shows the finiteness of the homomorphism $\mathcal{O}_{\mathfrak{Y},\overline{y}} \to \mathcal{O}_{\mathfrak{X},\overline{x}}$.

(i) Let U be an admissible open affinoid subset of $\mathfrak{Y}_\eta$ such that $f : f^{-1}(U) \to U$ is étale. It is enough to show that $\mathbb{K}_{\overline{y}}$ defines a point in $\mathrm{Spec}(\mathcal{O}(U))$, i.e. that $\mathbb{K}_{\overline{y}} \otimes_{\mathcal{O}(\mathfrak{Y})} \mathcal{O}(U) \neq 0$. Indeed, if $\mathbb{K}_{\overline{y}} \otimes_A \mathcal{O}(U) \neq 0$, then there exists a fields extension $\mathbb{L}/\mathbb{K}_{\overline{y}}$ and a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{O}(U) \\ \downarrow & & \downarrow \\ \mathbb{K}_{\overline{y}} & \longrightarrow & \mathbb{L}. \end{array} \tag{III.4.4.2}$$

Using the canonical isomorphism $\mathbb{L} \otimes_{\mathcal{O}(U)} \mathcal{O}(U) \otimes_A B \cong \mathbb{L} \otimes_{\mathcal{O}_{\mathfrak{Y},\overline{y}}} \mathcal{O}_{\mathfrak{Y},\overline{y}} \otimes_A B$, we deduce from (III.4.4.1) that

$$\mathbb{L} \otimes_{\mathcal{O}(U)} \mathcal{O}(f^{-1}(U)) = \mathbb{L} \otimes_{\mathcal{O}(U)} \mathcal{O}(U) \otimes_A B \cong \prod_{\overline{x} \in \mathfrak{X}(\overline{y})} \mathbb{L} \otimes_{\mathcal{O}_{\mathfrak{Y},\overline{y}}} \mathcal{O}_{\mathfrak{X},\overline{x}} = \prod_{\overline{x} \in \mathfrak{X}(\overline{y})} \mathbb{L} \otimes_{\mathbb{K}_{\overline{y}}} \mathbb{K}_{\overline{x}}. \tag{III.4.4.3}$$

We deduce that the canonical homomorphism

$$\mathbb{L} \to \mathbb{L} \otimes_{\mathbb{K}_{\overline{y}}} \mathbb{K}_{\overline{x}} \tag{III.4.4.4}$$

is étale, and thus $\mathbb{K}_{\overline{y}} \to \mathbb{K}_{\overline{x}}$ is a separable extension.

Now the admissible open immersion $U \hookrightarrow \mathfrak{Y}_\eta$ lifts to a formal morphism $\mathcal{U} \hookrightarrow \mathfrak{Y}' \to \mathfrak{Y}$, where $\mathfrak{Y}' \to \mathfrak{Y}$ is an admissible formal blow-up and $\mathcal{U} \hookrightarrow \mathfrak{Y}'$ is a formal open immersion. The geometric point $\overline{y}$ is a specialization of a rig-point $z \in U$. The latter defines a rig-point z' of $\mathfrak{Y}'$ by a property of admissible formal blow-ups [Bos14, 8.3, Prop. 5]. Then, as $\mathcal{U}_\eta = U$ is contained in the tube of $\overline{y}$, z' lies in $\mathcal{U}$ and induces a geometric point $\overline{y}' \to \mathcal{U}$ above $\overline{y}$. Let $\mathcal{V}$ be a formal affine open subset of $\mathcal{U}$ containing y'. Then, $\mathcal{O}_{\mathfrak{Y}',\overline{y}'} = \mathcal{O}_{\mathcal{V},\overline{y}'}$ and we have a commutative square

$$\begin{array}{ccccc} \mathcal{O}(\mathfrak{Y}) & \longrightarrow & \mathcal{O}(U) & \longrightarrow & \mathcal{O}(\mathcal{V}) \otimes_{\mathcal{O}_K} K \\ \downarrow & & & & \downarrow \\ \mathbb{K}_{\overline{y}} & & \longrightarrow & & \mathbb{K}_{\overline{y}'}, \end{array} \tag{III.4.4.5}$$

where $\mathbb{K}_{\overline{y}'}$ is the field of fractions of $\mathcal{O}_{\mathcal{V},\overline{y}'}$. It follows that $\mathbb{K}_{\overline{y}} \otimes_{\mathcal{O}(\mathfrak{Y})} \mathcal{O}(U) \neq 0$.

(ii) We use the following isomorphism deduced from (III.4.4.1) by tensoring with $\mathbb{K}_{\overline{y}}$

$$\mathbb{K}_{\overline{y}} \otimes_{\mathcal{O}(\mathfrak{Y})} \mathcal{O}(\mathfrak{X}) \xrightarrow{\sim} \prod_{\overline{x} \in \mathfrak{X}(\overline{y})} \mathbb{K}_{\overline{x}}. \tag{III.4.4.6}$$

If $\mathcal{O}(\mathfrak{X})^G \cong \mathcal{O}(\mathfrak{Y})$, then $(\prod_{\overline{x} \in \mathfrak{X}(\overline{y})} \mathbb{K}_{\overline{x}})^G \cong \mathbb{K}_{\overline{y}}$, and thus, for any $\overline{x} \in \mathfrak{X}(\overline{y})$, $\mathbb{K}_{\overline{x}}/\mathbb{K}_{\overline{y}}$ is Galois of group a subgroup of G. □

Definition III.4.5. We keep the notation and assumptions of III.4.2 and III.4.3. Let K' be a finite extension of K which is admissible for f.

(1) We say that the finite flat morphism f is *generically étale* if f is étale over a nonempty open admissible subset of C. If f is generically étale, then it has only a finite number of (closed) ramification points in C. Indeed, by the Jacobian criterion [**Abb10**, 6.4.21], the closed subset of X where f is not étale is the vanishing locus of a nonzero convergent power series in one variable, which is finite by the Weierstrass preparation theorem. In particular, f is étale over an open admissible subset of C°.

(2) We define S_f (resp. S'_f) to be the set of couples $\tau = (\overline{x}_\tau, \mathfrak{q}_\tau)$, where $\overline{x}_\tau$ is a geometric point of $\mathfrak{X}_{K'}$ above o and $\mathfrak{q}_\tau$ is a height 1 prime ideal of $B_\tau = \mathcal{O}_{\mathfrak{X}_{K'},\overline{x}_\tau}$ above the height 1 prime ideal $\mathfrak{p}$ (resp. $\mathfrak{p}'$) of $A = \mathcal{O}_{\mathscr{C},o}$. This is a nonempty finite set which is independent of the choice of the large enough extension K' of K which is admissible for f (see [**Bah20**, 4.22]).

(3) We assume that f is generically étale. Then, for each $\tau \in S_f$ (resp. S'_f) the induced homomorphism $A \to B_\tau$ is a morphism of $\widehat{\mathcal{C}}_K$ (III.4.4 (i)); we denote by V^h_τ (resp. V'^h_τ) the associated henselian $\mathbb{Z}^2$-valuation ring (III.3.1) and by $\mathbb{K}^h_\tau$ (resp. $\mathbb{K}'^h_\tau$) its field of fractions. With the notation at the end of III.3.1, we define the integers

$$d_{f,s} = \sum_{\tau \in S_f} v^\beta(c(B_\tau/A, \mathfrak{p}, \mathfrak{q}_\tau)), \tag{III.4.5.1}$$

$$d'_{f,s} = \sum_{\tau \in S'_f} v'^\beta(c(B_\tau/A, \mathfrak{p}', \mathfrak{q}_\tau)). \tag{III.4.5.2}$$

Proposition III.4.6. *We resume the notation and assumptions of* III.4.2 *and* III.4.3. *We further assume that the finite flat morphism $f : X \to C$ is generically étale. Let K' be a finite extension of K which is admissible for f. We denote by $\overline{x}_1, \dots, \overline{x}_N$ the geometric points of $\mathfrak{X}_{K'}$ above o and put $A = \mathcal{O}_{\mathscr{C},o}$, $B_j = \mathcal{O}_{\mathfrak{X}_{K'},\overline{x}_j}$, for $j = 1, \dots, N$. Then, the homomorphisms $A \to B_j$ induced by $\widehat{f}_{K'}$ are morphisms in $\widehat{\mathcal{C}}_{K'}$ and we have*

$$\sum_{j=1}^{N} (d_\eta(B_j/A) - 2\delta(B_j)) = d_{f,s} + d'_{f,s} - 2\deg(f). \tag{III.4.6.1}$$

Proof. By III.4.2 and III.4.3, the following couples

(III.4.6.2) $$(\mathscr{C}_{K'}/\mathscr{S}', o) \quad \text{and} \quad ((\mathfrak{X}_{K'} - \{\overline{x}_i,\ i \neq j\})/\mathscr{S}', \overline{x}_j)_{1 \leq j \leq N}$$

satisfy property (P) (III.3.1). Hence, A and the B_j are objects of the category $\widehat{\mathcal{C}}_{K'}$ (III.3.1). As $f_{K'}$ is étale over an nonempty admissible open subset of the tube $C^\circ_{K'}$ of o (III.4.5(1)), we see from III.4.4 that the homomorphisms $A \to B_j$ induced by $\widehat{f}_{K'}$ are indeed morphisms in $\widehat{\mathcal{C}}_{K'}$. As it is readily seen from the definitions that

(III.4.6.3) $$\delta(A) = 1 \quad \text{and} \quad \sum_{j=1}^{N} d_s(B_j/A) = d_{f,s} + d'_{f,s},$$

equation (III.4.6.1) follows from III.3.4. □

Proposition III.4.7. *We resume the notation and assumptions of III.4.2 and III.4.3. We further assume that X has trivial canonical sheaf, that f is étale over $C^{[r]}$ and $C^{[r']}$ and that*

(III.4.7.1) $$f^{-1}(C^{[r]}) = \coprod_{j=1}^{\delta_f} \Delta_j \quad \text{and} \quad f^{-1}(C^{[r']}) = \coprod_{j=1}^{\delta'_f} \Delta'_j,$$

where $\Delta_j = A(r/d_j, r/d_j)$ and $\Delta'_j = A(r'/d'_j, r'/d'_j)$ with the integer $d_j \geq 1$ (resp. $d'_j \geq 1$) the order of f on Δ_j (resp. Δ'_j) [Bah20, 4.2]. Let K' be a finite extension of K which is admissible for f. Denote by $\overline{x}_1, \ldots, \overline{x}_N$ the geometric points of $\mathfrak{X}_{K'}$ above o (III.4.3) and put $A = \mathcal{O}_{\mathscr{C},o}$, $B_j = \mathcal{O}_{\mathfrak{X}_{K'},\overline{x}_j}$, for $j = 1, \ldots, N$. Then, we have a nearby cycles formula

(III.4.7.2) $$\sum_{j=1}^{N} (d_\eta(B_j/A) - 2\delta(B_j) + |P_s(B_j)|) = \sigma + \delta_f - (\sigma' + \delta'_f),$$

where $|P_s(B_j)|$ denotes the cardinality of $P_s(B_j)$ and σ (resp. σ') is the total order of the derivative of the restriction $f_{|f^{-1}(C^{[r]})}$ (resp. $f_{|f^{-1}(C^{[r']})}$) of f [Bah20, 4.5].

