Introduction to motivic cohomology

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We present a distillation of the first two parts of the book on motivic cohomology by Mazza, Voevodsky, and Weibel [1], with a view towards the norm residue isomorphism theorem.

Contents

1 Motive-ation

Let A be an abelian group and let k be a field. In motivic cohomology, we define functors

$$H^{p,q}(-,A) : (\mathbf{Sm}/k)^{\mathrm{op}} \to \mathbf{Ab}$$

that restrict to interesting invariants, e.g. Milnor K-theory and étale cohomology, for various choices of p, q. Here q is interpreted as a weight, while p corresponds to the traditional indexing of cohomology groups. For example, for connected X we have

$$H^{0,0}(X,A) = A \qquad \text{and} \qquad H^{p,0}(X,A) = 0$$

for $p \neq 0$. For $q = 1$ and any X, we have

$$H^{1,1}(X,\mathbb{Z}) = \mathcal{O}^*(X), \quad H^{2,1}(X,A) = \mathrm{Pic}(X), \quad \text{and} \quad H^{p,1}(X,\mathbb{Z}) = 0$$

for $p \neq 1, 2$.

In fact, one can interpret all motivic cohomology groups as "higher Chow groups":

$$H^{p,q}(X,A) = \mathrm{CH}^q(X, 2q-p, A).$$

When $p = 2q$ and $A = \mathbb{Z}$, these give the ordinary Chow groups $\mathrm{CH}^q(X)$. This leads to important links between motivic cohomology and algebraic K-theory and intersection theory, which we will not pursue further here because our current goal is to understand the norm residue isomorphism theorem. Similarly, we will also not discuss the important relation to the derived category of motives, for though this may be relevant to our goal, this is merely an introduction to this topic of of motivic cohomology.

Recall the definition of Milnor K-theory.

Definition 1.1 (Milnor K-theory). *Let F be a field. The Milnor K-theory $K_n^M(F)$ is defined as the graded ring*

$$K_*^M(F) := \frac{T(F^*)}{(a \otimes (1-a))}$$

for $a \neq 0, 1$.

Here, $\otimes$ is multiplication operation, so sometimes we instead present $K_n^M(F)$ as the ring generated by symbols $\{a\}$ with $a \in F^*$, subject to the relations $\{a\}+\{b\} = \{ab\}$ and $\{a, 1-a\} = 0$, where $\{a, b\} := \{a\}\{b\}$. It follows from basic field axioms that $\{a, -a\} = 0$ and $\{a, b\} = -\{b, a\}$, and from there that if the sum of any nontrivial subset of the set $\{a_i\}_{i=1}^n$ is 0 or 1, then $\{a_1, a_2, \ldots, a_n\} = 0$.

We will see that Milnor K-theory is a special case of motivic cohomology. To be precise, we have the following theorem.

Theorem 1.2 ([1], Theorem 5.1). *For all $p \geq 0$, we have*

$$H^{p,p}(k, \mathbb{Z}) \cong K_p^m(k).$$

Next, we also have comparison theorems for étale cohomology. In fact, we will first work with *étale motivic cohomology*. Indeed, motivic cohomology is defined as the hypercohomology of certain complexes of sheaves $A(q)$ over the Zariski site of X. If we use the étale site instead, we can define étale motivic cohomology along the same lines as

$$H_L^{p,q}(X, A) := \mathbb{H}_{et}^p(X, A(q)|_{X_{et}}).$$

Then we have the following theorem.

Theorem 1.3 ([1], Theorem 10.2). *Let $(n, \operatorname{char} k) = 1$. Then*

$$H_L^{p,q}(X, \mathbb{Z}/n) = H_{et}^p(X, \mu_n^{\otimes q}) \quad \textit{for} \quad q \geq 0, p \in \mathbb{Z}.$$

Let $l \neq \operatorname{char} k$. Recall that the norm residue homomorphism is defined by treating elements $\{a\} \in K_1^M(k)/l$ as elements of $H_{et}^1(k, \mu_l)$ and identifying product of symbols on the left to cup products on the right. Then we have:

Theorem 1.4 (Norm residue isomorphism theorem). *The norm residue homomrphism*

$$K_n^M(k)/l \to H_{et}^n(k, \mu_l^{\otimes n})$$

is an isomorphism.

