Prismatization: introduction and overview

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The idea of today's talk is to give an overview of what I hope to cover in the seminar, and at the end sketch a rough schedule for the talks and get volunteers at least for the next few.

1. Background: prismatic cohomology

1.1 Deperfecting perfectoid spaces

Let's start with perfectoid rings. Everything in this section is essentially due to Scholze, with contributions from a few others (Bhatt, Kedlaya, Liu, Weinstein, and others). Roughly speaking, these are an analogue of perfect $\mathbb{F}_p$-algebras for $\mathbb{Z}_p$-algebras; slightly more specifically, they are p-complete $\mathbb{Z}_p$-algebras such that modulo p, the Frobenius is a bijection, and there is some element whose pth power divides p, plus some other technical conditions. (Note that for us, "perfectoid" will always mean "integral perfectoid." These are in many ways simpler than the adic notion, but exclude some examples while including others.) Examples include all perfect $\mathbb{F}_p$-algebras, including $\mathbb{F}_p$ itself, or the p-adic completions of $\mathbb{Z}_p[\zeta_{p^\infty}]$ or $\mathbb{Z}_p[p^{1/p^\infty}]$.

The main interest in perfectoid rings is the tilting correspondence, which relates characteristic 0 rings to characteristic p rings subject to the requirement that they be perfectoid: to any perfectoid $\mathbb{Z}_p$-algebra R, we can associate a characteristic p ring $R^\flat$ called its tilt, which can be thought of as replacing p with some other pseudouniformizer. (If R is already an $\mathbb{F}_p$-algebra, then $R^\flat = R$.) The tilting correspondence states that $A \mapsto A^\flat$ gives an equivalence of categories between perfectoid R-algebras and perfectoid $R^\flat$-algebras. Adding some reasonable technical conditions, one can further get an equivalence of étale sites and similar relationships.

This (and its various other incarnations) allows, at least in principle, the translation of characteristic 0 problems into characteristic p problems, which are often easier. One of the classical motivating results for the theory of perfectoid spaces is the Fontaine–Winterberger theorem, which states that the absolute Galois groups of $\mathbb{Q}_p(p^{1/p^\infty})^\wedge_p$ and $\mathbb{F}_p((T^{1/p^\infty}))^\wedge_T$ are naturally isomorphic. This is in fact an instance of a tilting correspondence: the second field turns out to be the tilt of the first, and the isomorphism of Galois groups follows from the equivalence of étale categories.

However, there are some barriers to this sort of method, most obviously that the restriction that the objects in question be perfectoid is a very restrictive one: these fields above are very large, so much so that it is not clear that this sort of result should apply to anything number-theoretic. One solution is to quotient everything by automorphism groups to get down to something more reasonably sized, and keep track of this group action; this is closely related to Scholze's theory of diamonds. We'll try to generalize in a different direction: where perfectoid spaces generalize perfect $\mathbb{F}_p$-algebras to $\mathbb{Z}_p$-algebras, we'll try and remove the adjective "perfect."

1.2 Prisms

First, we need to reframe perfectoid spaces into something we can more straightforwardly generalize. The right way to do this is suggested by the ring A_{inf}.

Fix a perfectoid ring R. Its tilt $R^\flat$ is in characteristic p, and so we can take the Witt vectors $W(R^\flat)$. We call this ring $A_{\text{inf}}(R)$, or just A_{inf} if R is clear. We can recover $R^\flat$ as $A_{\text{inf}}/(p)$; less obvious is that we can also recover R. In particular there is a surjection $A_{\text{inf}}(R) \to R$ whose kernel is a principal ideal (ξ), where ξ has certain properties (it is a "distinguished element") lifting p; the case $\xi = p$ recovers $R = R^\flat$, i.e. R is already characteristic p. Thus the distinguished elements of A_{inf} in a certain sense parametrize the untilts of $R^\flat$. More explicitly, the data of R is equivalent to a pair (A, I) where A is a ring with certain structures and properties (in practice, $A = A_{\text{inf}}$) and I is an ideal with certain properties (in practice, $I = (\xi)$). This is our model for a prism.

What are the necessary structures on A? Well, since $A_{\text{inf}} = W(R^\flat)$ is the Witt vectors of an $\mathbb{F}_p$-algebra, by functoriality it carries a Frobenius lifting the Frobenius on $R^\flat$, so A should be equipped with a lift of Frobenius. It turns out that for technical reasons it's better to instead equip A with a δ-structure, which is a function $\delta : A \to A$ such that the map $\phi : A \to A$ defined by $\phi(x) = x^p + p\delta(x)$ is a ring homomorphism. When A is p-torsionfree this is the same thing as specifying a Frobenius lift, but in general it behaves slightly better.