Proof. We put $X^{[r]} = f^{-1}(C^{[r]})$, $X^{[r']} = f^{-1}(C^{[r']})$ and denote by $\mathscr{C}^{[r]}$, $\mathscr{C}^{[r']}$, $\mathfrak{X}^{[r']}$, $\mathfrak{X}^{[r']}$, $\mathfrak{X}$ and $\widehat{f} : \mathfrak{X} \to \mathscr{C}$ the respective normalized integral models of $C^{[r]}$, $C^{[r']}$, $X^{[r]}$, $X^{[r']}$, X and $f : X \to C$ defined over $\mathcal{O}_{K'}$ (III.4.3). We have a Cartesian diagram

(III.4.7.3) $$\begin{array}{ccccc} X^{[r']} & \longrightarrow & X & \longleftarrow & X^{[r]} \\ \downarrow & \square & \downarrow f & \square & \downarrow \\ C^{[r']} & \longrightarrow & C & \longleftarrow & C^{[r]} \end{array}$$

where the horizontal arrows are inclusions, and the following decompositions into disjoint unions

(III.4.7.4) $$\mathfrak{X}^{[r]} = \coprod_{j} \widehat{\Delta}_j \quad \text{and} \quad \mathfrak{X}^{[r']} = \coprod_{j} \widehat{\Delta}'_j,$$

where the normalized integral model $\widehat{\Delta}_j = \mathrm{Spf}(\mathcal{O}^\circ(\Delta_j))$ of Δ_j is the formal annulus of radius $|\pi|^{r/d_j}$ with 0-thickness, defined over K', and is isomorphic to $\mathrm{Spf}(\mathcal{O}_{K'}\{T_j, T_j^{-1}\})$; and Δ'_j is similarly defined and isomorphic to $\mathrm{Spf}(\mathcal{O}_{K'}\{T'_j, T_j'^{-1}\})$. As $\mathfrak{X} \times_{\mathscr{C}} \mathscr{C}^{[r]}$ (resp. $\mathfrak{X} \times_{\mathscr{C}} \mathscr{C}^{[r]}$) is an affine formal model of $X_{K'}^{[r]}$ (resp. $X_{K'}^{[r']}$) (III.4.7.3), with a geometrically reduced special fiber, by [Bah20, 4.1(ii)], we have have a Cartesian diagram

(III.4.7.5)
$$\begin{array}{ccccc} \mathfrak{X}^{[r']} & \longrightarrow & \mathfrak{X} & \longleftarrow & \mathfrak{X}^{[r]} \\ \downarrow & \square & \downarrow{\scriptstyle \widehat{f}} & \square & \downarrow \\ \mathscr{C}^{[r']} & \longrightarrow & \mathscr{C} & \longleftarrow & \mathscr{C}^{[r]} \end{array}$$

The horizontal arrows above are formal open immersions. It follows that $\mathfrak{X}_{s'} - (\mathfrak{X}_{s'}^{[r]} \sqcup \mathfrak{X}_{s'}^{[r']})$ lies over the singular point $\mathscr{C}_{s'} - (\mathscr{C}_{s'}^{[r]} \sqcup \mathscr{C}_{s'}^{[r']}) = \{o\}$.

For each $1 \leq j \leq \delta_f$, we glue $\mathfrak{X}_{K'}$ and a formal closed disc $\mathfrak{D}_j = \mathrm{Spf}(\mathscr{O}_{K'}\{S_j\})$ along the boundary $\mathscr{C}_j^{[r]} = \mathrm{Spf}(\mathscr{O}_{K'}\{S_j, S_j^{-1}\})$ with gluing map $T_j \mapsto S_j^{-1}$. For each $1 \leq j \leq \delta'_f$, we also glue $\mathfrak{X}_{K'}$ and a formal closed disc $\mathfrak{D}'_j = \mathrm{Spf}(\mathscr{O}_{K'}\{S'_j\})$ along the boundary $\mathscr{C}_j^{[r']} = \mathrm{Spf}(\mathscr{O}_{K'}\{S'_j, S'^{-1}_j\})$ with gluing map $T'_j \mapsto S'_j$. The resulting formal relative curve

(III.4.7.6)
$$\mathfrak{Y}_{K'} = (\coprod_{j=1}^{\delta'_f} \mathfrak{D}'_j) \cup \mathfrak{X}_{K'} \cup (\coprod_{j=1}^{\delta_f} \mathfrak{D}_j)/\sim \;\to \mathscr{S}' = \mathrm{Spf}(\mathscr{O}_{K'}).$$

has smooth rigid fiber and contains $\mathfrak{X}_{K'}$ as a formal open subscheme. As $\mathfrak{X}_{K'}$ is normal, $\mathfrak{Y}_{K'}$ is also normal. Its special fiber $\mathfrak{Y}_{s'}$ is the gluing of $\mathfrak{X}_{s'}$ and $\delta_f + \delta'_f$ copies of $\mathbb{A}^1_k$; for each $1 \leq j \leq \delta_f$ (resp. $1 \leq j \leq \delta'_f$), the copy $\mathrm{Spec}(k[S_j])$ (resp. $\mathrm{Spec}(k[S'_j])$) is glued with $\mathrm{Spec}(k[T_j, T_j^{-1}]$ (resp. $\mathrm{Spec}(k[T'_j, T'^{-1}_j])$ by the identification $T_j \mapsto S_j^{-1}$ (resp. $T'_j \mapsto S'_j$). It follows that $\mathfrak{Y}_{s'}$ is a projective k-curve. Moreover, by construction, the singular locus of $\mathfrak{Y}_{s'}$ is contained in the set $\mathfrak{X}_{s'} - (\mathfrak{X}_{s'}^{[r]} \sqcup \mathfrak{X}_{s'}^{[r']})$. By Grothendieck's algebraization theorem, there exists a relative proper algebraic curve Y' over $S' = \mathrm{Spec}(\mathscr{O}_{K'})$ whose formal completion along its special fiber is $\mathfrak{Y}_{K'}$ [EGA III, 5.4.5]. As the rigid fiber $\mathfrak{Y}_{\eta'}$ of $\mathfrak{Y}_{K'}$ is smooth, so is the generic fiber $Y'_{\eta'}$ of Y'. Since the canonical sheaf of X is trivial, there exists a global section $\omega \in \Gamma(X_{K'}, \Omega^1_{X_{K'}/K'})$ inducing an isomorphism $\mathscr{O}_{X_{K'}} \xrightarrow{\sim} \Omega^1_{X_{K'}/K'}$. Thus we can write $df = f^\dagger \omega$, where $f^\dagger \in \Gamma(X_{K'}, \mathscr{O}_X)$. For each j, both $\omega|\Delta_j$ and dT_j (resp. $\omega|\Delta'_j$ and dT'_j) induce a basis of $\Omega^1_{X_{K'}/K'}$ on Δ_j (resp. Δ'_j); thus, we have $\omega|\Delta_j = u_j(T_j)dT_j$ and $\omega|\Delta'_j = u_j(T'_j)dT'_j$, for some $u_j(T_j) \in \Gamma(\Delta_j, \mathscr{O}_{\Delta_j})^\times$ and $u_j(T'_j) \in \Gamma(\Delta'_j, \mathscr{O}_{\Delta'_j})^\times$. Hence, we deduce that $f'(T_j) = u_j(T_j) f^\dagger|\Delta_j$ and $f'(T'_j) = u_j(T'_j) f^\dagger|\Delta'_j$. We choose a point y_j (resp. y'_j) in the generic fiber D_j (resp. D'_j) of $\mathfrak{D}_j$ (resp. $\mathfrak{D}'_j$) that is not in Δ_j (resp. Δ'_j). By the rigid Runge theorem [Ray94, 3.5.2], we can then approximate $\widehat{f} : \mathfrak{X}_{K'} \to \mathscr{C}_{K'}$ by the formal completion $\widehat{g} : \mathfrak{Y}_{K'} \to \widehat{\mathbb{P}}^1_{S'}$ of an algebraic morphism $g : Y' \to \mathbb{P}^1_{S'}$ satisfying $g^{-1}(\infty) \subset \{y_j, y'_j\}$, such that the induced morphism $g_{\eta'} : \mathfrak{Y}_{\eta'} \to \mathbb{P}^{1,\mathrm{rig}}_{K'}$ on rigid fibers has poles at most at the y_j and y'_j and, on each Δ_j (resp. Δ'_j), we have

(III.4.7.7)
$$|g_{\eta'} - f|_j < |f^\dagger|_j / |u_j^{-1}(T_j)|_{\sup} \quad (\text{resp.} \quad |g_{\eta'} - f|'_j < |f^\dagger|'_j / |u_j^{-1}(T'_j)|_{\sup}),$$

where $|\cdot|_j$ (resp. $|\cdot|'_j$) is defined as the sup-norm of the restriction to Δ_j (resp. Δ'_j). As for f, we have $dg_{\eta'}|X_{K'} = g^\dagger \omega$, for some $g^\dagger \in \Gamma(X_{K'}, \mathscr{O}_X)$, $g'_{\eta'}(T_j) = u_j(T_j) g^\dagger|\Delta_j$ and $g'_{\eta'}(T'_j) = u_j(T'_j) g^\dagger|\Delta'_j$. Since $Y'_{\eta'}$ is a smooth projective curve, and dg is a meromorphic section of the canonical sheaf $\Omega^1_{Y'_{\eta'}/K'}$ which is nonzero (by the equality $|g'_{\eta'}(T_j)|_{\sup} = |f'(T_j)|_{\sup}$ on Δ_j established just below

(III.4.7.10)), we have

$$2g(Y'_{\overline{\eta}'}) - 2|\pi_0(Y'_{\overline{\eta}'})| = \deg(\mathrm{div}(dg_{\eta'})), \tag{III.4.7.8}$$

where $g(Y'_{\overline{\eta}'})$ is the total genus of $Y'_{\overline{\eta}'}$, i.e. the sum of the genera of its connected components. Let us compute the right-hand side of (III.4.7.8). On Δ_j, taking the derivative of a power series expansion of $g_{\eta'} - f$ and using the strong triangle inequality gives

$$|g'_{\eta'}(T_j) - f'(T_j)|_{\sup} \leq |g_{\eta'} - f|_j. \tag{III.4.7.9}$$

Since $|g^\dagger - f^\dagger|_j \leq |u_j^{-1}(T_j)|_{\sup} |(g_{\eta'} - f)'(T_j)|_{\sup}$ and $|f^\dagger|_j \leq |u_j^{-1}(T_j)|_{\sup} |f'(T_j)|_{\sup}$, equations (III.4.7.7) and (III.4.7.9) yield both following inequalities

$$|g'_{\eta'}(T_j) - f'(T_j)|_{\sup} < |f'(T_j)|_{\sup} \quad \text{and} \quad |g^\dagger - f^\dagger|_j < |f^\dagger|_j. \tag{III.4.7.10}$$

Therefore, we also have $|g'_{\eta'}(T_j)|_{\sup} = |f'(T_j)|_{\sup}$ and $|g^\dagger|_j = |f^\dagger|_j$. On Δ'_j, the same argument gives $|g'_{\eta'}(T'_j)|_{\sup} = |f'(T'_j)|_{\sup}$ and $|g^\dagger|'_j = |f^\dagger|'_j$. Hence, at each point of the normalization $\widetilde{\mathfrak{Y}}_{s'}$ of $\mathfrak{Y}_{s'}$, $f^\dagger$ and $g^\dagger$ have the same order as defined in [Bah20, 2.20], and so do $f'(T_j)$ and $g'_{\eta'}(T_j)$ (resp. $f'(T'_j)$ and $g'_{\eta'}(T'_j)$). Denoting by $C_+(y)$ the fiber of a point $y \in \mathfrak{Y}_{s'}$ under the specialization map $\mathfrak{Y}_{\eta'} \to \mathfrak{Y}_{s'}$, it follows from [Bah20, 2.21], that, for each $x_j \in \mathfrak{X}_{s'} - (\mathfrak{X}_{s'}^{[r]} \sqcup \mathfrak{X}_{s'}^{[r']})$, we have $\deg(\mathrm{div}(g^\dagger)|C_+(x_j)) = \deg(\mathrm{div}(f^\dagger)|C_+(x_j))$. Hence, because we have

$$\mathrm{div}(dg_{\eta'})|C_+(x_j) = \mathrm{div}(g^\dagger)|C_+(x_j) + \mathrm{div}(\omega)|C_+(x_j), \tag{III.4.7.11}$$

and similarly for df and $f^\dagger$, we obtain $\deg(\mathrm{div}(dg_{\eta'})|C_+(x_j)) = \deg(\mathrm{div}(df|C_+(x_j))$. Moreover, as f is étale on $X_{K'}^{[r]}$, so is $g_{\eta'}$; hence, $\mathrm{div}(dg_{\eta'}|X_{K'})$ is supported in the tube of $\mathfrak{X}_{s'} - (\mathfrak{X}_{s'}^{[r]} \sqcup \mathfrak{X}_{s'}^{[r']})$. Therefore, we have

$$\deg(\mathrm{div}(dg_{\eta'}|X_{K'})) = \sum_{j=1}^{N} \deg(\mathrm{div}(df)|C_+(x_j)) = \sum_{j=1}^{N} d_\eta(B_j/A). \tag{III.4.7.12}$$