We can restate this in terms of motivic cohomology. Using the comparison theorems listed above it is simply the $p = q$ case of the following theorem.

Theorem 1.5. *Let X be a smooth variety over a field containing $1/l$. Then the change-of-topology map*

$$H^{p,q}(X, \mathbb{Z}/l) \to H_L^{p,q}(X, \mathbb{Z}/l)$$

is an isomorphism for all $p \leq q$.

Finally, we recall that the proof of this is achieved by induction with the $H90(n)$ property, which states that

$$H_L^{n+1,n}(k, \mathbb{Z}_{(l)}) = H_L^{n+1}(k, \mathbb{Z}_{(l)}(n)) = 0.$$

2 Correspondences and presheaves with transfers

To define the motivic complexes which will give us motivic cohomology, we will first enlarge the category of smooth varieties by considering correspondences. Presheaves with transfers are covariant functors on the category of correspondences. Motivic complexes are certain complexes of presheaves of transfers.

2.1 The category of correspondences

Let $X, Y \in \mathbf{Sm}_k$ be smooth separated schemes of finite type over k. The category of correspondences $\mathbf{Cor}_k$ has the same objects as $\mathbf{Sm}_k$ but has more morphisms, also known as correspondences. Very informally, one can think of $\mathbf{Cor}(X, Y)$ as a generalization of $\mathrm{Mor}(X, Y)$ to multi-valued morphisms.

Definition 2.1. *An **elementary correspondence** between a smooth connected scheme X/k to a separated scheme Y/k is an irreducible closed subset $W \subset X \times Y$ whose associated integral subscheme is finite and surjective over X.*

If X is not connected, then an elementary correspondence refers to one that is one from a connected component of X to Y.

*The group $\mathbf{Cor}(X, Y)$ of **finite correspondences** is the free abelian group generated by the elementary correspondences.*

Then given a closed subscheme $Z \subset X \times Y$ finite and surjective over X, we can associate the finite correspondence $\sum n_i W_i$ where W_i are the irreducible components of the support of Z surjective over a component of X with generic points ξ_i and $n_i = \mathrm{length}\, O_{Z,\xi_i}$.

Having defined the morphisms $\mathbf{Cor}(X, Y)$, we will now define the composition of correspondences $V \in \mathbf{Cor}_k(X, Y)$ and $W \in \mathbf{Cor}_k(Y, Z)$ as follows. Construct the cycle $[T] = (V \times Z) \cdot (X \times W)$ on $X \times Y \times Z$. Note that this involves using Serre's Tor formula. Then take its pushforward along the projection $p : X \times Y \times Z \to X \times Z$. We recall that the pushforward of a cycle W of X along some morphism $p : X \to Y$ it is finite along is defined as $f_* W = [k(W) : k(V)]\,\mathrm{im}(W)$.

Proposition 2.2. *The category $\mathbf{Sm}_k$ embeds into $\mathbf{Cor}_k$ where $f : X \to Y$ becomes the graph $\Gamma_f \subset X \times Y$.*

Indeed, looking at the base change

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times_k Y \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta_k} & Y \times_k Y \end{array}$$

shows that the separatedness of Y implies that Γ_f is a closed embedding. Furthermore, $\gamma_g \circ \Gamma_f = \Gamma_{g \circ f}$.

Furthermore, $\mathbf{Cor}_k$ is a symmetric monoidal category. Indeed, the tensor product is simply $X \otimes Y = X \times Y$. Given $V \in \mathbf{Cor}_k(X, X')$ and $W \in \mathbf{Cor}_k(Y, Y')$, we get the desired cycle $V \times W \in \mathbf{Cor}_k(X \otimes Y, X' \otimes Y')$.