We also have certain properties of A and I when they come from a perfectoid ring as above: in particular, we required that the Frobenius on $R^\flat$ be an isomorphism (this follows from it being one on R/p), and so its lift ϕ on $W(R^\flat)$ is also an isomorphism; we also have I principal, with generator ξ. There is also a technical condition which is at first glance somewhat mysterious: we always have $p \in (\xi, \phi(\xi))$. This turns out to follow from the distinguishedness of ξ. We can rephrase it without reference to ξ: we have $p \in I + \phi(I)A$.

This motivates the following definition: a *perfect orientable prism* is a pair (A, I), where A is a δ-ring and I is an ideal of A such that A is (p, I)-complete, the induced Frobenius ϕ on A is an isomorphism, I is principal, and $p \in I + \phi(I)A$.

The adjectives "perfect" and "orientable" here refer to ϕ being an isomorphism and I being principal respectively, and their presence suggests that we will relax these conditions. Indeed, the "perfect" condition can be dropped entirely; the requirement that I be principal can be weakened, but we still require that it be *locally* principal, i.e. define a Cartier divisor, in order to maintain reasonable behavior. Thus our general definition is: a *prism* is a pair (A, I) where A is a δ-ring, I is a locally principal ideal of A, A is (p, I)-complete, and $p \in I + \phi(I)A$. This "deperfects" the notion of perfectoid rings.

To fully justify this, we should state the following theorem.

Theorem 1. *The categories of perfect prisms and perfectoid rings are equivalent, via* $(A, I) \mapsto A/I$.

A simple example is $R = \mathbb{F}_p$. We can compute the corresponding prism directly: $A = W(R^\flat) = W(\mathbb{F}_p) = \mathbb{Z}_p$, and I is the kernel of the map $\mathbb{Z}_p \to \mathbb{F}_p$, i.e. (p). More generally, any perfect field (or even general algebra) k of characteristic p corresponds to the perfect prism $(W(k), (p))$.

More generally still, we could look at the class of prisms $(A, (p))$ for any A, perfect or not. These are called *crystalline* prisms; it turns out that this is possible (i.e. is a prism) if and only if A is p-torsionfree.

1.3 Prismatic cohomology

Since we were interested in perfectoid rings in the first place because of the tilting equivalence, it's natural to look for something similar for this "deperfected" version that we've defined. In particular, instead of looking at the category of perfectoid R-algebras for a perfectoid ring R, for a totally general R we can try and look at the category of prisms (A, I) together with a map $R \to A/I$ (since in the perfect case A/I carries all of the information of (A, I)). In this affine way we've been working with, we can just use the indiscrete topology on this category; in general one should use the flat topology. This gives a site $(R)_{\Delta}$, called the (absolute) prismatic site of R.

One can also fix a base prism (A, I) together with a map $A/I \to R$, and look at prisms (B, J) over (A, I) (i.e. with a map of δ-rings $A \to B$ sending I into J) with maps of A/I-algebras $R \to B/J$. This is generally what Bhatt-Scholze do, and it gives rise to the relative prismatic site $(R/A)_{\Delta}$.

There are several natural sheaves on each prismatic site: $\mathcal{O}_{\Delta}$, which sends $(B, J) \mapsto B$; $\mathcal{I}_{\Delta}$, which sends $(B, J) \mapsto J$; or $\overline{\mathcal{O}}_{\Delta}$, sending $(B, J) \mapsto B/J$. One can take the cohomology of these (especially the first) to get the relative prismatic complex

$$\Delta_{R/A} = R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta})$$

or the absolute prismatic complex

$$\Delta_R = R\Gamma((R)_{\Delta}, \mathcal{O}_{\Delta}).$$

One can do the same thing with $\overline{\mathcal{O}}_{\Delta}$ to get the Hodge-Tate complex $\overline{\Delta}_{R/A}$ (or $\overline{\Delta}_R$); in the relative case at least, it turns out that $\overline{\Delta}_{R/A} = \Delta_{R/A} A/I$, so the prismatic complex generally carries the most information.

For example, if $R = A/I$, then $(R/A)_{\Delta}$ is just the category of prisms over (A, I), and so has an initial object (A, I); so the prismatic cohomology is just A, and the Hodge-Tate complex is just A/I.

The main reason to be interested in prismatic cohomology is that it interpolates various other p-adic cohomology theories, such as de Rham cohomology, étale cohomology, crystalline cohomology, Hodge–Tate cohomology, etc. via various comparison theorems, which I will not write down for now. This is some sort of "motivic" analogue in the p-adic world, and indeed there are also connections between prismatic and syntomic cohomology, which forms an analogue of motivic cohomology.