We denote by Δ_j^- the annulus Δ_j seen as the boundary of the disc D_j, with coordinate $S_j = T_j^{-1}$. Since $g_{\eta'}$ is étale over Δ_j^- (resp. Δ'_j), $\mathrm{div}(dg_{\eta'})$ (resp. $\mathrm{div}(dg_{\eta'})$) is supported on $C_+(y_j)$ (resp. $C_+(y'_j)$). As $D_j - \Delta_j^- = C_+(y_j)$ (resp. $D'_j - \Delta'_j = C_+(y'_j)$), and $g'(T_j)$ and $f'(T_j)$ (resp. $g'(T'_j)$ and $f'(T'_j)$) have the same order σ_j (resp. σ'_j) on the annulus Δ_j (resp. Δ'_j), [Bah20, 2.21] again yields

$$\deg(\mathrm{div}(dg_{\eta'}|D_j - \Delta_j^-)) = \mathrm{ord}_{y_j}(g'_{\eta'}(T_j^{-1})) = -2 - \sigma_j. \tag{III.4.7.13}$$

$$\deg(\mathrm{div}(dg_{\eta'}|D'_j - \Delta'_j)) = \mathrm{ord}_{y'_j}(g'_{\eta'}(T'_j)) = \sigma'_j. \tag{III.4.7.14}$$

Summing (III.4.7.13) over the j's and adding (III.4.7.12), we find at last that the total degree is

$$\deg(\mathrm{div}(dg_{\eta'})) = \sum_{j=1}^{N} d_\eta(B_j/A) + \sigma' - \sigma - 2\delta_f. \tag{III.4.7.15}$$

Now, let $R\Psi$ be the nearby cycles functor associated to the proper structure morphism $Y' \to S'$ and let Λ be a finite field of characteristic different from p. Denoting by Z the closed subset $\mathfrak{X}_{s'} - (\mathfrak{X}_{s'}^{[r]} \sqcup \mathfrak{X}_{s'}^{[r']})$ of the special fiber $\mathfrak{Y}_{s'} \cong Y'_{s'}$, $i : Z \to Y'_{s'}$ the closed immersion and $j : U = Y'_{s'} - Z \to Y'_{s'}$ the inclusion of the complement, the long exact sequence of cohomology induced by the short exact sequence $0 \to j_!(\Lambda_{|U}) \to \Lambda \to i_*(\Lambda_{|Z}) \to 0$ of sheaves on $Y'_{s'}$ gives the following equality of Euler-Poincaré characteristics

$$\chi(Y'_{s'}, \Lambda) = \chi_c(U, \Lambda_{|U}) + \chi(Y'_{s'}, i_*(\Lambda_{|Z})), \tag{III.4.7.16}$$

where $\chi_c(\cdot)$ is the Euler-Poincaré characteristic with compact support. As the residue fields of the points in Z coincide with the algebraically closed field k, we get $\chi(Y'_{s'}, i_*(\Lambda_{|Z})) = \dim_\Lambda H^0_{\text{ét}}(Z, \Lambda_{|Z}) = |Z| = N$. As U is a disjoint union of $\delta_f + \delta'_f$ copies of $\mathbb{A}^1_k$, we see that

$$\chi_c(U, \Lambda_{|U}) = (\delta_f + \delta'_f)\chi_c(\mathbb{A}^1_k, \Lambda) = \delta_f + \delta'_f. \tag{III.4.7.17}$$

Since Y' is normal, the strict localization$Y'_{(\overline{x})}$ at any geometric point $\overline{x} \to Y'_{s'}$ is also normal; hence, $Y'_{(\overline{x})} \times \eta'$ is reduced. Moreover, as $Y'_{s'} \cong \mathfrak{Y}_{s'}$ is reduced, so is $Y'_{(\overline{x})} \times s' = (Y'_{s'})_{(\overline{x})}$. Therefore, applying [EGA IV, 18.9.8] to the flat local homomorphism $Y'_{(\overline{x})} \to S'$, we see that the Milnor tube $Y'_{(\overline{x})} \times \overline{\eta}'$ is connected. As $R^i\Psi(\Lambda_{|Y'_{\overline{\eta}'}})_{\overline{x}} = H^i_{\text{ét}}(Y'_{(\overline{x})} \times \overline{\eta}', \Lambda)$ [SGA 7, XIII, 2.1.4], the sheaf $R^0\Psi(\Lambda_{|Y'_{\overline{\eta}'}})$ is thus isomorphic to $\Lambda_{|Y'_{s'}}$ and $R^i\Psi(\Lambda_{|Y'_{\overline{\eta}'}}) = 0$ for $i > 1$ [SGA 7, I, Théorème 4.2]. Moreover, $R^1\Psi(\Lambda_{|Y'_{\overline{\eta}'}})$ is concentrated in the singular locus of $Y'_{s'}$ [SGA 7, XIII, 2.1.5], located in Z. It thus follows from (III.4.7.16) and (III.4.7.17) that

$$N + \delta_f + \delta'_f - \sum_{j=1}^{N} \dim_\Lambda H^1_{\text{ét}}(Y'_{(\overline{x}_j)} \times \overline{\eta}', \Lambda) = \chi(Y'_{s'}, R\Psi(\Lambda)). \tag{III.4.7.18}$$

By the proper base change theorem, we also have the equality

$$\chi(Y'_{s'}, R\Psi(\Lambda)) = \chi(Y'_{\overline{\eta}'}, \Lambda) = 2|\pi_0(Y'_{\overline{\eta}'})| - 2g(Y'_{\overline{\eta}'}). \tag{III.4.7.19}$$

It remains to link the cohomology group in (III.4.7.18) to $\delta(B_j)$ and $P_s(B_j)$ in the following way. As $\mathfrak{X}_{K'}$ is a formal open subscheme of $\mathfrak{Y}_{K'}$, we have $\mathcal{O}_{\mathfrak{X}_{K'}, \overline{x}_j} = \mathcal{O}_{\mathfrak{Y}_{K'}, \overline{x}_j}$. Then, [Bah20, 2.8] gives that

$$B_j/\mathfrak{m}_{K'} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{Y}_{s'}, \overline{x}_j} = \mathcal{O}_{Y'_{s'}, \overline{x}_j}. \tag{III.4.7.20}$$

Since, for $A \in \mathrm{Obj}(\mathcal{C}_{K'})$ (resp. $\mathrm{Obj}(\widehat{\mathcal{C}}_{K'})$), $P_s(A)$ identifies with the set of minimal prime ideals of $A/\mathfrak{m}_{K'}$, it then follows that

$$\delta(B_j) = \delta(\mathcal{O}_{Y', \overline{x}_j}) \quad \text{and} \quad |P_s(B_j)| = |P_s(\mathcal{O}_{Y', \overline{x}_j})|. \tag{III.4.7.21}$$

As the couple $((Y' - \{\overline{x}_i, i \neq j\})/S', \overline{x}_j)$ satisfy property (P) in III.3.1; then, [Kat87a, Prop. 5.9], in conjunction with (III.4.7.21), implies that

$$2\delta(B_j) - |P_s(B_j)| + 1 = \dim_\Lambda H^1_{\text{ét}}(Y'_{(\overline{x}_j)} \times \overline{\eta}', \Lambda). \tag{III.4.7.22}$$

Finally, combining (III.4.7.8), (III.4.7.15), (III.4.7.18), (III.4.7.19) and (III.4.7.22) yields (III.4.7.2), which concludes the proof. □

III.4.8. We resume the notation and assumptions of III.4.2 and III.4.3. We furthermore assume that the finite flat morphism $f : X \to A(r', r)$ is generically étale. We know from [Lüt93, 2.3] (see also [Bah20, 4.4]), via the semi-stable reduction theorem, that there exist a finite extension K' of K and a sequence of rational numbers $r' = r_0 > r_1 > \cdots > r_n > r_{n+1} = r$ in $v_K(K')$ such that $f^{-1}_{K'}(A^\circ_{K'}(r_{i-1}, r_i))$ is a finite disjoint union of open annuli $A^\circ_{K'}(r_{i-1}/d_{ij}, r_i/d_{ij})$ and the restrictions of f to the latter annuli are étale of the form

$$A^\circ_{K'}(r_{i-1}/d_{ij}, r_i/d_{ij}) \to A^\circ_{K'}(r_{i-1}, r_i), \quad \xi_{ij} \mapsto \xi_{ij}^{d_{ij}}(1 + h_{ij}(\xi_{ij})), \tag{III.4.8.1}$$

for some integers $d_{ij} \geq 1$, the order of f on the annulus $A^\circ_{K'}(r_{i-1}/d_{ij}, r_i/d_{ij})$, and functions h_{ij} on the same annulus satisfying $|h_{ij}|_{\sup} < 1$. A radius, i.e. an element of $[r, r'] \cap \mathbb{Q}$, which is different from all the r_i's is said to be *non-critical*; being non-critical is stable under base change. It follows that the assumption (III.4.7.1) is satisfied if we restrict f over a sub-annulus $A(t', t) \subset C$, where $t' \geq t$ are non-critical radii of f.

Lemma III.4.9. *We use the notation of III.4.6. If the morphism $f : X \to C$ is étale, then, for every $j = 1, \ldots, N$, we have $d_\eta(B_j/A) = 0$.*

Proof. As $\mathcal{O}(C_{K'})$ is reduced and the K'-affinoid algebra $\mathcal{O}(X_{K'})$ is finite over $\mathcal{O}(C_{K'})$, $\mathcal{O}(\mathfrak{X})$ is also a finite algebra over $\mathcal{O}(\mathscr{C})$ [BGR84, 6.4.1/6]. Moreover, since f is étale, $\mathcal{O}(\mathfrak{X})$ is rig-étale over $\mathcal{O}(\mathscr{C})$ [Abb10, 6.4.12]. It follows from [Abb10, 1.14.15] that $\mathcal{O}(\mathfrak{X})[1/\pi'] = \mathcal{O}(X_{K'})$ is étale over $\mathcal{O}(\mathscr{C})[1/\pi'] = \mathcal{O}(C_{K'})$, where π' is a unformizer of $\mathcal{O}_{K'}$. By [Bah20, 2.12], we also have

$$A_{K'} \otimes_{\mathcal{O}(C_{K'})} \mathcal{O}(X_{K'}) \cong K' \otimes_{\mathcal{O}_{K'}} (A \otimes_{\mathcal{O}(\mathscr{C})} \mathcal{O}(\mathfrak{X})) \cong K' \otimes_{\mathcal{O}_{K'}} (\prod_{j=1}^N B_j) = \prod_{j=1}^N B_{j,K'}. \tag{III.4.9.1}$$

Whence we deduce that, for each $j = 1, \ldots, N$, the extension of Dedekind domains $A_{K'} \to B_{j,K'}$ is étale and thus $d_\eta(B_j/A) = 0$. □

Lemma III.4.10. *The fiber $\widehat{f}_{s'}^{-1}(o)$ of the morphism $\widehat{f}_{s'} : \mathfrak{X}_{s'} \to \mathscr{C}_{s'}$ induced by the normalized integral model of f over $\mathcal{O}_{K'}$ lies in the non-smooth locus of $\mathfrak{X}_{s'}$.*

