Overview of prismatic cohomology

Avi Zeff

Last time, among other things, we spent some time motivating and discussing prisms as enlarging the category of perfectoid spaces. Today, we'll go in more depth to summarize the main points of prismatic cohomology; since we've already talked about some of the fundamentals we'll move through them relatively quickly today to get to some comparison theorems and the Nygaard filtration. The material for today is taken primarily from sections 2, 3, 4, 9, and 12 of [1].

We fix once and for all a prime number p.

1. δ-rings

Roughly speaking, a δ-ring is a ring A equipped with a lift of Frobenius $\phi : A \to A$, i.e. a homomorphism such that $\phi(x) = x^p + p\delta(x)$ for some function δ; for technical reasons it is better to specify the data of δ, which must satisfy certain equations (exercise: write these down) to ensure that ϕ is a homomorphism. In practice, we'll often assume that A is a $\mathbb{Z}_{(p)}$-algebra.

We can equip $\mathbb{Z}_{(p)}$ with a δ-structure: the Frobenius lift is just the identity, so $\delta(x) = \frac{x-x^p}{p}$. This is the initial object in the category of δ-rings. One can check that $\delta(p^n)$ is divisible by p^{n-1} but not p^n, so in particular $\delta^n(p^n)$ is a unit. It follows that for any nonzero δ-ring (A, δ'), we cannot have $p^n = 0$ in A for any n: if we did, the map of δ-rings $f : (\mathbb{Z}_{(p)}, \delta) \to (A, \delta')$ would have $f(\delta(p^n)) = \delta'(f(p^n)) = \delta'(0) = 0$, impossible since $\delta(p^n)$ is a unit.

One additional point worth discussing is categorical: the forgetful functor δ-**CRing** $\to$ **CRing**, sending $(A, \delta) \mapsto A$, preserves all limits and colimits, and so admits both a left and right adjoint. The right adjoint is given by the (p-typical) Witt vectors: for any ring R, there is a natural Frobenius lift on $W(R)$, given by the Witt vector Frobenius, which makes $W(R)$ naturally a δ-ring, and for any δ-ring A and any ring R we have $\mathrm{Hom}(A, R) = \mathrm{Hom}_\delta(A, W(R))$, where Hom_δ denotes homomorphisms preserving the δ-structure, i.e. commuting with δ (or ϕ).

The left adjoint is given by "free δ-rings": for example, the free δ-ring on one generator over $\mathbb{Z}_{(p)}$ is $\mathbb{Z}_{(p)}[x_0, x_1, \ldots]$ with $\delta(x_i) = x_{i+1}$. One can use these together with quotients to form pushouts of δ-rings.

We say that an element d of a δ-ring A is distinguished if $\delta(d)$ is a unit. For example: p in $\mathbb{Z}_p$; $x - p$ in $\mathbb{Z}_p[[x]]$; a generator of the map $W(R^\flat) \to R$ for R perfectoid; etc. It turns out that the property of being distinguished depends only on the ideal (d) rather than d itself.

We say that a δ-ring A is perfect if the Frobenius ϕ is an isomorphism. This implies that it is p-torsionfree.

In fact, the following categories are equivalent:

(1) the category of perfect p-complete δ-rings;

(2) the category of p-complete p-torsionfree rings A such that A/p is perfect;

(3) the category of perfect $\mathbb{F}_p$-algebras.

The functor from (1) to (2) is the forgetful functor, so this part is saying that a map of such rings automatically preserves the δ-structure (since the Frobenius automatically lifts); the functor from (2) to (3) is $A \mapsto A/p$, and in the other direction $R \mapsto W(R)$, so this is saying that every such ring as in (2) arises as the Witt vectors of a perfect $\mathbb{F}_p$-algebra and the δ-structure is unique.

2. Prisms

Definition. A prism is a pair (A, I) where A is a δ-ring and $I \subset A$ is an ideal defining a Cartier divisor on $\operatorname{Spec} A$, such that A is (p, I)-complete and $p \in I + \phi(I)A$.

