Syntomification

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Today we'll aim to define syntomic cohomology, building on Ivan's talk two weeks ago, explain why it deserves the name, and see how we can access it via a stacky approach, similar to (and building on) the prismatization.

1. Syntomic cohomology

Rather than the traditional approach, we'll define syntomic cohomology following [4] as the graded pieces of the motivic filtration on topological cyclic homology.

Explicitly, recall topological Hochschild homology THH from Ivan's talk earlier in the semester, which can be defined among other ways as sending an E_∞-ring spectrum to the universal S^1-equivariant E_∞-ring spectrum $\mathrm{THH}(A)$ equipped with a map $A \to \mathrm{THH}(A)$ (not necessarily equivariant). Fix a quasisyntomic ring A and consider the quasisyntomic site A_{qsyn} of A (we work for convenience with the quasisyntomic topology rather than the syntomic topology, but they yield equivalent cohomology theories). Then we get a sheaf of spectra $\mathrm{THH}(-;\mathbb{Z}_p)$ on A_{qsyn}. The natural action of the circle group S^1 on THH allows us to form $\mathrm{TC}^- = \mathrm{THH}^{hS^1}$ (homotopy invariants) and $\mathrm{TP} = \mathrm{THH}^{tS^1}$ (the Tate construction), together with a canonical map $\mathrm{can} : \mathrm{TC}^- \to \mathrm{TP}$. There is also a natural Frobenius on THH, given by composing the diagonal $\Delta_p : A \to (A \otimes \cdots \otimes A)^{tC_p}$ with the Tate construction applied to the map given by permuting factors $A \otimes \cdots \otimes A \to \mathrm{THH}(A)$, to get a map $A \to \mathrm{THH}(A)^{tC_p}$, which by the universal property factors through a map $\varphi_p : \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$. In particular, one can check that $(\mathrm{THH}(A)^{tC_p})^{hS^1} \simeq \mathrm{THH}(A)^{tS^1} = \mathrm{TP}(A)$, so φ_p induces a map

$$\varphi_p^{hS^1} : \mathrm{TC}^-(A) = \mathrm{THH}(A)^{hS^1} \to (\mathrm{THH}(A)^{tC_p})^{hS^1} \simeq \mathrm{TP}(A).$$

In particular, on π_0 we have $\pi_0 \mathrm{TC}^-(A) \simeq \pi_0 \mathrm{TP}(A)$ (indeed for any connective E_∞-ring spectrum A), so this induces a Frobenius endomorphism of $\pi_0 \mathrm{TC}^-(A) = \pi_0 \mathrm{TP}(A)$, lifting the Frobenius mod p in suitable cases.

The relationship to prismatic cohomology is as follows: define

$$\hat{\mathbb{\Delta}}_A = R\Gamma_{\mathrm{syn}}(A, \pi_0 \mathrm{TC}^-(-;\mathbb{Z}_p)) = R\Gamma_{\mathrm{syn}}(A, \pi_0 \mathrm{TP}^-(-;\mathbb{Z}_p)).$$

As the notation suggests, this agrees with the Nygaard-completed prismatic cohomology; one can check this on quasiregular semiperfectoid rings, which boils down to some explicit computations, and then the result follows in general as these give a basis for the quasisyntomic site. The above then gives the Frobenius on prismatic cohomology.

