Goal of Ch 2

Recall the following assertions

$BK(n)$: $K_n^M(k)/\ell \xrightarrow{\cong} H^n_{et}(k, \mu_\ell^{\otimes n})$

$\forall k$, $\mathrm{char}(k) \neq \ell$.

$H90(n)$: $H^{n+1}_{et}(k, \mathbb{Z}_{(\ell)}(n)) = 0$ $\forall k$, $\mathrm{char}(k) \neq \ell$

$BK(n) \Rightarrow H90(n)$ (Thm 1.7)

$\pi: (Sm/k)_{et} \longrightarrow (Sm/k)_{Zar}$

$L(n) := \tau^{\leq n} R\pi_*[\mathbb{Z}_\ell(n)]$

$\mathbb{Z}/\ell^\nu(n) := \tau^{\leq n} R\pi_*[\mathbb{Z}/\ell^\nu\mathbb{Z}(n)]$

Use adjointness: $id \longrightarrow R\pi_*\pi^*$

$\rightsquigarrow \alpha_n: \mathbb{Z}/\ell^\nu\mathbb{Z}(n) \longrightarrow R\pi_*[\mathbb{Z}/\ell^\nu\mathbb{Z}(n)]$

has image in good truncation

$\rightsquigarrow \alpha_n: \mathbb{Z}/\ell^\nu\mathbb{Z}(n) \longrightarrow L/\ell^\nu(n)$.

$BL(n)$: $\mathbb{Z}/\ell\mathbb{Z}(n) \xrightarrow{\alpha_n} L/\ell(n)$ is

quasi-isom $\forall k$, $\mathrm{char}(k) \neq \ell$.

Chap 1: $BL(n) \longrightarrow BK(n) \longrightarrow H90(n)$

Chap 2:

$$\begin{array}{ccccc} BL(n) & \longleftarrow & BK(n) & \longleftarrow & H90(n) \\ \downarrow & & \downarrow & & \downarrow \\ BL(n-1) & & BK(n-1) & & H90(n-1) \end{array}$$

idea: Work on a correct localization of category
Construct dimension shifting

Homotopy invariant presheaf

F presheaf of abelian groups

F homotopy invariant if $F(X) \xrightarrow{p^*} F(X\times\mathbb{A}^1)$.

$p: X\times\mathbb{A}^1 \longrightarrow X$

$\exists$ splitting by $i_0: X \longrightarrow X\times\mathbb{A}^1$

$\Rightarrow$ p^* injective, splits. p^* isom $\Leftrightarrow$ p^* surj.

Lemma. Consider $i_0, i_1: X \hookrightarrow \mathbb{A}^1$

F homotopic invariant $\Leftrightarrow$ $i_0^* = i_1^*$.

Proof. If $i_0^* = i_1^*$, consider the commutative diagram

$$\begin{tikzcd}
& F(X\times\mathbb{A}^1) \arrow[dl, "id"'] \arrow[d, "(id_X\times m)^*"] \arrow[r, "i_0^*"] & F(X) \arrow[d, "p^*"] \\
F(X\times\mathbb{A}^1) & F(X\times\mathbb{A}^1\times\mathbb{A}^1) \arrow[l, "(i_1\times 1_{\mathbb{A}^1})^*"'] \arrow[r, "(i_0\times 1_{\mathbb{A}^1})^*"] & F(X\times\mathbb{A}^1)
\end{tikzcd}$$

If we deal with complex of presheaves, we may want homotopy invariant cohomology instead.

Def. $i = 0, \ldots, n$, $\theta_i : \Delta^{n+1} \longrightarrow \Delta^n \times \mathbb{A}^1$

$$\theta_i(v_j) := \begin{cases} (v_j, 0), & j \leq i \\ (v_{j-1}, 1), & j > i \end{cases}$$

standard simplex decomposition.

Lemma. $i_0^*, i_1^* : C_* F(X \times \mathbb{A}^1) \longrightarrow C_* F(X)$ are chain homotopic.

$\rightsquigarrow$ Each $\mathbb{H}_n C_* F$ is homotopic invariant.

Proof. $h_i := F(\mathrm{id}_X \times \theta_i) : C_n(X \times \mathbb{A}^1) \longrightarrow C_{n+1}(X)$

$S_n := \sum_{i=0}^{n} (-1)^i h_i$ is a chain homotopy.