This second condition turns out to be equivalent to requiring that, after (ind-) Zariski localization, I is generated by a distinguished element.

We say that a prism (A, I) is perfect if A is perfect; bounded if $A[p^\infty] = A[p^n]$ for some n; orientable if I is principal, or oriented if we're given a generator; crystalline if $I = (p)$ (in which case it must also be orientable and bounded).

Lemma (Rigidity). *Let $(A, I) \to (B, J)$ be a map of prisms. Then $IB = J$.*

Given a ring and an ideal, we'd like to be able to take the quotient; in the case of perfect prisms, as discussed last time, this recovers a familiar notion.

Theorem. *The categories of (integral) perfectoid rings and perfect prisms are equivalent, via $(A, I) \mapsto A/I$.*

We can also ask about lifting maps of these quotients. When the source is perfect, this is possible:

Lemma. *Let (A, I) be a perfect prism. Then for any prism (B, J),* $\operatorname{Hom}(A/I, B/J) = \operatorname{Hom}((A, I), (B, J))$.

Together with the above theorem, this recovers the tilting equivalence!

Theorem. *Let R be a perfectoid ring, with tilt $R^\flat$. Then the categories of perfectoid R-algebras and perfectoid $R^\flat$-algebras are equivalent.*

Proof. By the theorem, we can uniquely write $R = A/I$; recall that the way this equivalence works is via $A = W(R^\flat)$, so we can write $R^\flat = A/p$. A perfectoid R-algebra is (by the theorem) the same thing as a perfect prism (B, J) together with a map $A/I \to B/J$, which by the lemma lifts uniquely to a map of δ-rings $A \to B$; then we can reduce modulo p to get a map $A/p = R^\flat \to B/p = (B/J)^\flat$. Conversely, since A/p is also perfect, we can pull the same trick in reverse: maps $A/p \to B/p$ lift uniquely to maps of δ-rings $A \to B$, which reduce modulo I to maps $R = A/I \to B/IB = B/J$ (by rigidity). $\square$

Since the lemma actually puts no restriction on (B, J), this even shows one way the tilting equivalence can be strengthened.

3. The prismatic site

Fix a (bounded) prism (A, I), and let X be a p-adic formal scheme over A/I. (Last time, we only talked about $X = \operatorname{Spec} R$.) We define the relative prismatic site, written $(X/A)_{\Delta}$, to be the category of prisms (B, IB) over (A, I) together with a map $\operatorname{Spf} B/IB \to X$ over A/I, i.e. the induced map $\operatorname{Spf} B/IB \to \operatorname{Spf} A/I$ factors through X.

We can also define the absolute prismatic site $(X)_{\Delta}$ without fixing a base prism, simply as the category of prisms (B, J) together with maps $\operatorname{Spf} B/J \to X$. Since the category of prisms has no initial object, this is not a special case of the above definition.

We can topologize either category by taking covers to be faithfully flat maps of prisms; in the case where X is affine, this is equivalent to the indiscrete topology. This justifies the term "site."

On either the relative or absolute prismatic sites, we have sheaves $\mathcal{O}_{\Delta}$ and $\overline{\mathcal{O}}_{\Delta}$ sending (B, J) to B and B/J respectively.

(In fact, one could choose a variety of different topologies, such as Zariski or étale, instead of flat; one gets different sites with change-of-topology maps between them, and pushing forward our sheaves recovers the same sheaves. In other words prismatic cohomology is in a sense insensitive to what topology we use. The reason to use the flat topology is its relationship to the quasisyntomic topology, which we may hear more about in future talks.)