Each of the above sheaves carries a natural filtration by truncations: for any integer n, we can take $\tau_{\geq n} \mathrm{THH}(-;\mathbb{Z}_p)$, and similarly for the other sheaves. In fact it turns out that all three are concentrated in even degrees, so it is natural to define

$$\begin{aligned}
\mathrm{Fil}^n \mathrm{THH}(A;\mathbb{Z}_p) &= R\Gamma_{\mathrm{syn}}(A, \tau_{\geq 2n} \mathrm{THH}(-;\mathbb{Z}_p)),\\
\mathrm{Fil}^n \mathrm{TC}^-(A;\mathbb{Z}_p) &= R\Gamma_{\mathrm{syn}}(A, \tau_{\geq 2n} \mathrm{TC}^-(-;\mathbb{Z}_p)),\\
\mathrm{Fil}^n \mathrm{TP}(A;\mathbb{Z}_p) &= R\Gamma_{\mathrm{syn}}(A, \tau_{\geq 2n} \mathrm{TP}(-;\mathbb{Z}_p)).
\end{aligned}$$

Thus $\hat{\Delta}_A = \mathrm{gr}^0 \mathrm{TC}^-(A;\mathbb{Z}_p) = \mathrm{gr}^0 \mathrm{TP}(A;\mathbb{Z}_p)$ carries a natural filtration $\mathcal{N}^{\geq *}\hat{\Delta}_A$, with nth graded piece $\mathcal{N}^n\hat{\Delta}_A$; this agrees with the Nygaard filtration.

Taking the first graded piece instead of the zeroth recovers another piece of information: the Breuil–Kisin twist $\hat{\Delta}_A\{1\} := \mathrm{gr}^1 \mathrm{TP}(A;\mathbb{Z}_p)[-2]$ is an invertible $\hat{\Delta}_A$-module, and for any $\hat{\Delta}_A$-module M we write $M\{n\}$ for $M \otimes_{\hat{\Delta}_A} \hat{\Delta}_A\{1\}^{\otimes n}$. Thus

$$\mathrm{gr}^n \mathrm{TP}(A;\mathbb{Z}_p) = \hat{\Delta}_A\{n\}[2n].$$

We can similarly understand the graded parts of the other two filtered sheaves: the grade parts of TC^- are the same as for TP but with the additional data of the Nygaard filtration,

$$\mathrm{gr}^n \mathrm{TC}^-(A;\mathbb{Z}_p) = \mathcal{N}^{\geq n}\hat{\Delta}_A\{n\}[2n],$$

and THH is similarly the same thing but for the graded pieces of the Nygaard filtration,

$$\mathrm{gr}^n \mathrm{THH}(A;\mathbb{Z}_p) = \mathcal{N}^n\hat{\Delta}_A\{n\}[2n].$$

Thus there are spectral sequences from the cohomology of the prismatic complex (with Breuil–Kisin twists), its Nygaard filtration, and the filtered pieces to the homotopy groups of $\mathrm{TP}(A;\mathbb{Z}_p)$, $\mathrm{TC}^-(A;\mathbb{Z}_p)$, and $\mathrm{THH}(A;\mathbb{Z}_p)$ respectively.

In particular, we can now build something new using the Frobenius map from above as well as the canonical map

$$\varphi, \mathrm{can} : \mathrm{TC}^-(A;\mathbb{Z}_p) \to \mathrm{TP}(A;\mathbb{Z}_p).$$

In particular, by definition topological cyclic homology TC is

$$\mathrm{TC}(A;\mathbb{Z}_p) = \mathrm{hofib}(\varphi - \mathrm{can} : \mathrm{TC}^-(A;\mathbb{Z}_p) \to \mathrm{TP}(A;\mathbb{Z}_p))$$

and both the Frobenius and the canonical map respect the filtrations above, giving rise to a filtration on TC

$$\mathrm{Fil}^n \mathrm{TC}(A;\mathbb{Z}_p) = \mathrm{hofib}(\varphi - \mathrm{can} : \mathrm{Fil}^n \mathrm{TC}^-(A;\mathbb{Z}_p) \to \mathrm{Fil}^n \mathrm{TP}(A;\mathbb{Z}_p))$$

with graded pieces

$$\mathrm{gr}^n \mathrm{TC}(A;\mathbb{Z}_p) = \mathrm{hofib}(\varphi - \mathrm{can} : \mathcal{N}^{\geq n}\hat{\Delta}_A\{n\}[2n] \to \hat{\Delta}_A\{n\}[2n]).$$