Proof. Since $f : X \to C$ is finite, the induced morphism $\widehat{f} : \mathfrak{X}_{K'} \to \mathscr{C}_{K'}$ is also finite [BGR84, 6.4.1/6]. Since $\mathscr{C}_{K'}$ is $\mathcal{O}_{K'}$-flat and f is surjective, $\widehat{f}$ is surjective. It follows that the induced morphism on special fibers $\widehat{f}_{s'} : \mathfrak{X}_{s'} \to \mathscr{C}_{s'}$ is also finite and surjective. We deduce that, for a (geometric) point x of $\mathfrak{X}_{s'}$ above $o \in \mathscr{C}_{s'}$, the morphism

$$(\mathfrak{X}_{s'})_{(x)} \to (\mathscr{C}_{s'})_{(o)} \tag{III.4.10.1}$$

on strict localizations is surjective. If x was smooth, then (III.4.10.1) would factor through the normalization map $(\widetilde{\mathscr{C}_{s'}})_{(o)} \to (\mathscr{C}_{s'})_{(o)}$ and the image of $(\mathfrak{X}_{s'})_{(x)} \to (\widetilde{\mathscr{C}_{s'}})_{(o)}$ would lie in one of the two connected components of $(\widetilde{\mathscr{C}_{s'}})_{(o)}$, contradicting the surjectivity of (III.4.10.1). □

Proposition III.4.11 (T. Saito [Sai]). *Let $Y \to S = \mathrm{Spec}(O_K)$ be a normal relative curve and x a closed point of the special fiber Y_s such that $Y - \{x\}$ is smooth over S. Then, we have*

$$\dim_\Lambda R^1\Psi(\Lambda)_x \geq |P_s(\mathcal{O}_{Y,x})| - 1, \tag{III.4.11.1}$$

where $P_s(\mathcal{O}_{Y,x})$ denotes the set of height 1 prime ideals of $\mathcal{O}_{Y,x}$ above the closed point of S.

Proof. (T. Saito) Since the formation of nearby cycles commutes with dominant base change of traits $S' \to S$ [SGA $4\frac{1}{2}$, Th. finitude, 3.7] and $Y_{s'} \xrightarrow{\sim} Y_s$, we may assume, by the local semi-stable reduction theorem [Sai87, 3.2.2, 4.9], that we have a proper S-morphism $g : W \to Y$ such that W is regular and semi-stable, the exceptional divisor $E = g^{-1}(x)$ is a strict normal crossing divisor and g induces an isomorphism $W - E \xrightarrow{\sim} Y - \{x\}$ (W is the MN-model of Y in the terminology of *loc. cit.*, which exists by [Lip69]). The special fiber W_s identifies with $E \cup \widetilde{Y_s}$ and $P_s(\mathcal{O}_{Y,x}) = \pi^{-1}(x) = E \cap \widetilde{Y_s}$, where $\pi : \widetilde{Y_s} \to Y_s$ is the normalization map.

As g is proper, the base change morphism

$$R\Psi_{Y/S}(Rg_*\Lambda) \to Rg_{s*}(R\Psi_{W/S}(\Lambda)) \tag{III.4.11.2}$$

is an isomorphism. Observe that g induces an isomorphism on the generic fibers. Hence, by the proper base change theorem, the spectral sequence of the hypercohomology of the functor $\Gamma(E,-)$ with respect to the complex $R\Psi_{W/S}(\Lambda)$ yields a spectral sequence

$$E_2^{p,q} = H^p(E, R^q\Psi_{W/S}(\Lambda)) \Rightarrow R^{p+q}\Psi_{Y/S}(\Lambda)_x. \tag{III.4.11.3}$$

For any $q \geq 2$, $R^q\Psi_{W/S}(\Lambda) = 0$ [**SGA 7**, I, Théorème 4.2] and $R^1\Psi_{W/S}(\Lambda)$ is concentrated at the non-smooth locus of W_s [**SGA 7**, XIII, 2.1.5], which is $P_s(\mathcal{O}_{Y,x}) \cup \mathrm{Sing}(E)$. Therefore, (III.4.11.3) yields the exact sequence

$$R^1\Psi_{Y/S}(\Lambda)_x \to \bigoplus_{w\in P_s(\mathcal{O}_{Y,x}) \,\cup\, \mathrm{Sing}(E)} R^1\Psi_{W/S}(\Lambda)_w \to H^2(E,\Lambda) \to 0. \tag{III.4.11.4}$$

On the one hand, the Λ-dimension of the top cohomology group $H^2(E,\Lambda)$ is the number irreducible components of E [**SGA 4**, IX, 4.7]; as E is connected, by induction on this number, one sees that it is $\leq |\mathrm{Sing}(E)|+1$. On the other hand, as $R^1\Psi_{W/S}(\Lambda)$ coincides with the corresponding tame nearby cycles [**Ill94**, 3.5], we know from [**SGA 7**, I, Théorème 3.3] that the Λ-dimension of every $R^1\Psi_{W/S}(\Lambda)_w$ is 1. Then, (III.4.11.1) follows from (III.4.11.4). ☐

Corollary III.4.12 (T. Saito [**Sai**]). *We keep the notation and assumption of* III.4.11.

1. *We have the following inequalities*

$$\dim_\Lambda R^1\Psi(\Lambda)_x \geq \delta(\mathcal{O}_{Y,x}) \geq |P_s(\mathcal{O}_{Y,x})|-1 \geq 0. \tag{III.4.12.1}$$

2. *The following conditions are equivalent :*
   (i) *Y is smooth over S;*
   (ii) *Y_s is smooth over k;*
   (iii) *$\delta(\mathcal{O}_{Y,x}) = 0$;*
   (iv) *$R^1\Psi(\Lambda)_x = 0$.*

Proof. 1. We know from Kato [**Kat87a**, 5.9] that

$$\dim_\Lambda R^1\Psi(\Lambda)_x + |P_s(\mathcal{O}_{Y,x})|-1 = 2\delta(\mathcal{O}_{Y,x}). \tag{III.4.12.2}$$

With (III.4.11.1), this implies (III.4.12.1).

2. Since $Y \to S$ is flat, (i) and (ii) are equivalent. The latter is also equivalent to Y_s being normal, that is $\delta(\mathcal{O}_{Y,x}) = 0$. Finally, it is well-known that (i) implies (iv), and the implication (iv) $\Rightarrow$ (iii) follows from (III.4.12.1). ☐

III.5. Variation of conductors of a morphism to a rigid annulus

III.5.1. Let $0 \leq r < r'$ be rational numbers and put $C = A(r',r)$ and $C^\circ = A^\circ(r',r)$. Let X be a smooth and connected K-affinoid space and $f : X \to C$ a finite flat morphism which is generically étale. Let G be a finite group with a right action on X such that f is invariant under G. By [**BGR84**, 6.3.3/3], $\mathcal{O}(X)^G$ is a K-affinoid closed sub-algebra of $\mathcal{O}(X)$ and $\mathcal{O}(X)$ is finite over $\mathcal{O}(X)^G$. We assume that $\mathcal{O}(X)^G \cong \mathcal{O}(C)$, i.e. $X/G = \mathrm{Sp}(\mathcal{O}(X)^G) \cong C$. In this section, we recall and extend the results of [**Bah20**, §7] in the above setting.

III.5.2. We use the notation of III.4.5. Let K' be a finite extension of K which is admissible for f. The right action of G on X induces a canonical action of G on $\mathfrak{X}_{K'}$ from which we deduce an action of G on S_f (resp S'_f) given as follows : for $g \in G$ and $\tau = (\overline{x}_\tau, \mathfrak{q}_\tau) \in S_f$ (resp. S'_f), we put $g \cdot \tau = (g \circ \overline{x}_\tau, g_{\#}^{-1}(\mathfrak{q}_\tau))$, where $g_{\#}$ is the induced isomorphism

$$\mathcal{O}_{\mathfrak{X}_{K'}, g\circ\overline{x}_\tau} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}_{K'},\overline{x}_\tau}. \tag{III.5.2.1}$$

Lemma III.5.3. (1) *The above action of G on the set S_f (resp. S'_f) is transitive.*

(2) *For every $\tau \in S_f$ (resp. S'_f), the extension of fields of fractions $\mathbb{K}^h_\tau/\mathbb{K}^h$ (resp. $\mathbb{K}'^h_\tau/\mathbb{K}'^h$) induced by V^h_τ/V^h (resp. V'^h_τ/V'^h) is finite and Galois of group G_τ (resp. G'_τ) isomorphic to the stabilizer of τ under the action of G on S_f (resp. S'_f).*

(3) *Let H be a subgroup of G. The quotient $Y = X/H = \mathrm{Sp}(\mathcal{O}(X)^H)$ is a smooth K-rigid space and the moprhism $f_H : X/H \to C$ induced by f is finite, flat and generically étale. The canonical maps $g_S : S_f \to S_{f_H}$ and $g'_S : S'_f \to S'_{f_H}$ induced by the quotient morphism $X \to Y$ are surjective and their fibers are the orbits of H under the action of G on S_f and S'_f respectively.*

Proof. The items (1) and (3) are respectively [Bah20, 7.3] and [Bah20, 7.11] extended from the target space D to C. The proofs go through verbatim. Note that the $\overline{x}$ in *loc. cit.* corresponds to our geometric point o (III.4.2). Item (2) follows from III.4.4 (see also [Bah20, 7.4]). Indeed, as the normalized integral model $\widehat{f}_{K'} : \mathfrak{X}_{K'} \to \mathscr{C}_{K'}$ of f over $\mathcal{O}_{K'}$ (III.4.3) is finite and $\mathscr{C}_{K'}$ is normal and integral, the isomorphism $\mathcal{O}(X)^G \cong \mathcal{O}(C)$ implies that $\mathcal{O}(\mathfrak{X}_{K'})^G \cong \mathcal{O}(\mathscr{C}_{K'})$, and thus III.4.4(ii) applies. □

III.5.4. As in [Bah20, 7.5], for $\tau \in S_f$, the ramification theory of $\mathbb{Z}^2$-valuation rings gives rise to a $\mathbb{Q}$-valued class function $a^\alpha_{G_\tau}$ and a $\mathbb{Z}$-valued class function $\mathrm{sw}^\beta_{G_\tau}$ on G_τ [Bah20, 6.16]. The functions $a^\alpha_{G_\tau}$ are conjugate to each other as τ varies in S_f; so are the functions $\mathrm{sw}^\beta_{G_\tau}$ for τ ranging in S_f. They induce on G the class functions

$$a^\alpha_{f,K'} = \mathrm{Ind}^G_{G_\tau} a^\alpha_{G_\tau} \quad \text{and} \quad \mathrm{sw}^\beta_{f,K'} = \mathrm{Ind}^G_{G_\tau} \mathrm{sw}^\beta_{G_\tau} \tag{III.5.4.1}$$

which are independent of the choice of $\tau \in S_f$. In the same way, for $\tau \in S'_f$, we also have functions $a^\alpha_{G'_\tau}$ and $\mathrm{sw}^\beta_{G'_\tau}$ on G'_τ inducing $a'^\alpha_{f,K'}$ and $\mathrm{sw}'^\beta_{f,K'}$ on G respectively.

Lemma III.5.5. *Let L be a finite extension of K'. Then, we have*

$$a^\alpha_{f,L} = a^\alpha_{f,K'} \quad (\text{resp.} \quad a'^\alpha_{f,L} = a'^\alpha_{f,K'}), \tag{III.5.5.1}$$

$$\mathrm{sw}^\beta_{f,L} = \mathrm{sw}^\beta_{f,K'} \quad (\text{resp.} \quad \mathrm{sw}'^\beta_{f,L} = \mathrm{sw}'^\beta_{f,K'}). \tag{III.5.5.2}$$

In particular, $a^\alpha_{f,K'}$ (resp. $a'^\alpha_{f,K'}$) and $\mathrm{sw}^\beta_{f,K'}$ (resp $\mathrm{sw}'^\beta_{f,K'}$) are independent of the choice of the extension K' of K which is admissible for f; we denote them by a^α_f (resp. a'^α_f) and sw^β_f (resp sw'^β_f).