Lemma. If $\exists \{h_X\}$,

$$\begin{tikzcd} & F(X) \arrow[dl, "0"'] \arrow[d, "h_X"] \arrow[dr, "\mathrm{id}"] & \\ F(X) & F(X \times \mathbb{A}^1) \arrow[l, "F(i_0)"'] \arrow[r, "F(i_1)"] & F(X). \end{tikzcd}$$

$C_* F$ is chain contractible. i.e, $\mathrm{id} \simeq 0$.

Cor. $P_* : C_* \mathbb{Z}_{tr}(X \times \mathbb{A}^1) \longrightarrow C_* \mathbb{Z}_{tr}(X)$ is homotopic equivalence.

Define homotopy on $\mathrm{Hom}_k(X, Y)$ is not good (not transitive)

Define on Cor_k or (Cor_k, R) instead.

Def. $f \sim g \in Cor_k(X,Y)$ if $\exists$
$H \in Cor_k(X \times \mathbb{A}^1, Y)$, i_0^*, i_1^* : $X \times \mathbb{A}^1 \times Y \to X \times Y$

$(i_0^*)_*(H) = f$, $(i_1^*)_*(H) = g$.

This defines a equivalence relation, compatible with addition.

$f \in Cor_k(X,Y)$ is $\mathbb{A}^1$-homotopy equivalence if $\exists$
$g \in Cor_k(Y,X)$, s.t $fg \sim id_Y$, $gf \sim id_X$.

Ex. $H_0(C_* \mathbb{Z}_{tr}(Y)(X)) = Cor_k(X,Y)/\sim$.
$p: X \times \mathbb{A}^1 \longrightarrow X$ is $\mathbb{A}^1$-homotopic equivalence
i_0 is homotopic inverse.

Lemma. If $f \in Cor(X,Y)$ is $\mathbb{A}^1$-homotopic equivalence with homotopy inverse g,
$f_*: C_*\mathbb{Z}_{tr}(X) \longrightarrow C_*\mathbb{Z}_{tr}(Y)$ is chain homotopy equivalence with homotopy inverse g_*.

Nisnevich topology:

$\{ s_i : P_i \longrightarrow U \}$ is a covering if $\forall u \in U$,

$\exists i$, $x \in P_i$, s.t $s_i(x) = u$, $k(x) \cong k(u)$,
and each of s_i is étale.

Example. $\{ \mathbb{A}^1 - \{0\} \xrightarrow{t \mapsto t^2} \mathbb{A}^1,\ \mathbb{A}^1 - \{a\} \xrightarrow{id} \mathbb{A}^1 \}$
is a covering iff a is a square.

This defines a Grothendieck topology. $\mathbb{Z}_{tr}(X)$ is a sheaf.
If $\{ s_i : P_i \to U \}$ a covering, $\exists$ i s.t.
$s_i : P_i \to U$ is birational $\Rightarrow$ $\exists$ local section.

Ex. A sheaf on $(\mathrm{Spec}\, k)_{Nis}$ is a way to give each l/k finite separable a $F(l)$. No relation.

Recall $PST(k)$ is $Psh(Cor_k)$. $F \in Sh_{Nis}(Cor_k)$ if it is in $PST(k)$ and the underlying presheaf is a sheaf.

Let $F \in Sh_{Nis}(Cor_k)$. Consider standard flasque resolution $F \to E^*(F)$. $E^*(F)$ also has transfer.
$\Rightarrow$ $H^n(\cdot, F)$ also in $PST(k)$.
Note $H^0(X, F) = F(X) = \mathrm{Hom}_{Sh_{Nis}(Cor_k)}(\mathbb{Z}_{tr}(X), F)$.
If F injective in $Sh_{Nis}(Cor_k)$, $F \to E^0(F)$ splits.
$\Rightarrow$ $H^*(X, F)$ is a component of $H^*(X, E^0(F))$
$\Rightarrow$ $H^i(X, F) = 0 \quad \forall\, i > 0$
$\Rightarrow$ $H^n(X, F) = \mathrm{Ext}^n_{Sh_{Nis}(Cor_k)}(\mathbb{Z}_{tr}(X), F)$.
This also hold for every complex of $Sh_{Nis}(Cor_k)$, or with coefficient R.

Consider $\mathbb{D} = D(Sh_{Nis}(Cor_k))$ (or with coefficient R).

Recall we have derived tensor product $\otimes^L$ and internal hom <u>$R\mathcal{H}om$</u>. There's an adjunction.