We define the complexes of sheaves $\mathbb{Z}_p(n)$ by

$$\mathbb{Z}_p(n)(A) = \mathrm{gr}^n \mathrm{TC}(A;\mathbb{Z}_p)[-2n] = \mathrm{hofib}(\varphi - \mathrm{can} : \mathcal{N}^{\geq n}\hat{\Delta}_A\{n\} \to \hat{\Delta}_A\{n\}).$$

(As the notation suggests, we can recover $\mathbb{Z}_p(n)(A)$ as the cohomology $R\Gamma_{\mathrm{syn}}(A, \mathbb{Z}_p(n))$ of the sheaf $\mathbb{Z}_p(n)$ on the quasisyntomic site of A given as the homotopy fiber of $\varphi - \mathrm{can}$ on the graded components of the sheaves defined above.) Thus there is a spectral sequence

$$E_2^{ij} = H^{i-j}(\mathbb{Z}_p(-j)(A)) \implies \pi_{-i-j} \mathrm{TC}(A;\mathbb{Z}_p).$$

The complexes $\mathbb{Z}_p(n)$ will be our syntomic complexes.

2. WHAT?

Why is this something we should call syntomic cohomology? Admittedly it does involve the cohomology of sheaves on the (quasi)syntomic site, but so do many things, including prismatic cohomology; what makes this especially noteworthy or connects it to "classical" syntomic cohomology (whatever that might mean)?

For one version of classical syntomic cohomology, say in the case where A is an $\mathbb{F}_p$-algebra, we would first associate to A a sheaf $\mathcal{O}^{\mathrm{crys}}$ (or more properly a system of sheaves for lifting towards characteristic 0), which carries a Frobenius action taking image in $p^n\mathcal{O}^{\mathrm{crys}}$ for some n (involving the data of the system). Then we could consider the *divided Frobenius* φ/p^n and take the fiber $\ker\left(\frac{\varphi}{p^n} - 1 : \mathcal{O}^{\mathrm{crys}}_n \to \mathcal{O}^{\mathrm{crys}}\right)$, which once we lift to characteristic 0 we call the syntomic sheaf $\mathbb{Z}_p(n)$. (I am eliding a lot of details; for one thing this shouldn't really be on $\mathcal{O}^{\mathrm{crys}}$, but rather an nth twist, but let's not get into it.)

How could we do a characteristic 0 version of this story? Well, our standard trick is to replace something like the pair $(\mathbb{Z}_p, (p))$ with a more general prism (B, I). In this case it's important that $(\mathbb{Z}_p, (p))$ is perfect to have good behavior of the Frobenius, so we'll assume that (B, I) is perfect, i.e. $B/I =: R$ is a perfectoid ring, which means $I = (\xi)$ is principal. Then going through we expect similar behavior: we replace the crystalline sheaf $\mathcal{O}^{\mathrm{crys}}$ by the (completed) prismatic complex $\hat{\Delta}_A$ for A and R-algebra, and then on the nth stratum of $\hat{\Delta}_A$, namely $\mathcal{N}^{\geq n}\hat{\Delta}_A$, we hope to have the Frobenius $\mathcal{N}^{\geq n}\hat{\Delta}_A\{n\} \to \hat{\Delta}_A\{n\}$ taking image in $\xi^n\hat{\Delta}_A\{n\}$. Indeed this is what happens: since $(B, (\xi))$ is perfect, we can choose a trivialization $B \simeq B\{1\}$ (corresponding to choosing a generator ξ), with the twist corresponding to dividing by ξ. Thus the restriction of Frobenius $\hat{\Delta}_A \to \hat{\Delta}_A$ to $\mathcal{N}^{\geq n}\hat{\Delta}_A$ is ξ^n times $\varphi : \mathcal{N}^{\geq n}\hat{\Delta}_A\{n\} \to \hat{\Delta}_A\{n\}$. In particular the former Frobenius has image in $\xi^n\hat{\Delta}_A$, and so we can view the latter, φ, as really being Frob/ξ^n, and can as the identity restricted to this twisted filtered part. Thus the construction of the syntomic complexes here identifies with the more classical construction via divided Frobenii.