One can also relate prismatic sheaves to étale sheaves as follows. We have a functor $\nu : (X/A)_{\Delta} \to \mathbf{Sch}_X$ sending (B, J) to the map $\operatorname{Spf} B/J \to X$, which induces a map $\nu_* : \operatorname{Sh}((X/A)_{\Delta}) \to \operatorname{Sh}(X_{\text{ét}})$. Explicitly, if F is a sheaf on the prismatic site, $\nu_* F$ sends an étale cover $U \to X$ to the colimit of $F((B, J, \operatorname{Spf} B/J \to X))$ for $\operatorname{Spf} B/J \to X$ factoring through U. In particular, we can use this to define the commutative algebra objects

$$\Delta_{X/A} := R\nu_* \mathcal{O}_{\Delta} \in D(X_{\text{ét}}, A),$$

$$\overline{\Delta}_{X/A} := R\nu_* \overline{\mathcal{O}}_{\Delta} \in D(X_{\text{ét}}, \mathcal{O}_X),$$

with

$$\overline{\Delta}_{X/A} \simeq \Delta_{X/A} \otimes^{\mathbb{L}}_A A/I.$$

4. Hodge–Tate comparison

For any A/I-module M, we can define $M\{i\} = M \otimes_{A/I} I^i/I^{i+1}$. We then get a triangle

$$\overline{\Delta}_{X/A}\{i+1\} \to \Delta_{X/A} \otimes_A I^i/I^{i+2} \to \overline{\Delta}_{X/A}\{i\},$$

and taking cohomology gives a Bockstein differential

$$\beta_I : H^i(\overline{\Delta}_{X/A}\{i\}) \to H^{i+1}(\overline{\Delta}_{X/A}\{i+1\}).$$

Varying i, since $\overline{\Delta}_{X/A}$ is a commutative algebra object these assemble to a graded ring $H^*(\overline{\Delta}_{X/A}\{*\})$. Assume that X is smooth. The zeroth term $H^0(\overline{\Delta}_{X/A})$ is by definition an $\mathcal{O}_X$-algebra, with structure map $\eta^0_X : \mathcal{O}_X \to H^0(\overline{\Delta}_{X/A})$. By the universal property of Kähler differentials together with a strict graded commutativity result, this extends to a map of differential graded A/I-algebras

$$\eta^*_X : \Omega^*_{X/(A/I)} \to H^*(\overline{\Delta}_{X/A}\{*\}).$$

Theorem (Hodge–Tate comparison). *The map η_X^* is an isomorphism.*

In particular, it follows that $\overline{\Delta}_{X/A}$ is a perfect complex, with ith cohomology given by $\Omega^*_{X/(A/I)}$.

This is proven by passing first to characteristic p, where we first prove a comparison theorem with crystalline cohomology and then relate this back to Hodge–Tate cohomology (via the Cartier isomorphism), and then by comparing prismatic cohomology over different base prisms, in particular over crystalline prisms vs. general bounded ones.

5. Étale comparison

Another cohomology theory we might hope to recover is mod p cohomology of the generic fiber. In particular, suppose that X is a p-adic formal scheme over a base ring R. Then we can form its adic generic fiber $X_\eta := X \times_{\mathrm{Spf}\, R} \mathrm{Spa}(R[1/p], R)$. We take $\mu : X_{\eta,\text{ét}} \to X_{\text{ét}}$ to be the nearby cycles functor, with $\mu_* : \mathrm{Sh}(X_{\eta,\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$ being the pushforward. (I'm not sure what the difference between this and just taking the pushforward of the inclusion of the generic fiber is, maybe one of you can explain it; I think what is happening is that because we're in the formal schemes world, the generic fiber doesn't exist in the usual sense, and so we have to do this more complicated nearby cycles thing to get something genuinely on $X_{\text{ét}}$.)

On the one hand, we want to study the étale cohomology of X_η with coefficients in $\mathbb{F}_p$. On the other, prismatic cohomology gives us an object on $X_{\text{ét}}$; the relationship is exactly what one would hope, except that we have to modify prismatic cohomology somewhat.