Examples

1. $\mathbf{Cor}_k(\mathrm{Spec}\, k, X)$ is generated by the 0-cycles of X.
2. $\mathbf{Cor}_k(X, \mathrm{Spec}\, k)$ is generated by the irreducible components of X.
3. Let x be a closed point of X and consider it as a correspondence in $\mathbf{Cor}(\mathrm{Spec}(k), X)$. Then the composition $\mathrm{Spec}\, k \to X \to \mathrm{Spec}\, k$ is given by the degree $[k(x) : k] \in \mathbb{Z} \cong \mathbf{Cor}_k(\mathrm{Spec}\, k, \mathrm{Spec}\, k)$ and the composition $X \to \mathrm{Spec}\, k \to X$ is given by $X \times x \subset X \times X$.

4. Take $W \in \mathbf{Cor}_k(\mathbb{A}^1, X)$ and two k-points $s, t : \operatorname{Spec} k \to \mathbb{A}^1$. Then the zero-cycles $W \circ \Gamma_s$ and $W \circ \Gamma_t$ are rationally equivalent.

2.2 Presheaves with transfers

Definition 2.3. *A **presheaf with transfers** is a contravariant additive functor $F : \mathbf{Cor}_k \to \mathbf{Ab}$.*

Additivity gives a map

$$\mathbf{Cor}_k(X, Y) \otimes F(Y) \to F(X).$$

Thus there are extra "transfer maps" $F(Y) \to F(X)$ coming from $\mathbf{Cor}_k(X, Y)$. We define $\mathbf{PST}(k)$ to be the functor category $\mathbf{Ab}^{\mathbf{Cor}_k}$.

Proposition 2.4. $\mathbf{PST}(k)$ *is an abelian category with enough injectives and projectives.*

This only uses the fact that $\mathbf{Cor}_k$ is a small category.

Examples

Example 2.5. *The constant presheaf A on $\mathbf{Sm}_k$ can be extended to a presheaf with transfers. Indeed, for $W \in \mathbf{Cor}(X, Y)$ with X, Y connected, the corresponding homomorphism $A \to A$ is multiplication by the degree of W over X.*

Example 2.6. *$\mathcal{O}^*$ and $\mathcal{O}$, at least for X normal. Use the norm and trace maps. The same holds for the subsheaves $\mu_n \subset \mathcal{O}^*$ and $k \subset \mathcal{O}$.*

$$\begin{tikzcd}
\mathcal{O}^*(Y) \arrow[rr] \arrow[dr] & & \mathcal{O}^*(X) \\
& \mathcal{O}^*(W) \arrow[ur, "N"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}(Y) \arrow[rr] \arrow[dr] & & \mathcal{O}(X) \\
& \mathcal{O}(W) \arrow[ur, "\mathrm{Tr}"'] &
\end{tikzcd}$$

Example 2.7. $\mathrm{CH}^i(-)$*, the Chow groups. We define the maps*

$$\phi_W : \mathrm{CH}^i(Y) \to \mathrm{CH}^i(X) \quad \textit{by} \quad \phi_W(\alpha) = q_*(W \cdot p^*\alpha).$$

Example 2.8. *Representable functors: $h_X(-)$. These are denoted by $\mathbb{Z}_{tr}(X)$; we will now investigate them further.*

2.3 Representable functors of $\mathbf{Cor}_k(X)$

Take $X \in \mathrm{Ob}(\mathbf{Cor}_k(X))$. We denote

$$\mathbb{Z}_{tr}(X) \coloneqq h_X(-).$$

By Yoneda, $\mathbb{Z}_{tr}(X)$ is a projective object in $\mathbf{PST}(k)$ – we are working with presheaves, not sheaves!

Note that $\mathbb{Z}_{tr}(\operatorname{Spec} k)$ is just the constant sheaf $\mathbb{Z}$ on $\mathbf{Sm}_k$, with the transfer maps constructed in the example from the previous subsection. Let (X, x) be a pointed scheme. We define

$$\mathbb{Z}_{tr}(X, x) \coloneqq \operatorname{coker}[x_* : \mathbb{Z} \to \mathbb{Z}_{tr}(X)].$$

The structure map $X \to \operatorname{Spec} k$ provides a splitting, so

$$\mathbb{Z}_{tr}(X) \cong \mathbb{Z} \oplus \mathbb{Z}_{tr}(X, x).$$

We can extend this to decompose a product in the following way. (Out of laziness we screenshot the following from Voevodsky's lectures.)