Maybe a more convincing demonstration is to see that the complexes $\mathbb{Z}_p(n)$ have some of the good properties we want from syntomic cohomology. Some of these are:

- For $n < 0$, the complexes $\mathbb{Z}_p(n)$ should have vanishing cohomology.
- For $n = 0$, we expect $\mathbb{Z}_p(0)$ to just be the constant sheaf $\mathbb{Z}_p$.
- For $n = 1$, we expect $\mathbb{Z}_p(1)$ to be the Tate module $T_p\mathbb{G}_{\mathrm{m}}$.
- There should be a spectral sequence from syntomic cohomology to étale K-theory, coming from an identification of syntomic cohomology with graded pieces of étale K-theory.

Are those true for our definition?

We'll go in reverse order: the main thing is the comparison to étale K-theory. There is a trace map $\mathrm{tr} : K(A) \to \mathrm{TC}(A)$; it turns out that in many cases, especially in the p-adic world, this is a (p-adic) equivalence (in particular, whenever A is Henselian with respect to (p), i.e. roots of polynomial equations modulo p can be lifted to roots in A). An important special case is when A is quasiregular semiperfectoid, since these give a basis for the quasisyntomic topology. Since we defined syntomic cohomology as the graded pieces of the motivic filtration on TC, we get the desired identification with graded pieces of p-adic étale K-theory, and thus a spectral sequence from syntomic cohomology to K-theory. In particular we can think of syntomic cohomology as a p-adic version of motivic cohomology.

This now lets us do some of the computations, which we also see agree with the expectations. By definition, $\mathbb{Z}_p(n) = \mathrm{gr}^n \mathrm{TC}(-; \mathbb{Z}_p)[-2n]$ is concentrated in (cohomological) degrees 0 and 1: on quasiregular semiperfectoid rings A, it is just the truncation of $\mathrm{TC}(A; \mathbb{Z}_p)$ between $2n-1$ and $2n$, shifted by $-2n$ into the correct degrees. Thus for example in the case $n = 0$ it suffices to check that locally $\pi_{-1} \mathrm{TC}(-; \mathbb{Z}_p)$ vanishes on quasiregular semiperfectoid rings; then by the comparison to p-adic étale K-theory the degree 0 portion is just $\mathbb{Z}_p$, which is then the whole thing. Similarly by comparison to K-theory one can check the statement about $\mathbb{Z}_p(1) = T_p\mathbb{G}_\mathrm{m}$, and the fact that $\pi_i \mathrm{TC}(A; \mathbb{Z}_p) = 0$ for $i < -1$ implies that $\mathbb{Z}_p(n) = 0$ for $n < 0$, since then the truncation to $[2n-1, 2n]$ gives zero.

3. Syntomification

Now that we are convinced this is a reasonable object to call syntomic cohomology, we want to geometrize it, just as we did for prismatic cohomology. Recall that for any (bounded) p-adic formal scheme X, we can associate to it the prismatization X^{Δ}, whose R-points classify Cartier–Witt divisors (I, α) on R together with a map $\operatorname{Spec} \operatorname{Cone}(\alpha) \to X$. When $X = \operatorname{Spf} \mathbb{Z}_p$, we call this stack $(\operatorname{Spf} \mathbb{Z}_p)^{\Delta} =: \Sigma$, and we also constructed the larger stack Σ', where we require I to only be admissible rather than invertible, and two embeddings $j_\pm : \Sigma \hookrightarrow \Sigma'$, corresponding to a 'canonical' and 'Frobenius' embedding, giving two disjoint loci isomorphic to Σ in Σ'. Finally we glued Σ' along these loci to get a stack Σ''. We can do the same thing over X to get stacks $X^{\Delta'}$ and $X^{\Delta''}$ respectively, which Bhatt calls $X^{\mathcal{N}}$ (which we want to think of as related to prisms together with the Nygaard filtration) and X^{syn} (related to syntomic cohomology).