Theorem (Étale comparison). *With notation as above, suppose that R is perfectoid, corresponding to the perfect prism $(A, (d))$. Then there is a canonical identification*

$$R\mu_*\mathbb{F}_p \simeq (\Delta_{X/A}[1/d]/p)^{\phi=1},$$

where ϕ denotes the Frobenius on $\Delta_{X/A}$. More generally, for any $n > 1$ we have an identification

$$R\mu_*\mathbb{Z}/p^n \simeq (\Delta_{X/A}[1/d]/p^n)^{\phi=1}.$$

In particular, if $X = \mathrm{Spf}\, S$ is affine, the left-hand side is $R\Gamma(\mathrm{Spec}\, S[1/p], \mathbb{Z}/p^n)$, and so this lets us understand the étale cohomology.

One can also (apparently) get an identification even without inverting d: if X is a p-adic formal R-scheme for $R = A/(d)$ perfectoid, then

$$\mathbb{Z}/p^n \simeq (\Delta_{X/A}/p^n)^{\phi=1}.$$

6. The Nygaard filtration

Suppose that $R = A/(d)$ is a perfectoid ring. By the Hodge–Tate comparison, we have an increasing multiplicative filtration $\mathrm{Fil}_i\, \overline{\Delta}_{R/A}$ on $\overline{\Delta}_{R/A}$ whose associated gradeds recover the Kähler differentials (or shifts of exterior powers of the cotangent complex). We might wonder if this comes from some filtration on $\Delta_{R/A}$; the answer is yes, up to a twist.

We define the Nygaard filtration $\mathrm{Fil}_N^i A$ to be the set of $x \in A$ such that $\phi(x) \in d^i A$. In particular, $\mathrm{Fil}_N^1 = \phi^{-1}(I)A$.

This is a decreasing multiplicative filtration on A. The main property is that, up to a twist, it lifts the above filtration: the image of $\mathrm{Fil}_N^i A$ under $\frac{\phi}{d^i}$ and reduction modulo d agrees with $\mathrm{Fil}_i \overline{\mathbb{\Delta}}_{R/A}$ on $\overline{\mathbb{\Delta}}_{R/A} = A/d = R$.

Now, Hodge–Tate comparison does not require the perfectoid condition, so we would like this to hold more generally. However, to define this filtration we need to have this initial prism $(A, (d))$, which we don't have for fully arbitrary R. It turns out that the right condition to impose is that R be quasiregular semiperfectoid. Here the "quasiregular" part should be thought of as a technical condition, and "semiperfectoid" means essentially that it is a quotient of a perfectoid ring. In particular if R is quasiregular semiperfectoid, it admits a map from some perfectoid $A/(d)$ for $(A, (d))$ a perfect prism; if there is a prism (B, J) with $B/J = R$, then it follows that this map lifts uniquely to a map of prisms $(A, (d)) \to (B, J)$, so by rigidity $J = (d)$. It turns out that such a prism always exists: namely under these conditions, $B = \mathbb{\Delta}_R = \mathbb{\Delta}_{R/A}$, the prismatic cohomology of R, is a δ-ring whose quotient by d recovers R.

Therefore we can do all the above with R quasiregular semiperfectoid, replacing throughout A by $\mathbb{\Delta}_R$.

Why does this improve our situation? Because it turns out that we can think of the prismatic complex and its Nygaard filtration as living on the quasisyntomic site of X (in this case $\mathrm{Spf}\, R$), and quasiregular semiperfectoid rings give a basis for this site. In other words we can assume X is locally of the form $\mathrm{Spf}\, R$ for R quasiregular semiperfectoid, and then define the Nygaard filtration for all X by descent.

One can also do all this in the relative setting, rather than absolute. The filtration is critical for the comparison to syntomic cohomology and the topological connections, which are beyond the scope of this introductory lecture.

References

[1] Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. *arXiv preprint arXiv:1905.08229*, 2019.