Proof. This is [Bah20, 7.6] extended from D to C. The proof goes through verbatim. □

III.5.6. We keep the notation of III.5.2. We modify a_f^α and sw_f^β in the following way. We put

$$\widetilde{a}_f^\alpha = \frac{|G|}{|S_f|} a_f^\alpha \quad \text{and} \quad \widetilde{\mathrm{sw}}_f^\beta = \frac{|G|}{|S_f|} \mathrm{sw}_f^\beta. \tag{III.5.6.1}$$

We consider similar modifications $\widetilde{a}_f'^\alpha$ and $\widetilde{\mathrm{sw}}_f'^\beta$ of $a_f'^\alpha$ and $\mathrm{sw}_f'^\beta$ respectively, by replacing S_f with S'_f.

III.5.7. We recall that there is a discriminant function $\partial_f : [r', r] \cap \mathbb{Q}_{\geq 0} \to \mathbb{Q}$ associated to the morphism f [Bah20, 4.13-17] (there, it was denoted by ∂_f^α). Lütkebohmert [Lüt93, 2.3, 2.6] proved the following variational result for ∂_f.

Proposition III.5.8. *Let $r = r_{n+1} < r_n < \cdots < r_1 < r_0 = r'$ be a sequence containing all critical radii of f (III.4.8) and $\Delta_i = \coprod_j \Delta_{ij}$ the associated decomposition of $\Delta_i = f^{-1}(A^\circ(r_{i-1}, r_i))$ into a disjoint union of open annuli such that the restriction $f|\Delta_i$ is étale. Then, the function ∂_f is affine on $]r_i, r_{i-1}[\cap\mathbb{Q}$ and its right slope at $t \in [r_i, r_{i-1}[\cap\mathbb{Q}$ is*

$$\frac{d}{dt}\partial_f(t^+) = \sigma_i - d + \delta_f(i), \tag{III.5.8.1}$$

where σ_i is the total order of the derivative of $f|\Delta_i$ [Bah20, 4.5] and $\delta_f(i)$ is the number of connected components of Δ_i (i.e. the number of Δ_{ij}'s).

Proof. Although this result is stated in [Lüt93, 2.6] for the closed unit disc D instead of C, i.e. for $r = \infty$ and $r' = 0$, the proof, detailed in [Bah20, 4.23], works for the above more general statement. □

Proposition III.5.9. *Assume that $f : X \to C$ is étale. Then, the discriminant function ∂_f is convex.*

Proof. With the notation of proof of III.4.7. For every $j = 1, \ldots, N$, we have from (III.4.7.22)

$$1 - \dim_\Lambda H^1_{\text{ét}}(Y'_{(\overline{x}_j)} \times \overline{\eta}', \Lambda) = -2\delta(B_j) + |P_s(B_j)|. \tag{III.5.9.1}$$

As $\overline{x}_j$ is singular (III.4.10), the stalk $R^1\Psi_{Y'/S'}(\Lambda)_{\overline{x}_j} = H^1_{\text{ét}}(Y'_{(\overline{x}_j)} \times \overline{\eta}', \Lambda)$ is nonzero (III.4.12). It follows that $-2\delta(B_j) + |P_s(B_j)| \leq 0$. By III.4.9, (III.4.7.2) and (III.5.8.1), we deduce that

$$\frac{d}{dt}\partial_f(r+) - \frac{d}{dt}\partial_f(r'-) \leq 0. \tag{III.5.9.2}$$

By III.4.8, we can apply this to the restriction of f above the annulus $A(t, t')$ for all non-critical radii $t < t'$, hence the convexity of ∂_f. □

Proposition III.5.10. *We denote by $\langle \cdot, \cdot \rangle$ the usual pairing of class functions on G* [Ser98, 2.2, Remarques]. *Let H be a subgroup of G and ∂_{f_H} the discriminant function* (II.4) *associated to the quotient morphism $f_H : X/H \to C$ induced by f.*

(i) *We have the following identities*

$$\partial_{f_H}(r) = \langle \widetilde{a}_f^\alpha, \mathbb{Q}[G/H] \rangle, \tag{III.5.10.1}$$

$$\partial_{f_H}(r') = \langle \widetilde{a}_f'^\alpha, \mathbb{Q}[G/H] \rangle, \tag{III.5.10.2}$$

where the representation $\mathbb{Q}[G/H] = \mathrm{Ind}_H^G 1_H$ of G stands in abusively for its character.

(ii) *We assume that X/H has* trivial canonical sheaf *and that f_H satisfies the decomposition hypothesis* (III.4.7.1). *Then, we also have the following identity for the right and left derivatives of $\partial^{\alpha}_{f_H}$ at r and r' respectively*

$$\frac{d}{dt}\partial_{f_H}(r+) - \frac{d}{dt}\partial_{f_H}(r'-) = \langle \widetilde{\mathrm{sw}}^{\beta}_f, \mathbb{Q}[G/H]\rangle + \langle \widetilde{\mathrm{sw}}'^{\beta}_f, \mathbb{Q}[G/H]\rangle, \tag{III.5.10.3}$$

Proof. We first note that, as G acts transitively on S_f, we have $|G_\tau|= |G|/|S_f|$. By Frobenius reciprocity, we have the following identities

$$\langle a^{\alpha}_f, \mathbb{Q}[G/H]\rangle = \langle a^{\alpha}_{G_\tau}, \mathbb{Q}[G/H]|G_\tau\rangle, \tag{III.5.10.4}$$

$$\langle \mathrm{sw}^{\beta}_f, \mathbb{Q}[G/H]\rangle = \langle \mathrm{sw}^{\beta}_{G_\tau}, \mathbb{Q}[G/H]|G_\tau\rangle. \tag{III.5.10.5}$$

Let R be a set of representatives in G of the double cosets $G_\tau\backslash G/H$. From [Ser98, §7.3, Prop. 22], for $\tau \in S_f$, we have the identity

$$\mathbb{Q}[G/H]|G_\tau = \bigoplus_{\sigma\in R} \mathrm{Ind}^{G_\tau}_{H_\sigma} 1_{H_{\sigma,\tau}}, \tag{III.5.10.6}$$

where $H_{\sigma,\tau} = \sigma H\sigma^{-1} \cap G_\tau$. If $\sigma \in R$ and $g\sigma h$ is another representative of the double coset $G_\tau\sigma H$, then $H_{g\sigma h,\tau} = H_{\sigma,\tau}$. Hence, $H_{\sigma,\tau}$ depends only on the double coset $G_\tau\sigma H$ and the sum (III.5.10.6) is taken over $G_\tau\backslash G/H$. Therefore, we have

$$\langle a^{\alpha}_f, \mathbb{Q}[G/H]\rangle = \sum_{\sigma\in G_\tau\backslash G/H} \langle a^{\alpha}_{G_\tau}, \mathbb{Q}[G_\tau/H_{\sigma,\tau}]\rangle, \tag{III.5.10.7}$$

$$\langle \mathrm{sw}^{\beta}_f, \mathbb{Q}[G/H]\rangle = \sum_{\sigma\in G_\tau\backslash G/H} \langle \mathrm{sw}^{\beta}_{G_\tau}, \mathbb{Q}[G_\tau/H_{\sigma,\tau}]\rangle. \tag{III.5.10.8}$$

From [Bah20, (6.18.4) and (6.18.5)], we get

$$|G_\tau|\langle a^{\alpha}_{G_\tau}, \mathbb{Q}[G_\tau/H_{\sigma,\tau}]\rangle = v^{\alpha}(\mathfrak{d}_{V^h(\sigma,\tau)/V^h}), \tag{III.5.10.9}$$

$$|G_\tau|\langle \mathrm{sw}^{\beta}_{G_\tau}, \mathbb{Q}[G_\tau/H_{\sigma,\tau}]\rangle = v^{\beta}(\mathfrak{d}_{V^h(\sigma,\tau)/V^h}) - \frac{|G_\tau|}{|H_{\sigma,\tau}|} + 1, \tag{III.5.10.10}$$

where $V^h(\sigma,\tau) = (V^h_\tau)^{H_{\sigma,\tau}}$. The subgroup $\sigma^{-1}H_{\sigma,\tau}\sigma$ of $\sigma^{-1}G_\tau\sigma = G_{\sigma^{-1}\cdot\tau}$ is $H_{\mathrm{id},\sigma^{-1}\cdot\tau}$. Then, σ^{-1} yields an isomorphism of $\mathbb{Z}^2$-valuation rings $V^h(\sigma^{-1}\cdot\tau) \xrightarrow{\sigma} V^h_\tau$, via (III.5.2.1), which induces an isomorphism

$$V^h(\mathrm{id}, \sigma^{-1}\cdot\tau) = (V^h(\sigma^{-1}\cdot\tau))^{\sigma^{-1}H_{\sigma,\tau}\sigma} \xrightarrow{\sim} V^h(\sigma,\tau). \tag{III.5.10.11}$$

By III.5.3 (3), the map $C : G_\tau\backslash G/H \to S_{f_H}$, $G_\tau\sigma H \mapsto g_S(\sigma^{-1}\cdot\tau)$ is well-defined since, for $g \in G_\tau$ and $h \in H$, $(g\sigma h)^{-1}\cdot\tau = h^{-1}\cdot\tau$. Moreover, as G acts transitively on S_f (III.5.3 (1)) and g_S is surjective, C is also surjective. If $C(G_\tau\sigma H) = C(G_\tau\sigma' H)$, then there exists $h \in H$ such that $\sigma^{-1}\cdot\tau = h\sigma'^{-1}\cdot\tau$; so $\sigma h\sigma'^{-1} \in G_\tau$ and thus $\sigma' \in G_\tau\sigma H$. Hence, C is also injective, hence a bijection. We also have $V^h(\mathrm{id},\sigma^{-1}\cdot\tau) = V^h(g_S(\sigma^{-1}\cdot\tau))$. It follows that, if $\sigma,\sigma' \in R$ represent double cosets such that $g_S(\sigma^{-1}\cdot\tau) = g_S(\sigma'^{-1}\cdot\tau)$, then $V^h(\mathrm{id},\sigma^{-1}\cdot\tau) = V^h(\mathrm{id},\sigma'^{-1}\cdot\tau)$.