2. Prismatization

Since it is such a powerful theory (I haven't really shown this, but let's accept it for now), one might want to take prismatic cohomology with other coefficients. This leads directly to studying various kinds of well-behaved sheaves on the prismatic site of R (or more generally of a p-adic formal scheme X satisfying certain technical conditions). The key observation of Bhatt-Lurie and Drinfeld is that this is actually equivalent to studying (derived) sheaves on a certain stack, called the *prismatization* of X, written either X^{Δ} or WCart_X depending whose paper you read. (This is for the absolute site; one can also define relative versions.)

As for étale cohomology, even the point should carry interesting information, in some ways the most interesting (étale cohomology of $\mathrm{Spec}\, k$ recovers the representation theory of $\mathrm{Gal}(\bar{k}/k)$). In this case, the relevant point is when $X = \mathrm{Spf}\, \mathbb{Z}_p$; its prismatization, called Σ or WCart, can be defined quite explicitly and is studied in depth in [1] and [4]. Roughly speaking, we can think of Σ as the stack sending a p-nilpotent ring R to the set of prism structures on $W(R)$. In the case where R is perfect of characteristic p, this is the same thing as classifying untilts of R. For general X, we can think of the R-points of X^{Δ} as classifying prism structures on $W(R)$ over X, i.e. prisms $(W(R), I)$ together with a map $\mathrm{Spec}\, W(R)/I \to X$; when R is perfectoid of characteristic p, this classifies untilts to X, analogous to the "diamondization" functor. (It's also worth noting that the words "derived" and "animated" should be sprinkled in generously to all these statements.)

3. Refinements

However, it turns out that these stacks can be refined further. In particular, prismatic cohomology comes equipped with a filtration called the Nygaard filtration; one can define a version of the prismatization stacks which keeps track of this data, called $X^{\Delta'}$ and in particular Σ' in the case $X = \mathrm{Spf}\, \mathbb{Z}_p$. Just as derived sheaves on X^{Δ} are supposed to correspond to sheaves on the prismatic site of X, in this case derived sheaves on $X^{\Delta'}$ should correspond to certain algebraic gadgets called gauges on the prismatic site of X, which are very roughly sheaves together with a filtration with certain properties. There is a natural map $X^{\Delta} \to X^{\Delta'}$ which on sheaves can be thought of as forgetting this extra structure, and another map which can be thought of as pulling it back by Frobenius; this gives two copies of X^{Δ} as open substacks of $X^{\Delta'}$. Gluing these together gives a third stack $X^{\Delta''}$, derived sheaves on which correspond to gauges together with an isomorphism with the Frobenius pullback; these are called F-gauges, and are conjectured to be in some sense the "right" coefficients for prismatic cohomology, at least in the derived setting. Bhatt and Lurie have a forthcoming paper with some results justifying this.

4. Schedule

I think it's probably worth having a little more discussion of prismatic cohomology than we've done today; we could easily spend a whole semester on it, but I would prefer not to, so maybe we'll condense it down and skip most proofs. I'd propose something like the following lectures:

- Prismatic cohomology: focusing on definitions, statements of theorems, maybe some proofs or sketches of proofs for particularly interesting or important things. One thing which I think might be fun to fit in is the relationship to THH and the more topological side; if so this might need to be two lectures.

- Prismatic cohomology: focusing on applications, such as André's flatness lemma, the direct summand theorem, crystalline Galois representations, the various related cohomology theories in p-adic geometry, maybe the work of Anschütz–le Bras on prismatic Dieudonné theory. All of these are well beyond one lecture of material to discuss completely, I'm picturing a discussion of the relationship with prismatic cohomology and some of the ideas at play. This talk can probably be skipped as needed.

After this, I'm hoping to start on Drinfeld's prismatization paper [**?**]. Some topics, which may or may not end up corresponding to a single lecture each, are:

- C-stacks and g-stacks
- Witt vector schemes and modules
- The stack Σ
- Bhatt-Lurie's version of it (see [1])
- The stack Σ' (at least two lectures)
- The stack Σ''

Another lecture we may want in there somewhere (perhaps preceding the Bhatt-Lurie lecture) is something about the animated or derived worlds:

- Animated prisms and derived prismatization

From there, we're free to go in other directions. Some possible topics for talks (possibly multiple each):

- X^{Δ}, $X^{\Delta'}$, and $X^{\Delta''}$ for other X than Spf $\mathbb{Z}_p$
- prismatic F-gauges
- ???

References

[1] Bhargav Bhatt and Jacob Lurie. Absolute prismatic cohomology. *arXiv preprint arXiv:2201.06120*, 2022.

[2] Bhargav Bhatt and Jacob Lurie. The prismatization of p-adic formal schemes. *arXiv preprint arXiv:2201.06124*, 2022.

[3] Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. *arXiv preprint arXiv:1905.08229*, 2019.

[4] Vladimir Drinfeld. Prismatization. *arXiv preprint arXiv:2005.04746*, 2020.