DEFINITION 2.12. If (X_i, x_i) are pointed schemes for $i = 1, \dots, n$ we define $\mathbb{Z}_{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$, or $\mathbb{Z}_{tr}(X_1 \wedge \cdots \wedge X_n)$, to be:

$$\operatorname{coker}\left(\bigoplus_i \mathbb{Z}_{tr}(X_1 \times \cdots \hat{X}_i \cdots \times X_n) \xrightarrow{id \times \cdots \times x_i \times \cdots \times id} \mathbb{Z}_{tr}(X_1 \times \cdots \times X_n)\right).$$

LEMMA 2.13. *The presheaf $\mathbb{Z}_{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$ is a direct summand of $\mathbb{Z}_{tr}(X_1 \times \cdots \times X_n)$. In particular, it is a projective object of* **PST**.

Moreover, the following sequence of presheaves with transfers is split-exact:

$$0 \to \mathbb{Z} \xrightarrow{\{x_i\}} \oplus_i \mathbb{Z}_{tr}(X_i) \to \oplus_{i,j} \mathbb{Z}_{tr}(X_i \times X_j) \to \cdots$$
$$\cdots \to \oplus_{i,j} \mathbb{Z}_{tr}(X_1 \times \cdots \hat{X}_i \cdots \hat{X}_j \cdots \times X_n) \to \oplus_i \mathbb{Z}_{tr}(X \times \cdots \hat{X}_i \cdots \times X_n) \to$$
$$\to \mathbb{Z}_{tr}(X_1 \times \cdots \times X_n) \to \mathbb{Z}_{tr}(X_1 \wedge \cdots \wedge X_n) \to 0.$$

By convention, $\mathbb{Z}_{tr}((X, x)^{\wedge q})) = \mathbb{Z}$ for $q = 0$ and 0 for $q < 0$.

Consider the pointed scheme $(\mathbb{G}_m, 1)$. We will be interested in the presheaf with transfers $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})$.

2.4 Simplicial structure

Before continuing, we recall the construction of the algebraic simplex

$$\Delta^n = \operatorname{Spec} k[x_0, \dots, x_n]/(x_0 + \cdots + x_n - 1).$$

Recall that a simplicial object of a category C is a functor $F : \Delta^{op} \to C$. The algebraic simplices Δ^k glue together to form a *cosimplicial* scheme $\Delta^\bullet$. The structure is determined by the face maps

$$d^i : \Delta^n \to \Delta^{n+1} \quad \text{induced by} \quad x_i = 0 \text{ and shifting coordinates.}$$

and degeneracy maps

$$s^i : \Delta^{n+1} \to \Delta^n \quad \text{induced by} \quad x_i \mapsto x_i + x_{i+1} \text{ and shifting coordinates.}$$

Then if F is a presheaf of abelian groups on $\mathbf{Sm}_k$, then $F(U \times \Delta^\bullet)$ is a simplicial abelian group. Then

$$C_\bullet F : U \mapsto F(U \times \Delta^\bullet)$$

is a simplicial presheaf with transfers. Now recall the **Moore complex**, which takes in a simplicial abelian group and gives a chain complex of abelian groups using the alternating sum of the face maps. For example, if X is a topological space, then $\mathbb{Z}[\operatorname{sing} X]$ is a simplicial abelian group, and the homology of its Moore complex $C_*(\mathbb{Z}[\operatorname{sing} X])$ is the singular homology of X. In our situation, $C_* F(U)$ is the complex of abelian groups

$$\cdots \to F(U \times \Delta^2) \xrightarrow{d_0 - d_1 + d_2} F(U \times \Delta^1) \xrightarrow{d_0 - d_1} F(U) \to 0.$$