Combining all this with (III.5.10.7), (III.5.10.8), (III.5.10.9) and (III.5.10.10) yields

$$\langle \widetilde{a}^{\alpha}_f, \mathbb{Q}[G/H]\rangle = \sum_{\tau'\in S_{f_H}} v^{\alpha}(\mathfrak{d}_{V^h_{\tau'}/V^h}) = v^{\alpha}\left(\prod_{\tau'\in S_{f_H}} \mathfrak{d}_{V^h_{\tau'}/V^h}\right) = \partial_{f_H}(r), \tag{III.5.10.12}$$

$$\langle \widetilde{\mathrm{sw}}^{\beta}_f, \mathbb{Q}[G/H]\rangle = v^{\beta}\left(\prod_{\tau'\in S_{f_H}} \mathfrak{d}_{V^h_{\tau'}/V^h}\right) - \deg(f_H) + |S_{f_H}|. \tag{III.5.10.13}$$

Equation (III.5.10.12) yields the identities (III.5.10.1). The same arguments work for $\widetilde{a}'^{\alpha}_f$ and $\widetilde{\mathrm{sw}}'^{\beta}_f$, producing identities similar to (III.5.10.12) and (III.5.10.13), yielding also (III.5.10.2).
From (III.5.10.13) and the corresponding formula for $\widetilde{\mathrm{sw}}'^{\beta}_f$, we see by (III.4.5.1) and (III.4.5.2) that

$$\langle \widetilde{\mathrm{sw}}^{\beta}_f, \mathbb{Q}[G/H]\rangle = d_{f_H,s} - \deg(f_H) + |S_{f_H}|. \tag{III.5.10.14}$$

$$\langle \widetilde{\mathrm{sw}}'^{\beta}_f, \mathbb{Q}[G/H]\rangle = d'_{f_H,s} - \deg(f_H) + |S'_{f_H}|. \tag{III.5.10.15}$$

Whence we deduce that

$$\langle \widetilde{\mathrm{sw}}^{\beta}_f + \widetilde{\mathrm{sw}}'^{\beta}_f, \mathbb{Q}[G/H]\rangle = d_{f_H,s} + d'_{f_H,s} - 2\deg(f_H) + |S_{f_H}| + |S'_{f_H}|. \tag{III.5.10.16}$$

As $|S_{f_H}|+|S'_{f_H}|$ is also the sum over $j = 1, \dots, N$ of the integers $|P_s(B_j)|$ (notation of III.4.6), by III.4.6, applied to f_H, we now see that

$$\langle \widetilde{\mathrm{sw}}^{\beta}_f + \widetilde{\mathrm{sw}}'^{\beta}_f, \mathbb{Q}[G/H]\rangle = \sum_{j=1}^{N}(d_{\eta}(B_j/A) - 2\delta(B_j) + |P_s(B_j)|). \tag{III.5.10.17}$$

Now, if X/H has a trivial canonical sheaf and f_H satisfies the decomposition hypothesis (III.4.7.1), then we can apply to it the formula (III.4.7.2). The latter, combined with III.5.8.1 applied to f_H, and (III.5.10.17), yields

$$\langle \widetilde{\mathrm{sw}}^{\beta}_f + \widetilde{\mathrm{sw}}'^{\beta}_f, \mathbb{Q}[G/H]\rangle = \sigma + \delta_{f_H} - (\sigma' + \delta'_{f_H}) = \frac{d}{dt}\partial_{f_H}(r+) - \frac{d}{dt}\partial_{f_H}(r'-). \tag{III.5.10.18}$$

□

III.5.11. Let $r \le t \le r'$ be a rational number, put $C^{[t]} = A(t,t)$, $X^{[t]} = f^{-1}(C^{[t]})$ and denote by $f^{[t]} : X^{[t]} \to C^{[t]}$ the morphism induced by f. Let K' be a finite extension of K such that the normalized integral models $\mathscr{C}^{[t]}_{K'}$, $\mathfrak{X}^{[t]}_{K'}$ and $\widehat{f}^{[t]}_{K'} : \mathfrak{X}^{[t]}_{K'} \to \mathscr{C}^{[t]}_{K'}$ of $C^{[t]}$, $X^{[t]}$ and $f^{[t]}$ respectively are defined over $\mathcal{O}_{K'}$ (III.4.3). Let $\mathfrak{p}^{(t)}$ be the generic point of the special fiber $\mathscr{C}^{[t]}_{s'}$ of $\mathscr{C}^{[t]}_{K'}$ and $\overline{\mathfrak{p}}^{(t)} \to \mathscr{C}^{[t]}_{s'}$ a geometric generic point. The latter defines a geometric point of $\mathscr{C}^{[t]}_{K'}$ [**Bah20**, 2.5]. Let $\mathfrak{X}^{[t]}_{K'}(\overline{\mathfrak{p}}^{(t)})$ be the set of geometric generic points of $\mathfrak{X}^{[t]}_{K'}$ above $\overline{\mathfrak{p}}^{(t)}$. The natural right action of G on X induces an action of G on $\mathfrak{X}^{[t]}_{K'}$, hence an action of G on $\mathfrak{X}^{[t]}_{K'}(\overline{\mathfrak{p}}^{(t)})$. The latter action is transitive. For $\overline{\mathfrak{q}}^{(t)} \in \mathfrak{X}^{[t]}_{K'}(\overline{\mathfrak{p}}^{(t)})$, the natural map

$$A_{\overline{\mathfrak{p}}^{(t)}} = \mathcal{O}_{\mathscr{C}^{[t]}_{K'},\overline{\mathfrak{p}}^{(t)}} \to A_{\overline{\mathfrak{q}}^{(t)}} = \mathcal{O}_{\mathfrak{X}^{[t]}_{K'},\overline{\mathfrak{q}}^{(t)}} \tag{III.5.11.1}$$

is a finite homomorphism of henselian discrete valuation rings [**Bah20**, 2.8, 2.12]. The induced extension of fields of fractions $\mathbb{K}_{\overline{\mathfrak{q}}^{(t)}}/\mathbb{K}_{\overline{\mathfrak{p}}^{(t)}}$ is Galois of group $G_{\overline{\mathfrak{q}}^{(t)}}$ isomorphic to the stabilizer of $\overline{\mathfrak{q}}^{(t)}$ under the action of G on $\mathfrak{X}^{[t]}_{K'}(\overline{\mathfrak{p}}^{(t)})$ (*cf.* proof of III.4.4 (ii)). As the normalized integral models have (geometrically) reduced special fibers, the extension $\mathbb{K}_{\overline{\mathfrak{q}}^{(t)}}/\mathbb{K}_{\overline{\mathfrak{p}}^{(t)}}$ has ramification index 1. Moreover, its residue extension $\kappa(\overline{\mathfrak{q}}^{(t)})/\kappa(\overline{\mathfrak{p}}^{(t)})$ is a monogenic extension of discrete valuation fields, with trivial residue extension as k is algebraically closed. To properly justify this claim, one picks arbitrary specializations of $\overline{\mathfrak{p}}^{(t)}$ and $\overline{\mathfrak{q}}^{(t)}$ to (geometric) closed points of the special fibers of $\mathscr{C}^{[t]}_{K'}$ and $\mathfrak{X}^{[t]}_{K'}$ respectively, above one another, and applies [**Bah20**, 3.18], followed by [**Ser68**, III, §6, Prop. 12].

Let $\mathbb{K}'_{\overline{\mathfrak{p}}^{(t)}}$ be the maximal unramified sub-extension of $\mathbb{K}_{\overline{\mathfrak{q}}^{(t)}}/\mathbb{K}_{\overline{\mathfrak{p}}^{(t)}}$. Then, $\mathbb{K}_{\overline{\mathfrak{q}}^{(t)}}/\mathbb{K}'_{\overline{\mathfrak{p}}^{(t)}}$ is of type (II), i.e. it has ramification index 1 and a purely inseparable and monogenic residue extension $\kappa(\overline{\mathfrak{q}}^{(t)})/\kappa'(\overline{\mathfrak{p}})$. Let $A'_{\overline{\mathfrak{p}}^{(t)}}$ be the valuation ring of $\mathbb{K}'_{\overline{\mathfrak{p}}^{(t)}}$ and $G'_{\overline{\mathfrak{q}}^{(t)}}$ the Galois group of $\mathbb{K}_{\overline{\mathfrak{q}}^{(t)}}/\mathbb{K}'_{\overline{\mathfrak{p}}^{(t)}}$. Then, the $A'_{\overline{\mathfrak{p}}^{(t)}}$-algebra $A_{\overline{\mathfrak{q}}^{(t)}}$ is monogenic (so is the $A_{\overline{\mathfrak{p}}^{(t)}}$-algebra $A'_{\overline{\mathfrak{p}}^{(t)}}$) [Ser68, III, §6, Prop. 12]. (Note that *loc. cit.*, with the same proof, applies here because $\kappa(\overline{\mathfrak{q}}^{(t)})/\kappa'(\overline{\mathfrak{p}})$, despite being purely inseparable, is monogenic.) Then, just as in the classical setting, we have an *Artin class function* $a_{G'_{\overline{\mathfrak{q}}^{(t)}}} : G'_{\overline{\mathfrak{q}}^{(t)}} \to \mathbb{Z}$ on $G'_{\overline{\mathfrak{q}}^{(t)}}$ defined as follows. Let b be a generator of $A_{\overline{\mathfrak{q}}^{(t)}}$ over $A'_{\overline{\mathfrak{p}}^{(t)}}$, denote by $v_{\overline{\mathfrak{q}}^{(t)}} : \mathbb{K}^{\times}_{\overline{\mathfrak{q}}^{(t)}} \to \mathbb{Z}$ its normalized valuation map associated to $A_{\overline{\mathfrak{q}}^{(t)}}$ and put

(III.5.11.2) $$a_{G'_{\overline{\mathfrak{q}}^{(t)}}}(\sigma) = -v_{\overline{\mathfrak{q}}^{(t)}}(\sigma(b)-b) \quad \text{if} \quad \sigma \neq 1,$$

(III.5.11.3) $$a_{G'_{\overline{\mathfrak{q}}^{(t)}}}(1) = -\sum_{\sigma\neq 1} a_{G'_{\overline{\mathfrak{q}}^{(t)}}}(\sigma).$$

This definition is independent of the chosen generator a because [Ser68, IV, §1, Lemme 1]

(III.5.11.4) $$v_{\overline{\mathfrak{q}}^{(t)}}(\sigma(b)-b) = \min\{v_{\overline{\mathfrak{q}}^{(t)}}(\sigma(x)-x) \mid x \in A_{\overline{\mathfrak{q}}^{(t)}}\}.$$

Since, for $\overline{\mathfrak{q}}^{(t)}$ varying in $\mathfrak{X}^{[t]}_{K'}(\overline{\mathfrak{p}}^{(t)})$, the subgroups $G'_{\overline{\mathfrak{q}}^{(t)}} \subset G_{\overline{\mathfrak{q}}^{(t)}}$ of G, as well as the functions $a_{G'_{\overline{\mathfrak{q}}^{(t)}}}$, are all conjugate, we have a well-defined class function

(III.5.11.5) $$\widetilde{a}_{f^{[t]}} = \mathrm{Ind}^{G}_{G'_{\overline{\mathfrak{q}}^{(t)}}}(|G'_{\overline{\mathfrak{q}}^{(t)}}|\cdot a_{G'_{\overline{\mathfrak{q}}^{(t)}}}).$$

III.5.12. We let ℓ be a prime number different form the residue characteristic p of K, fix $\overline{\mathbb{Q}}_\ell$ a separable closure of $\mathbb{Q}_\ell$ and denote by $R_{\overline{\mathbb{Q}}_\ell}(G)$ the Grothendieck group of finitely generated $\overline{\mathbb{Q}}_\ell[G]$-modules. We denote by $\mathcal{I}([r,r']\cap\mathbb{Q})$ the set of intervals of $[r',r]$ bounded by rational numbers and *which are not singletons*. For $[t,t'] \in \mathcal{I}([r,r']\cap\mathbb{Q})$ (with $t \neq t'$), we denote the restriction of f over $A(t,t')$ by

(III.5.12.1) $$f^{[t,t']} : f^{-1}(A(t',t)) \to A(t',t).$$

Then, by III.5.6, we have well-defined associated class functions on G

(III.5.12.2) $$\widetilde{a}^{\alpha}_{f^{[t,t']}},\ \widetilde{a}'^{\alpha}_{f^{[t,t']}},\ \widetilde{\mathrm{sw}}^{\beta}_{f^{[t,t']}} \text{ and } \widetilde{\mathrm{sw}}'^{\beta}_{f^{[t,t']}}.$$

