Overview of the Langlands program

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1 The Langlands correspondence for GL_n

Let F be either a local or global field. The Langlands correspondence predicts a correspondence between certain Galois representations into $\mathrm{GL}_n(\mathbb{C})$ and admissible representations of $\mathrm{GL}_n(F)$ (if F is local) or automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ (if F is global). Unfortunately, the "Galois representations" here aren't literally representations of Gal_F, so we'll hand-wave away some definitions.

Theorem 1 (Local reciprocity)**.** If F is local, there is a bijection between irreducible admissible representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_n(\mathbb{C})$-conjugacy classes of certain irreducible representations $\rho : W_F' \to \mathrm{GL}_n(\mathbb{C})$, where W_F' is the "Weil-Deligne group." This bijection should be compatible with taking L-functions and local class field theory.

Conjecture 1 (Global reciprocity)**.** If F is global, there is a bijection between cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_n(\mathbb{C})$-conjugacy classes of certain irreducible representations $\rho : L_F \to \mathrm{GL}_n(\mathbb{C})$ (where we don't know what the Langlands group L_F is supposed to be). This bijection should be compatible with taking L-functions and the local Langlands correspondence.

2 The Langlands dual group

If we replace GL_n (on the automorphic side) by another connected reductive group G, it makes sense to talk about admissible representations of $G(F)$ or automorphic representations of $G(\mathbb{A}_F)$. However, we must replace $\mathrm{GL}_n(\mathbb{C})$ (on the Galois side) by a different group, namely the **Langlands dual group** ${}^LG$.

Let $(X, \Phi, X^\vee, \Phi^\vee)$ denote the root datum for $G_{F^{\mathrm{sep}}}$ with respect to some maximal split torus $T \subset G_{F^{\mathrm{sep}}}$. We can take the dual root datum $(X^\vee, \Phi^\vee, X, \Phi)$ and get a unique connected reductive group $\widehat{G}$ over $\mathbb{C}$, which will be the identity component of ${}^LG$. Here are some examples:

G	$\widehat{G}$
GL_n	GL_n
SL_n	PGL_n
PGL_n	SL_n
Sp_{2n}	SO_{2n+1}
SO_{2n+1}	Sp_{2n}
SO_{2n}	SO_{2n}

To get the Langlands dual group ${}^LG$, we must take a semidirect product: ${}^LG := \widehat{G}(\mathbb{C}) \rtimes \mathrm{Gal}_F$. It takes some time to define this action, so I'll just provide some properties:

- If G is split, the action is trivial, so that ${}^L G \cong \widehat{G}(\mathbb{C}) \times \mathrm{Gal}_F$.
- (Satake isomorphism) If the action of Gal_F on $\widehat{G}(\mathbb{C})$ is unramified at some place v (this happens when $G_v := G_{F_v}$ is unramified, i.e. quasi-split over F_v and split over some unramified extension), then there is a bijection between unramified representations of G_v and "Fr_v-semisimple" $\widehat{G}(\mathbb{C})$-conjugacy classes in $\widehat{G}(\mathbb{C}) \rtimes \mathrm{Fr}_v$ (where Fr_v is any lift).

For G other than GL_n, the Galois side of the Langlands correspondence becomes "certain Galois representations into ${}^L G$ compatible with the projection to Gal_F." If G is split, these are the same as Galois representations into $\widehat{G}(\mathbb{C})$. More generally, if the action of Gal_F on $\widehat{G}$ factors through some $\mathrm{Gal}(E/F)$, then we can replace ${}^L G$ with its quotient $\widehat{G}(\mathbb{C}) \rtimes \mathrm{Gal}(E/F)$.

Here are some examples of Langlands dual groups for nonsplit G:

- $G = \mathrm{Res}_{E/F} H$ for split H over F: In this case, ${}^L G \cong \widehat{H}(\mathbb{C})^{\sigma : E \hookrightarrow F^{\mathrm{sep}}} \rtimes \mathrm{Gal}_F$, where Gal_F permutes the factors.
- G an inner form of a split G': An inner form of G' is a form corresponding to an element of $\mathrm{im}(H^1(\mathrm{Gal}_F, G^{*\mathrm{ad}}) \to H^1(\mathrm{Gal}_F, \mathrm{Aut}(G))$, i.e. we can pick the Galois descent data such that the automorphisms are all inner. Examples of inner twists of GL_n include multiplicative groups of n^2-dimensional central simple algebras. Taking Langlands duals doesn't distinguish between inner forms, so ${}^L G \cong {}^L G^* \cong \widehat{G'} \times \mathrm{Gal}_F$.

The local and global Langlands correspondences for general G are harder to formulate but have a similar shape: there is a surjective finite-to-one map from admissible/automorphic representations of G to certain "Galois representations" into ${}^L G$ that is compatible with taking L-functions.

3 Functoriality

A map of L-groups ${}^L G \to {}^L G'$ sends a Galois representation into ${}^L G$ to a Galois representation into ${}^L G'$. When G' is quasisplit, the "admissible" Galois representations appearing in the Langlands correspondence should be sent to admissible Galois representations. If we're to believe the Langlands correspondence, then we should be able to transfer automorphic representations of G to automorphic representations of G' in a way that is compatible with taking L-functions. This is Langlands' **principle of functoriality**. Over a local field, in the unramified case, this is already true thanks to the Satake isomorphism: the map ${}^L G \to {}^L G'$ sends the Satake parameter of an unramified representation of G to the Satake parameter of an unramified representation of G'.