In general, recall that the Dold-Kan correspondence gives an equivalence between the category of simplicial abelian groups and nonnegative chain complexes. This functor

$$A_\bullet \to NA_*$$

sends a simplicial abelian group not to its Moore complex, but to its *normalized complex*. Here, $C_*^{DK}(A)_n$ consists of the subgroup of $C_*(A)_n$ killed by d_i for $i < n$, and the differential $d_n : C_*^{DK}(A)_n \to C_*^{DK}(A)_{n-1}$ is given by $(-1)^n d_n$. The normalized complex is homotopically equivalent to the Moore complex (and isomorphic to the quotient of the Moore complex by degenerate simplices). For instance, in the case of a constant presheaf with transfers A, we have the Moore complex

$$C_*(A) : \cdots \to A \xrightarrow{\mathrm{id}} A \xrightarrow{0} A \to 0$$

while $C_*^{DK}(A)$ consists of just A in degree 0.

2.5 Homotopy invariant presheaves

Definition 2.9. *A presheaf F is **homotopy invariant** if for every X, the map $p^* : F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism.*

Note that this is equivalent to p^* being surjective. We can check that an equivalent condition is that for all X, we have

$$i_0^* = i_1^* : F(X \times \mathbb{A}^1) \to F(X).$$

Furthermore, if F is any presheaf, we have that $i_0^*, i_1^* : C_*F(X \times \mathbb{A}^1) \to C_*F(X)$ are chain homotopic. From this we deduce that if F is a presheaf, then the homology presheaves

$$H_n C_* F : X \mapsto H_n C_* F(X)$$

are homotopy invariant for all n.

Definition 2.10. *Two finite correspondences from X to Y are $\mathbb{A}^1$**-homotopic** if they are the restrictions along $X \times 0$ and $X \times 1$ of an element of* **Cor**$(X \times \mathbb{A}^1, Y)$.

This is an equivalence relation on **Cor**(X, Y). Note that it is not one if we just look at morphisms of schemes! With this definition though, we define $f : X \to Y$ to be an $\mathbb{A}^1$-homotopy equivalence in the expected way.

3 Motivic complexes and motivic cohomology

3.1 The motivic complex

Definition 3.1. *For $q \in \mathbb{Z}_{\geq 0}$, the **motivic complex** $\mathbb{Z}(q)$ is defined as the following complex of presheaves with transfers.*

$$\mathbb{Z}(q) := C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q].$$

We can change coefficients to $A \in$ **Ab** by setting $A(q) = \mathbb{Z}(q) \otimes A$.

For example when $q = 0$, applying this to a scheme Y we just get

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

which is quasi-isomorphic to just $\mathbb{Z}$. When $q = 1$, the complex looks like

$$\cdots \to \mathbf{Cor}(Y \times \Delta^2, \mathbb{G}_m) \to \mathbf{Cor}(Y \times \Delta^1, \mathbb{G}_m) \to \mathbf{Cor}(Y, \mathbb{G}_m) \to 0 \to \cdots$$

(The first 0 is degree 2, and the degrees are increasing.)

Proposition 3.2. *The presheaf with transfers $\mathbb{Z}_{tr}(Y)$ is a sheaf in the Zariski topology.*

Proof. Let $\{U_1, U_2\}$ be a covering of U. We have to show the exactness of

$$0 \to \mathbf{Cor}_k(U, Y) \to \mathbf{Cor}_k(U_1, Y) \oplus \mathbf{Cor}_k(U_2, Y) \to \mathbf{Cor}_k(U_1 \cap U_2, Y))$$

We may assume that U is connected and irreducible. Note that if two closed subschemes of $U_i \times Y$ agree on a dense open subset of $X \times Y$, then they must be the same. Thus $\mathbf{Cor}_k(U, Y)$ injects into $\mathbf{Cor}_k(U_i, Y)$. In the second place, if Z_1 and Z_2 are elementary correspondences that coincide on $(U_1 \cap U_2) \times Y$, then we can simply take their union $Z_1 \cup Z_2 \subset \mathbf{Cor}_k(U \times Y)$ that restricts to both of them.