By III.5.10 (i), the rational number $\langle \widetilde{a}^{\alpha}_{f^{[t,t']}}, \mathbb{Q}[G/H]\rangle$ (resp. $\langle \widetilde{a}'^{\alpha}_{f^{[t,t']}}, \mathbb{Q}[G/H]\rangle$) depends only on t (resp. t'), not on the interval $[t,t']$. In fact, by taking $\tau = (\overline{x}_\tau, \mathfrak{q}_\tau) \in S_{f^{[t,t']}}$ (resp. $\tau = (\overline{x}_{\tau'}, \mathfrak{q}_{\tau'}) \in S'_{f^{[t,t']}}$) and letting $\overline{\mathfrak{q}}^{(t)}$ (resp. $\overline{\mathfrak{q}}^{(t')}$) be the geometric generic point of $\mathfrak{X}^{[t]}_{s'}$ (resp. $\mathfrak{X}^{[t]}_{s'}$) above $\overline{\mathfrak{p}}^{(t)}$ (resp. $\overline{\mathfrak{p}}^{(t')}$) in III.5.11 corresponding to $\mathfrak{q}_\tau$ (resp. $\mathfrak{q}_{\tau'}$) [Bah20, 3.21], we see that $G_\tau = G_{\overline{\mathfrak{q}}^{(t)}}$ and $G_{\tau'} = G_{\overline{\mathfrak{q}}^{(t')}}$ (III.4.7.5). Then, by [Bah20, (6.19.1)], we have (III.5.11.5)

(III.5.12.3) $$a^{\alpha}_{G_\tau}|G'_{\overline{\mathfrak{q}}^{(t)}} = a_{G'_{\overline{\mathfrak{q}}^{(t)}}} \quad \text{and} \quad a^{\alpha}_{G_{\tau'}}|G'_{\overline{\mathfrak{q}}^{(t)}} = a_{G'_{\overline{\mathfrak{q}}^{(t')}}};$$

(III.5.12.4) $$\widetilde{a}_{f^{[t]}} = \widetilde{a}^{\alpha}_{f^{[t,t']}} \quad \text{and} \quad \widetilde{a}_{f^{[t']}} = \widetilde{a}'^{\alpha}_{f^{[t,t']}},$$

where the last equation follows from (III.5.12.3) and [Ser98, §7.2, Rem. 3]. To $\chi \in R_{\overline{\mathbb{Q}}_\ell}(G)$, we associate the functions

(III.5.12.5) $$\widetilde{a}_f(\chi,\cdot) : [r,r']\cap\mathbb{Q} \to \mathbb{Q}, \quad t \mapsto \langle \widetilde{a}_{f^{[t]}}, \chi\rangle,$$

(III.5.12.6) $$\widetilde{\mathrm{sw}}_f^\beta(\chi,\cdot):\ \mathcal{I}([r,r']\cap\mathbb{Q})\to\mathbb{Q},\quad [t,t']\mapsto\langle\widetilde{\mathrm{sw}}^\beta_{f^{[t,t']}}+\widetilde{\mathrm{sw}}'^\beta_{f^{[t,t']}},\chi\rangle.$$

Proposition III.5.13. *Let H be a subgroup of G.*

(i) *The function $\widetilde{a}_f(\mathbb{Q}[G/H],\cdot)$ above is continuous, convex and piecewise linear with finitely many slopes which are all integers.*

(ii) *We assume that X/H has* trivial canonical sheaf. *Then, for rational numbers $r\le t<t'\le r'$, the difference of the right and left derivatives of $\widetilde{a}_f(\mathbb{Q}[G/H],\cdot)$ at t and t' respectively is*

(III.5.13.1) $$\frac{d}{dt}\widetilde{a}_f(\mathbb{Q}[G/H],t+)-\frac{d}{dt}\widetilde{a}_f(\mathbb{Q}[G/H],t'-)=\widetilde{\mathrm{sw}}^\beta_f(\mathbb{Q}[G/H],[t,t']).$$

Proof. We know form [Bah20, (4.17.4), (4.22.2)] that

(III.5.13.2) $$\partial_{f_H^{[t,t']}}(t)=\partial_{f_H}(t)\quad\text{and}\quad\partial_{f_H^{[t,t']}}(t')=\partial_{f_H}(t')$$

Hence, (i) follows from (III.5.13.2), (III.5.12.4), III.5.10(i), III.5.8 and III.5.9. By III.4.8, the decomposition hypothesis (III.4.7.1) is satisfied by f_H for all but a finite number of radii. As we are computing left and right derivatives (of a piecewise linear function), (III.5.10.3) applies and, with (III.5.10)(i), (III.5.12.4) and (III.5.13.2), yield (ii). □

Theorem III.5.14. *We assume that, for every subgroup $H\subset G$, the quotient X/H has a trivial canonical sheaf. Let $\chi\in R_{\overline{\mathbb{Q}}_\ell}(G)$. Then, the function $\widetilde{a}_f(\chi,\cdot)$ (III.5.12.5) is continuous and piecewise linear with finitely many slopes which are all integers. For rational numbers $r\le t<t'\le r'$, the difference of the right and left derivatives of $\widetilde{a}_f(\chi,\cdot)$ at t and t' respectively is*

(III.5.14.1) $$\frac{d}{dt}\widetilde{a}_f(\chi,t+)-\frac{d}{dt}\widetilde{a}_f(\chi,t'-)=\widetilde{\mathrm{sw}}^\beta_f(\chi,[t,t']).$$

Proof. Mutatis mutandis, the proof is the same as the one for [Bah20, Theorem 7.16] with the key statement [Bah20, 7.13] replaced by III.5.13 above. □

III.5.15. Let Λ be a finite extension of $\mathbb{Q}_\ell$ inside $\overline{\mathbb{Q}}_\ell$ and $\overline{\Lambda}$ its residue field. By [Ser98, 16.1, Théorème 33], we have a surjective homomorphism $d_G:R_\Lambda(G)\to R_{\overline{\Lambda}}(G)$, the Cartan homomorphism. Let $\overline{\chi}\in R_{\overline{\Lambda}}(G)$ and $\chi\in R_\Lambda(G)$ a pre-image of $\overline{\chi}$ by d_G. Then, for rational numbers $r\le t<t'\le r'$, we put

(III.5.15.1) $$\widetilde{a}_f(\overline{\chi},t)=\widetilde{a}_f(\chi,t)\quad\text{and}\quad\widetilde{\mathrm{sw}}^\beta_f(\overline{\chi},[t,t'])=\widetilde{\mathrm{sw}}^\beta_f(\chi,[t,t']).$$

These quantities are independent of the chosen pre-image χ : the proof is the same as the one given for the analogous statement [Bah20, 7.18].

Corollary III.5.16. *We resume the assumptions of III.5.14. Let $\overline{\chi}\in R_{\overline{\Lambda}}(G)$. Then, the function*

(III.5.16.1) $$\widetilde{a}_f(\overline{\chi},\cdot):\ [r,r']\cap\mathbb{Q}_{\ge 0}\to\mathbb{Q},\quad t\mapsto\widetilde{a}_f(\overline{\chi},t)$$

is continuous and piecewise linear with finitely many slopes which are all integers. For rational numbers $r\le t<t'\le r'$, the difference of the right and left derivatives of $\widetilde{a}_f(\overline{\chi},\cdot)$ at t and t' respectively is

(III.5.16.2) $$\frac{d}{dt}\widetilde{a}_f(\overline{\chi},t+)-\frac{d}{dt}\widetilde{a}_f(\overline{\chi},t'-)=\widetilde{\mathrm{sw}}^\beta_f(\overline{\chi},[t,t']).$$

Remark III.5.17. For χ (resp. $\overline{\chi}$) the image in $R_{\overline{\mathbb{Q}}_\ell}(G)$ (resp. $R_{\overline{\Lambda}}(G)$) of a representation of G, we expect the function $\widetilde{a}_f(\chi,\cdot)$ (resp. $\widetilde{a}_f(\overline{\chi},\cdot)$) to be convex (*cf.* III.5.9).

III.6. Swan conductor of a lisse torsion sheaf on a rigid annulus

III.6.1. Let $\overline{\Lambda}$ be a finite field of characteristic $\ell \neq p$ and $\psi : \mathbb{F}_p^\times \to \overline{\Lambda}^\times$ a nontrivial character. Let $0 \leq r < r'$ be rational numbers and $C = A(r', r)$ the closed sub-annulus of the closed unit disc D defined by $r \leq v_K(\xi) \leq r'$, where ξ is the coordinate of D. Let $\mathcal{F}$ be a lisse sheaf of $\overline{\Lambda}$-modules on C. By [deJ95, 2.10], $\mathcal{F}$ corresponds to a connected Galois étale cover $f : X \to C$ and a continuous finite dimensional $\overline{\Lambda}$-representation $\rho_\mathcal{F}$ of $G = \mathrm{Aut}(X/C)$. We denote by $\chi_\mathcal{F}$ the image of $\rho_\mathcal{F}$ in $R_{\overline{\Lambda}}(G)$.

III.6.2. We use the notation of III.5.11. Let $r \leq t \leq r'$ be a rational number and $\overline{\mathfrak{q}}^{(t)}$ an element of $\mathfrak{X}_{K'}^{[t]}(\overline{\mathfrak{p}}^{(t)})$. The stabilizer of $\overline{\mathfrak{q}}^{(t)}$ under the action of G on $\mathfrak{X}_{K'}^{[t]}(\overline{\mathfrak{p}}^{(t)})$ is isomorphic to the Galois group $G_{\overline{\mathfrak{q}}^{(t)}}$ of a finite extension of a henselian discrete valuation field which is of type (II) over an unramified sub-extension, with residue extension $\kappa(\overline{\mathfrak{p}}^{(t)}) \to \kappa(\overline{\mathfrak{q}}^{(t)})$. The ramification theory of Abbes and Saito, applied to the $G_{\overline{\mathfrak{q}}^{(t)}}$-representation $M_{\overline{\mathfrak{q}}^{(t)}} = \rho_\mathcal{F}|G_{\overline{\mathfrak{q}}^{(t)}}$ yields a rational number $\mathrm{sw}^{\mathrm{AS}}_{G_{\overline{\mathfrak{q}}^{(t)}}}(M_{\overline{\mathfrak{q}}^{(t)}})$, the *Swan conductor of* $M_{\overline{\mathfrak{q}}^{(t)}}$ [Bah20, (8.21.2)], and (a power of) a logarithmic differential form $\mathrm{CC}_\psi(M_{\overline{\mathfrak{q}}^{(t)}})$, the *characteristic cycle of* $M_{\overline{\mathfrak{q}}^{(t)}}$ [Hu15, 4.12], which in our setting of type (II) extension, lies in fact in $(\Omega^1_{\kappa(\overline{\mathfrak{p}}^{(t)})})^{\otimes m_t}$, where $m_t = \dim_{\overline{\Lambda}}(M_{\overline{\mathfrak{q}}^{(t)}}/(M_{\overline{\mathfrak{q}}^{(t)}})^{(0)})$, with $(M_{\overline{\mathfrak{q}}^{(t)}})^{(0)}$ denoting the part of $M_{\overline{\mathfrak{q}}^{(t)}}$ fixed by the tame inertia subgroup of $G_{\overline{\mathfrak{q}}^{(t)}}$ [Hu15, 10.5].

The residue field $\kappa(\overline{\mathfrak{p}}^{(t)})$ coincides with $\mathcal{O}_{\mathscr{C}^{[t]}_{s'},\overline{\mathfrak{p}}^{(t)}}$. We denote by $\mathrm{ord}_{0^{(t)}} : \kappa(\overline{\mathfrak{p}}^{(t)})^\times \to \mathbb{Z}$ (resp. $\mathrm{ord}_{\infty^{(t)}} : \kappa(\overline{\mathfrak{p}}^{(t)})^\times \to \mathbb{Z}$) the normalized valuation map defined by $\mathrm{ord}_{0^{(t)}}(\xi) = 1$ (resp. $\mathrm{ord}_{\infty^{(t)}}(\xi) = -1$). We again denote by $\mathrm{ord}_{0^{(t)}} : (\Omega^1_{\kappa(\overline{\mathfrak{p}}^{(t)})})^{\otimes m_t} - \{0\} \to \mathbb{Z}$ the multiplicative extension of $\mathrm{ord}_{0^{(t)}}$ defined by $\mathrm{ord}_{0^{(t)}}(b\mathrm{d}a) = \mathrm{ord}_{0^{(t)}}(b)$, for any $a, b \in \kappa(\overline{\mathfrak{p}}^{(t)})^\times$ such that $\mathrm{ord}_{0^{(t)}}(a) = 1$; we also denote by $\mathrm{ord}_{\infty^{(t)}}$ the similar extension of $\mathrm{ord}_{\infty^{(t)}}$ to $(\Omega^1_{\kappa(\overline{\mathfrak{p}}^{(t)})})^{\otimes m_t} - \{0\}$. The rational numbers

$$\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, t) = \mathrm{sw}^{\mathrm{AS}}_{G_{\overline{\mathfrak{q}}^{(t)}}}(M_{\overline{\mathfrak{q}}^{(t)}}), \tag{III.6.2.1}$$

$$\varphi_s(\mathcal{F}, t) = -\,\mathrm{ord}_{0^{(t)}}(\mathrm{CC}_\psi(M_{\overline{\mathfrak{q}}^{(t)}})) - m_t. \tag{III.6.2.2}$$

are well-defined, independently of the chosen $\overline{\mathfrak{q}}^{(t)} \in \mathfrak{X}_{K'}^{[t]}(\overline{\mathfrak{p}}^{(t)})$, as can be seen a posteriori from III.6.3.