Here are some known cases of global functoriality (over a number field F):

- Cyclic base change: For E/F cyclic of prime degree, the map ${}^L\,\mathrm{GL}_n \to {}^L\,\mathrm{Res}_{E/F}\mathrm{GL}_n$ is the diagonal map $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})^{\sigma : E \hookrightarrow \overline{F}}$.
- Cyclic automorphic induction: For E/F cyclic of prime degree, the map ${}^L\,\mathrm{Res}_{E/F}\mathrm{GL}_n \to {}^L\,\mathrm{GL}_{n[E:F]}$ is given by the representation of $\mathrm{GL}_n(\mathbb{C})^{\sigma : E \hookrightarrow F^{\mathrm{sep}}} \rtimes \mathrm{Gal}(E/F)$ obtained as the tensor product of the standard representation of $\mathrm{GL}_n(\mathbb{C})$ and the regular representation of $\mathrm{Gal}(E/F)$.
- Jacquet-Langlands (inner forms of GL_n): For an n^2-dimensional central simple algebra A, the map ${}^L A^\times \to {}^L\,\mathrm{GL}_n$ is the identity.
- Symmetric power functoriality: Consider the map ${}^L\,\mathrm{GL}_2 \to {}^L\,\mathrm{GL}_{k+1}$ corresponding to the kth symmetric power representation of GL_2. Functoriality is known for $k \leqslant 4$.

4 The global Langlands correspondence over function fields

When F is a global function field, we gain access to tools from geometry. Let F be the function field of some smooth projective curve $X/\mathbb{F}_q$. For convenience, let G be a connected split reductive group over F, which must then be the base change of a split reductive group over $\mathbb{F}_q$, so that we can speak of the group scheme G over X (in particular, at very point closed $v \in |X|$, G has an $\mathcal{O}_v$-structure).

Both sides of the global Langlands correspondence have geometric incarnations. Because of this geometry, we prefer to use ℓ-adic coefficients instead of $\mathbb{C}$-coefficients. Galois representations come from ℓ-adic sheaves on X or even finite products X^I. Automorphic forms can be studied thanks to Weil's uniformization: for any closed finite subscheme $N \subset X$, there is a natural bijection $G(F)\backslash G(\mathbb{A}_F)/K_N \cong \mathrm{Bun}_{G,N}(\mathbb{F}_q)$, where $K_N \subset G(\mathbb{A}_F)$ is the open compact subgroup $\ker(G(\mathbb{O}) \to G(\mathcal{O}_N))$ and $\mathrm{Bun}_{G,N}$ is the moduli stack of G-bundles on X with a trivialization of the restriction to N (i.e. level N structure). Thus, automorphic forms stabilized by K_N are certain functions on $\mathrm{Bun}_{G,N}(\mathbb{F}_q)$.

We now state the main theorem of Vincent Lafforgue, who proved a global Langlands parametrization of cuspidal automorphic forms on $G(\mathbb{A}_F)$ by "global Langlands parameters," which are $\widehat{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy classes of maps $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ defined on a finite extension of $\mathbb{Q}_\ell$, continuous, and semisimple (i.e. the Zariski closure of the image is reductive). We fix a lattice $\Xi \in Z(F)\backslash Z(\mathbb{A})$, so that the spaces $C_c^{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell)$ are finite dimensional.

Theorem 2 (Automorphic $\implies$ Galois)**.** There is a canonical decomposition of $C_c(K_N\backslash G(\mathbb{A})/K_N, \overline{\mathbb{Q}}_\ell)$-modules

$$C_c^{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/K_N\Xi, \overline{\mathbb{Q}}_\ell) = \bigoplus_\sigma \mathfrak{H}_\sigma,$$

where the σ are global Langlands parameters unramified outside N. This parametrization is compatible with the Satake isomorphism at places v outside N in the following sense: for an irreducible representation V of $\widehat{G}$, the Hecke operator $T(h_{V,v})$ acts on $\mathfrak{H}_\sigma$ by multiplication by the scalar $\chi_V(\sigma(\mathrm{Fr}_v))$, where χ_V is the character of V and Fr_V is any Frobenius lift.

The main geometric ingredients of the proof are geometric Satake and moduli stacks of shtukas, which we would really like to have in the mixed characteristic situation.

5 The geometric Langlands correspondence

The geometric Langlands program further geometrizes both sides of the Langlands correspondence over a function field. Instead of a finite field, we work with a curve X and a reductive group G over an algebraically closed field of characteristic 0, and for now, we only consider the unramified situation. On the Galois side, we have $\widehat{G}$-local systems, which correspond to everywhere unramified Galois representations. On the automorphic side, we view automorphic forms as traces of Frobenii associated to ℓ-adic sheaves; since we are in characteristic 0, the Riemann-Hilbert correspondence tells us we can replace ℓ-adic sheaves with D-modules.

Nowadays, the geometric Langlands correspondence reads

$$\text{D-Mod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\widehat{G}}(X)),$$

where both sides are stable ∞-categories (here, $\mathrm{LocSys}_{\widehat{G}}(X)$ is a DG-stack).