□

The same result holds for $A_{tr}(Y)$, and also for $A(q)$ because the latter is a direct summand of the former. In fact, these are also sheaves in the étale topology, which we will use when considering étale motivic cohomology.

3.2 Motivic cohomology groups

Definition 3.3. *The* ***motivic cohomology groups*** *$H^{p,q}(X, A)$ are defined to be the hypercohomology of the motivic complexes $A(q)$ with respect to the Zariski topology:*

$$H^{p,q}(X, A) = \mathbb{H}^p_{Zar}(X, A(q)).$$

Recall that to compute the hypercohomology, we take an injective resolution of the complex of sheaves, apply the global section functor, and compute cohomology. In general, we have two spectral sequences for hypercohomology:

$$E_1^{r,s} = R^s f A^r \Rightarrow \mathbb{R}^{r+s} f A^\bullet \quad \text{and} \quad E_2^{r,s} = R^r f(H^s(A^\bullet)) \Rightarrow \mathbb{R}^{r+s} f A^\bullet.$$

Using the second, the fact that $H^s(A(q))$ vanishes for $s > q$, and Grothendieck's vanishing theorem, we have that $H^{p+q}(X, A) = 0$ when $p > q + \dim X$.

3.3 Weight 1

There is a quasi-isomorphism

$$\mathbb{Z}(1) \xrightarrow{\cong} \mathcal{O}^*[-1].$$

Thus we have the following table.

$H^{-2,2}$	$H^{-1,2}$	$H^{0,2}$	$H^{1,2}$	$H^{2,2}$	$H^{3,2}$
0	0	0	$\mathcal{O}^*(X)$	$\mathrm{Pic}(X)$	0
0	0	$\mathbb{Z}(X)$	0	0	0
0	0	0	0	0	0

FIGURE 4.1. Weight q motivic cohomology

Idea of proof. We define the functor $M^*(\mathbb{P}^1; 0, \infty) : (\mathbf{Sm}/k)^{op} \to \mathbf{Ab}$ sending a scheme X to the group of rational functions on $X \times \mathbb{P}^1$ which are regular in a neighborhood of $X \times \{0, \infty\}$ and equal to 1 on $X \times \{0, \infty\}$. Going from f to $D(f)$ gives a morphism of sheaves $M^*(\mathbb{P}^1; 0, \infty) \to \mathbb{Z}_{tr}(\mathbb{G}_m)$. In fact, we have an exact sequence

$$0 \to C_*(M^*(\mathbb{P}^1; 0, \infty)) \to \mathbb{Z}(1)[1] \to C_*(\mathcal{O}^*) \to 0.$$

Then we show using simplicial methods that $C_*(M^*(\mathbb{P}^1; 0, \infty))$ is acyclic. The result follows. $\square$

We are also interested in the complex $\mathbb{Z}/l(1)$. The results are the following.

Proposition 3.4. *For $(l, \operatorname{char} k) = 1$ and X smooth, we have $H^{p,1}(X, \mathbb{Z}/l) = 0$ for $p \neq 0, 1, 2$ and*

$$H^{0,1}(X, \mathbb{Z}/l) = \mu_l(X), \quad H^{1,1}(X, \mathbb{Z}/l) = H^1_{et}(X, \mu_l), \quad H^{2,1}(X, \mathbb{Z}/l) = \operatorname{Pic}(X)/l \operatorname{Pic}(X).$$

The first statement follows from a universal coefficient theorem. We will discuss the rest more when we discuss étale motivic cohomology.

3.4 Milnor K-theory and the diagonal

We have the following comparison theorem.

Theorem 3.5. *For any field F and any n we have*

$$H^{n,n}(\operatorname{Spec} F, \mathbb{Z}) \cong K^M_n(F).$$

The proof is quite long and complicated, but doesn't use anything particularly advanced.

4 Étale motivic cohomology

4.1 Definitions

Recall that the check that a presheaf $F : (\mathbf{Sm}/k)^{\mathrm{op}} \to \mathbf{Ab}$ is an étale sheaf, it suffices to check the following conditions.