Proposition III.6.3. *We use the notation of* III.5.15. *The following statements hold.*

(1) *For a rational number* $r \leq t < r'$, *we have*

$$\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, t) = \widetilde{a}_f(\chi_\mathcal{F}, t). \tag{III.6.3.1}$$

(2) *For rational numbers* $r \leq t < t' \leq r'$, *we have*

$$\varphi_s(\mathcal{F}, t) - \varphi_s(\mathcal{F}, t') = \widetilde{\mathrm{sw}}^\beta_f(\chi_\mathcal{F}, [t, t']). \tag{III.6.3.2}$$

Proof. For rational numbers $r \leq t < t' \leq r'$, let $(\overline{x}_\tau, \mathfrak{q}_\tau)$ (resp. $(\overline{x}_{\tau'}, \mathfrak{q}_{\tau'})$) be an element of $S_{f^{[t,t']}}$ (resp. $S_{f^{[t,t']}}$) (III.5.12) and denote by $\overline{\mathfrak{q}}^{(t)}$ (resp. $\overline{\mathfrak{q}}^{(t')}$) the geometric generic point of the normalized integral model of $f^{-1}(A(t', t))$ corresponding to $\mathfrak{q}_\tau$ (resp. $\mathfrak{q}_{\tau'}$) (III.5.11). The geometric point $\overline{x}_\tau$ is above the origin point $o^{(t)}$ defined by the coordinate ξ with respect to the outer radius t while $\overline{x}_{\tau'}$ is above the point at infinity $\infty^{(t')}$ defined by ξ with respect to the inner radius t'. As $\mathrm{d}\frac{1}{\xi} = -\xi^{-2}\mathrm{d}\xi$, a straightforward computation shows that, for any $\omega \in (\Omega^1_{\kappa(\overline{\mathfrak{p}}^{(t')})})^{\otimes m_{t'}} - \{0\}$, we have

$$\operatorname{ord}_{\infty^{(t')}}(\omega) = -\operatorname{ord}_{0^{(t')}}(\omega) - 2m_{t'}. \tag{III.6.3.3}$$

We also have $G_{\overline{\mathfrak{q}}^{(t)}} = G_\tau$ and $G_{\overline{\mathfrak{q}}^{(t')}} = G_{\tau'}$. Then, it follows from [Bah20, 8.24] and (III.5.12.4) that

$$\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, t) = \langle |G_\tau| a^\alpha_{G_\tau}, \chi_{\mathcal{F}}|G_\tau\rangle = \langle \widetilde{a}^\alpha_{f^{[t,t']}}, \chi_{\mathcal{F}}\rangle = \widetilde{a}_f(\chi_{\mathcal{F}}, t), \tag{III.6.3.4}$$

$$\varphi_s(\mathcal{F}, t) = -\operatorname{ord}_{0^{(t)}}(\mathrm{CC}_\psi(M_{\overline{\mathfrak{q}}^{(t)}})) - m_t = \langle |G_\tau| \mathrm{sw}^\beta_{G_\tau}, \chi_{\mathcal{F}}|G_\tau\rangle = \langle \widetilde{\mathrm{sw}}^\beta_{f^{[t,t']}}, \chi_{\mathcal{F}}\rangle. \tag{III.6.3.5}$$

Now applying (III.6.3.3) to $\omega = \mathrm{CC}_\psi(M_{\overline{\mathfrak{q}}^{(t')}}$, we get also from [Bah20, 8.24]

$$\begin{aligned} \varphi_s(\mathcal{F}, t') &= -\operatorname{ord}_{0^{(t')}}(\mathrm{CC}_\psi(M_{\overline{\mathfrak{q}}^{(t')}})) - m_{t'} = \operatorname{ord}_{\infty^{(t')}}(\mathrm{CC}_\psi(M_{\overline{\mathfrak{q}}^{(t')}})) + m_{t'} \\ &= -\langle |G_{\tau'}| \mathrm{sw}^\beta_{G_{\tau'}}, \chi_{\mathcal{F}}|G_\tau\rangle = -\langle \widetilde{\mathrm{sw}}'^\beta_{f^{[t,t']}}, \chi_{\mathcal{F}}\rangle. \end{aligned} \tag{III.6.3.6}$$

Putting (III.6.3.5) and (III.6.3.6) together with (III.5.12.6) yields (III.6.3.2). □

Corollary III.6.4. *We use the notation of* (III.5.15.1). *For rational numbers $r \leq t < t' < t'' \leq r'$, we have*

$$\widetilde{\mathrm{sw}}^\beta_f(\chi_{\mathcal{F}}, [t, t']) + \widetilde{\mathrm{sw}}^\beta_f(\chi_{\mathcal{F}}, [t', t'']) = \widetilde{\mathrm{sw}}^\beta_f(\chi_{\mathcal{F}}, [t, t'']). \tag{III.6.4.1}$$

Theorem III.6.5. *The function $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot) : [r, r'] \cap \mathbb{Q} \to \mathbb{Q}$ is continuous and piecewise linear with finitely many slopes which are all integers. Moreover, for rational numbers $r \leq t < t' \leq r'$, the difference of the right and left derivatives of $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$ at t and t' respectively is*

$$\frac{d}{dt}\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, t+) - \frac{d}{dt}\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, t'-) = \varphi_s(\mathcal{F}, t) - \varphi_s(\mathcal{F}, t'). \tag{III.6.5.1}$$

Proof. Since f is étale, for every subgroup H of G, X/H is also étale, hence has trivial canonical sheaf. Then, the theorem follows from III.5.16 and III.6.3. □

Remark III.6.6. If $\mathcal{F}$ is the restriction to the annulus C of a lisse sheaf of $\overline{\Lambda}$-modules on D, then the functions $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$ and $\varphi_s(\mathcal{F}, \cdot)$ coincide on $[r, r']$ with the similarly denoted functions in [Bah20, 1.9, 9.3]. Hence, if t' is a non-critical radius for f, we deduce from the main result [Bah20, 1.9] that

$$\frac{d}{dt}\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, t+) - \frac{d}{dt}\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, t'-) = \varphi_s(\mathcal{F}, t) - \varphi_s(\mathcal{F}, t'), \tag{III.6.6.1}$$

which recovers III.6.5.1.

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Titre: Variation du conducteur de Swan d'un faisceau étale ℓ-adique sur un disque ou une couronne rigide

Mots clés: Disques et couronnes rigides, Faisceau étale ℓ-adique, Conducteur de Swan, Cycle caractéristique, Cycles évanescents.

Résumé: Ce mémoire de thèse, qui contient deux parties liées, a pour thème la ramification des faisceaux étales ℓ-adiques sur un disque ou une couronne rigide. Soient K un corps de valuation discrète complet de corps résiduel algébriquement clos de caractéristique $p > 0$, D le disque unité fermé rigide et C une couronne fermée dans D centrée en l'origine. À un faisceau étale de $\mathbb{F}_\ell$-modules $\mathcal{F}$ sur D ou C, ramifié en au plus un nombre fini de points rigides de D ou C, on associe une fonction conducteur de Swan $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}$, construite avec la théorie de ramification d'Abbes et Saito, qui, pour une variable $t \in \mathbb{Q}_{\geq 0}$, mesure la ramification de $\mathcal{F}$ le long de la fibre spéciale du modèle entier normalisé de la sous-couronne de rayon t et d'épaisseur nulle (quand elle est définie). On montre que cette fonction est continue, affine par morceaux et possède un nombre fini de pentes qui sont toutes entières. Dans la première partie, $\mathcal{F}$ est un faisceau lisse sur D; on montre alors que la dérivée à droite de $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$ en t est l'ordre du cycle caractéristique de $\mathcal{F}$ en t. Dans la seconde partie, $\mathcal{F}$ est un faisceau étale sur C, possiblement ramifié en un nombre fini de points rigides; on exprime alors la variation des pentes de $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$ entre deux rayons distincts t et t' suffisamment proches comme la différence des ordres des cycles caractéristiques de $\mathcal{F}$ en t et t'. Pour établir ces résultats, on démontre d'abord une formule de cycles évanescents reliant la dérivée de la fonction discriminant associée à un revêtement étale de D ou C, étudiée par W. Lütkebohmert, à des invariants, introduits par K. Kato, pour des homomorphismes entre anneaux locaux formels induits par le revêtement. Ensuite, on applique à des anneaux de valuation construits à partir de ces anneaux locaux formels la théorie de ramification de Kato pour les $\mathbb{Z}^2$-anneaux de valuation et on déduit des résultats de variation pour des conducteurs issus de cette théorie de ramification. On exploite enfin un lien crucial, connu grâce à un travail de H. Hu, entre cette dernière et la théorie de ramification d'Abbes et Saito pour établir la variation de $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$.

Title: Variation of the Swan conductor of an ℓ-adic étale sheaf on a rigid disc or annulus.

Keywords: Rigid discs and annuli, ℓ-adic étale sheaf, Swan conducteur, Characteristic cycle, Vanishing cycles.

Abstract: This thesis, which has two related parts, is on the theme of ramification theory for étale ℓ-adic sheaves. Let K be a complete discrete valuation field with algebraically closed residue field of characteristic $p > 0$, D the closed rigid unit disc and C a closed sub-annulus of D centered around the origin. By the ramification theory of Abbes and Saito, to an étale sheaf of $\mathbb{F}_\ell$-modules on D or C, ramified at most at a finite number of rigid points of D or C, we associate a Swan conductor function $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}$ which, for a variable t, measures the ramification of $\mathcal{F}$ along the special fiber of the normalized integral model of the sub-annulus of radius t with 0-thickness (when the latter is defined). We prove that this function is continuous, piecewise linear and has finitely many slopes which are all integers. In the first part, $\mathcal{F}$ is assumed to be a lisse étale sheaf on D. Then, the right derivative of $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$ at t is shown to be the order of the characteristic cycle of $\mathcal{F}$ at t. In the second part, $\mathcal{F}$ is an étale sheaf on C possibly ramified at a finite number of rigid points. Then, the variation of the slopes of $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$ between two distinct radii t and t' is shown to be the difference of the orders of the characteristic cycles of $\mathcal{F}$ at t and t'. These results are established by connecting works of K. Kato, W. Lütkebohmert and H. Hu. First, we prove a vanishing cycles formula linking the derivative of the discriminant function of an étale cover of D or C, studied by Lütkebohmert, to invariants attached by Kato to homomorphisms of formal étale local rings induced by the cover. Then, we apply Kato's ramification theory for $\mathbb{Z}^2$-valuation rings to two-dimensional valuation rings constructed from these formal local rings and we deduce the variation of associated conductors. Lastly, we exploit a crucial link, elucidated by a work of Hu, between Kato's theory and the Abbes-Saito ramification theory to establish the variation of $\mathrm{sw}_{\mathrm{AS}}(\mathcal{F}, \cdot)$.

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