Explicitly, the (expected) theorem is as follows, part (1) of which we've already seen.

Theorem. *(1) There is a natural equivalence of categories between $D(X^{\Delta})$ and the ∞-category of* prismatic crystals on X, *i.e. sheaves of $\mathcal{O}_{\Delta}$-modules on the prismatic site of X.*

(2) There is a natural equivalence of categories between $D(X^{\mathcal{N}})$ and the ∞-category of prismatic gauges on X, *i.e. filtered modules over $\mathcal{O}_{\Delta}$ equipped with the Nygaard filtration.*

(3) There is a natural equivalence of categories between $D(X^{\mathrm{syn}})$ and the ∞-category of prismatic F-gauges on X, *i.e. an F-gauge $\mathcal{E}$ together with an isomorphism $\varphi^*\mathcal{E}[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_{\Delta}]$.*

Why is this related to syntomic cohomology? Well, let's recall the Breuil–Kisin twists $\mathcal{O}_{\Delta}\{n\}$ on the prismatic site. By the above theorem, these translated to twists $\mathcal{O}_{X^{\Delta}}\{n\}$ of the structure sheaf on the prismatization of X. However, they carry some additional structure: they are compatible with the Nygaard filtration and so extend to twists of $\mathcal{O}_{X^{\mathcal{N}}}$

whose restriction to either copy of X^{Δ} gives the corresponding twists there—we saw this a couple talks ago in the case $X = \operatorname{Spf} \mathbb{Z}_p$.

In fact, more is true: we also have compatibility with Frobenius, via an isomorphism $\varphi^* \mathcal{O}_{\Delta}\{1\}[1/\mathcal{I}_{\Delta}] \simeq \mathcal{O}_{\Delta}\{1\}[1/\mathcal{I}_{\Delta}]$ coming from the Bhatt–Lurie construction of the Breuil–Kisin twist as a chain of quotients by powers of $\mathcal{I}_{\Delta}$; this means that the extension $\mathcal{O}_{X^{\mathcal{N}}}\{1\}$ of $\mathcal{O}_{X^{\Delta}}\{1\}$ to $X^{\mathcal{N}}$ further descends to a sheaf $\mathcal{O}_{X^{\mathrm{syn}}}\{1\}$ on X^{syn}, which is an invertible $\mathcal{O}_{X^{\mathrm{syn}}}$-module. By the theorem (as well as the discussion above), we can think of this as a prismatic F-gauge on X.

On the other hand, we build X^{syn} by gluing $X^{\mathcal{N}}$ along the two embedded copies of X^{Δ}, via a canonical map and the Frobenius:

$$\begin{array}{ccc} X^{\Delta} & \xrightarrow{\text{can}} & X^{\mathcal{N}} \\ \downarrow{\scriptstyle\varphi} & & \downarrow \\ X^{\mathcal{N}} & \longrightarrow & X^{\mathrm{syn}} \end{array}$$

is a pushout diagram, and so

$$\begin{array}{ccc} \mathcal{O}_{X^{\mathrm{syn}}} & \longrightarrow & \mathcal{O}_{X^{\mathcal{N}}} \\ \downarrow & & \downarrow{\scriptstyle\text{can}} \\ \mathcal{O}_{X^{\mathcal{N}}} & \xrightarrow{\varphi} & \mathcal{O}_{X^{\Delta}} \end{array}$$

is a pullback diagram. Thus $\mathcal{O}_{X^{\mathrm{syn}}}\{n\}[2n] = \operatorname{fib}(\varphi - \text{can} : \mathcal{O}_{X^{\mathcal{N}}}\{n\}[2n] \to \mathcal{O}_{X^{\Delta}}\{n\}[2n])$ is identified with the graded components of $\mathrm{TC}(A; \mathbb{Z}_p)$, namely $\mathbb{Z}_p(n)$. In other words the syntomic complexes appear naturally as the Breuil–Kisin twists on X^{syn}!