1. For any surjective étale morphism $U \to X$,

   $$0 \to F(X) \to F(U) \rightrightarrows F(U \times_X U) \to 0$$

   is exact.

2. $F(X \coprod Y) = F(X) \oplus F(Y)$.

We naturally have a notion of étale sheaves with transfers, which form a category $\mathbf{Sh}_{et}(\mathbf{Cor}_k)$.

Proposition 4.1. *For any scheme T/k, we have that $\mathbb{Z}_{tr}(T)$ is an étale sheaf.*

We already proved this for the Zariski topology. This proof is a little more elaborate, because the fiber product here is not just the intersection. We need to use some results on flat pullback of cycles and faithfully flat descent. Similar to before, this result implies that the motivic complexes $A(n)$ are complexes of étale sheaves.

Now we can define étale motivic cohomology, where the L stands for Lichtenbaum.

Definition 4.2. *The **étale motivic cohomology groups** $H^{p,q}_L(X, A)$ are defined to be the hypercohomology of the motivic complexes $A(q)$ with respect to the Zariski topology:*

$$H^{p,q}_L(X, A) = \mathbb{H}^p_{et}(X, A(q)|_{X_{et}}).$$

Remark. There is another natural way to define étale motivic cohomology using a certain triangulated category. These agree in some but not all cases.

4.2 Computations

For $q = 0$ we get $H_L^{p,0}(X, A) \cong H_{et}^p(X, A)$. We will now consider the cohomology of the complexes $\mathbb{Z}/n(1)$ both for the Zariski topology and the étale topology. Recall that there is a quasi-isomorphism of Zariski sheaves

$$\mathbb{Z}(1) \cong \mathcal{O}^*[-1].$$

Tensoring with $\mathbb{Z}/l$ we have $\mathbb{Z}/l(1) \cong \mathcal{O}^*[-1] \otimes^L \mathbb{Z}/l$, which is the complex $[\mathcal{O}^* \xrightarrow{l} \mathcal{O}^*]$ in degrees 0 and 1. She

These are étale sheaves, and we may take any n with $(n, \operatorname{char} k) \neq 1$ and thus obtain we have a quasi-isomorphism of complexes of étale sheaves

$$\mathbb{Z}/n(1) \cong \mu_n.$$

Now recall the claimed results

$$H^{0,1}(X, \mathbb{Z}/l) = \mu_l(X), \quad H^{1,1}(X, \mathbb{Z}/l) = H_{et}^1(X, \mu_l), \quad H^{2,1}(X, \mathbb{Z}/l) = \operatorname{Pic}(X)/l \operatorname{Pic}(X).$$

Consider the change of topology map

$$H^{*,1}(X, \mathbb{Z}/l) \to H_L^{*,1}(X, \mathbb{Z}/l) = H_{et}^1(X, \mu_l).$$

Please read the proof of Corollary 4.9 in [1] to see how to use this to show the claimed results.

Back to étale motivic cohomology, the quasi-isomorphism of complexes of étale sheaves $\mathbb{Z}/n(1) \cong \mu_n$ yields that for $q = 1$, we have $H_L^{p,q}(X, \mathbb{Z}/n) \cong H_{et}^1(X, \mu_n)$. Finally, for all q, there is the following theorem.

Theorem 4.3. *For* $(n, \operatorname{char} k) = 1$*, we have*

$$H_L^{p,q}(X, \mathbb{Z}/n) = H_{et}^p(X, \mu_n^{\otimes q}) \quad \textit{for } q \geq 0, p \in \mathbb{Z}.$$

A reformulation of this theorem states that $\mu_n^{\otimes q} \to \mathbb{Z}/n(q)$ is a quasi-isomorphism of complexes of étale sheaves. The proof of this involves $\mathbb{A}^1$-homotopy and various other concepts we have not introduced here.

References

[1] C. Mazza, V. Voevodsky, C. Weibel, *Lecture notes on motivic cohomology* `https://sites.math.rutgers.edu/~weibel/MVWnotes/prova-hyperlink.pdf`