This also suggests that the "right" way to generalize syntomic cohomology and still get interesting things (beyond in the trivial way of allowing arbitrary sheaves on the quasisyntomic site) is by replacing the Breuil–Kisin twists $\mathcal{O}_{X^{\mathrm{syn}}}\{n\}$ with other complexes on X^{syn}. In other words, prismatic F-gauges give the right theory of coefficients for syntomic cohomology, with fundamental examples such as Breuil–Kisin twists recovering the classical examples.

The special case $X = \operatorname{Spf} \mathbb{Z}_p$, so that $X^{\mathrm{syn}} = \Sigma''$, provides some evidence for the claim that this is the "right" theory. In particular we have the following expected theorem of Bhatt–Lurie [1]:

Theorem. *Let E be any perfect prismatic F-gauge on $X = \operatorname{Spf} \mathbb{Z}_p$, i.e. a perfect complex on X^{syn}.*[1] *We define the syntomic cohomology with coefficients in E to be $R\Gamma(\mathbb{Z}_p^{\mathrm{syn}}, E) := R\operatorname{Hom}(\mathcal{O}, E)$, computed in the category of prismatic F-gauges. It satisfies the following properties:*

(1) For any perfect F-gauge E, the syntomic cohomology $R\Gamma(\mathbb{Z}_p^{\mathrm{syn}}, E)$ is a perfect object of $D(\mathbb{Z}_p)$.

(2) There is a natural symmetric monoidal "étale realization" functor from the ∞-category of perfect prismatic F-gauges to $D^b_{\mathrm{cons}}(\operatorname{Spec} \mathbb{Q}_p, \mathbb{Z}_p) \simeq D^b_{\text{fd, cts}}(\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p), \mathbb{Z}_p)$ such

[1] Note that "perfect" here means in the sense of complexes, and is unrelated to perfect prisms.

that $T(\mathcal{O}\{1\}) = \mathbb{Z}_p(1)$ *and for any perfect prismatic* F*-gauge* E*, each cohomology group of* $T(E)[1/p]$ *is a crystalline representation.*

(3) Let $E^* = R\operatorname{Hom}(E, \mathcal{O})$ *be the dual of* E*. Then there is a natural fiber sequence*

$$R\Gamma(\mathbb{Z}_p^{\mathrm{syn}}, E) \to R\Gamma(\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p), T(E)) \to R\operatorname{Hom}_{\mathbb{Z}_p}(R\Gamma(\mathbb{Z}_p^{\mathrm{syn}}, E^*\{1\}[2]), \mathbb{Z}_p),$$

where the first map is induced by T *and the second is induced by the classical Tate duality isomorphism*

$$R\Gamma(\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p), T(E)) \simeq R\operatorname{Hom}_{\mathbb{Z}_p}(R\Gamma(\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p), T(E^*)(1)[2]), \mathbb{Z}_p)$$

together with (the dual of the map induced by) T*.*

Thus on Spf $\mathbb{Z}_p$, we think of prismatic F-gauges as corresponding to the "crystalline part" of the derived category of Galois representations (though T is not fully faithful, so there is some extra structure). Via geometrization, we can think of $R\Gamma(\mathbb{Z}_p^{\mathrm{syn}}, E)$ as the cohomology of the corresponding sheaf $\mathcal{F}$ to E on Σ'', so that for example property (3) is a version of Poincaré duality for Σ'' (suggesting that we should think of it as one-dimensional).

References

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