$\mathbb{A}^1$-weak equivalence

$\mathcal{E} \subset \mathcal{D}^-$ a full additive subcategory is thick if

① $A, B \in \mathcal{E} \Rightarrow A \oplus B \in \mathcal{E}$

② Given $A \to B \to C \xrightarrow{+1}$ distinguished triangle.

Two of A, B, C in $\mathcal{E} \Rightarrow$ the third in $\mathcal{E}$

We say $f \in W_{\mathcal{E}}$ if $\mathrm{cone}(f) \in \mathcal{E}$

$\exists\ \mathcal{D}^-/\mathcal{E} := \mathcal{D}^-[W_{\mathcal{E}}^{-1}]$.

Def. $\mathcal{E}_{\mathbb{A}}$ is the thick subcategory s.t

$\mathrm{cone}(\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}_{tr}(X)) \in \mathcal{E}_{\mathbb{A}}$

$\mathcal{E}_{\mathbb{A}}$ is closed under direct sums.

f is $\mathbb{A}^1$-weak equivalence if $f \in W_{\mathcal{E}_{\mathbb{A}}}$

$DM_{Nis}^{eff,-}(k, R) := \mathcal{D}^-[W_{\mathbb{A}}^{-1}]$.

Remark. In fact, $\mathcal{E}_{\mathbb{A}} = \{ E \mid C_*(E) \text{ is acyclic} \}$.

Def. $F \overset{f}{\underset{g}{\rightrightarrows}} G$ are $\mathbb{A}^1$-homotopic if

$\exists\ h : F \otimes^L R_{tr}(\mathbb{A}^1) \longrightarrow G$ restricts to f, g.

Lemma. If so, $f = g$ in $DM_{Nis}^{eff,-}$.

Lemma. $\forall\ F \in Sh_{Nis}(Cor(k,R))$,
$F \to C_*(F)$ is a $\mathbb{A}^1$-weak equivalence

Def. $L \in D^-$ is $\mathbb{A}^1$-local if $Hom_{D^-}(\cdot, L)$ sends $\mathbb{A}^1$-weak equivalence to bijection.

Lemma. If L is $\mathbb{A}^1$-local,
$Hom_{DM^{eff,-}_{Nis}}(K, L) \cong Hom_{D^-}(K, L)$.

Lemma. Suppose F is a homotopic invariant presheaf. If $F(E) = 0\ \forall\ E/k$ field, $F_{Zar} = 0$
Also $F_{Nis} = 0$.

Lemma. $H^*_{Nis}(\cdot, F_{Nis})$ preserves homotopic invariance if k is perfect.

Prop. Let k be a perfect field. If F is a homotopic invariant sheaf with transfers.
$H^n_{Zar}(X, F) \cong H^n_{Nis}(X, F)$.

Proof. $\pi_N : (Sm/k)_{Nis} \to (Sm/k)_{Zar}$

Leray Spectral sequence:
$$H^p_{Zar}(X, R^q\pi_{N*}F) \Rightarrow H^{p+q}_{Nis}(X, F).$$
Want to show $H^q_{Nis}(S, F) = 0\ \forall\ q > 0$, S local.

We show that $H^q_{Nis}(E, F) = 0$ $\forall$ field E/k.
fields are Henselian local, which has no higher Nis cohomology.

This result extends to C a bounded above complex of $Sh_{Nis}(Cor(k, R))$ with homotopy invariant cohomology sheaves.

$$\Rightarrow \mathbb{H}^n_{Zar}(X, C) = \mathbb{H}^n_{Nis}(X, C) = \mathrm{Ext}^n_{Sh_{Nis}(Cor(k,R))}(R_{tr}(X), C)$$

$$= \mathrm{Hom}_{D^-}(R_{tr}(X), C[n])$$

$$= \mathrm{Hom}_{DM^{eff,-}_{Nis}}(R_{tr}(X), C[n]).$$

Example of such C: $\mathbb{Z}(n)$, $R\pi_{et\to N} F$, where $F \in Sh_{et}(Cor(k, R))$ and homotopic invariant.

If k perfect, $R\pi_{*et\to N} F \overset{?}{=} R\pi_* F$

Good properties of $DM^{eff,-}_{Nis}$: Let $M(X)$ = class of $\mathbb{Z}_{tr}(X)$.

1. $M(X) \cong C_* \mathbb{Z}_{tr}(X)$. $M(X) \otimes M(Y) \cong M(X \times Y)$.
   $M(X) \cong M(X \times \mathbb{A}^1)$.

2. $H^{n,i}(X, \mathbb{Z}) \cong \mathrm{Hom}_{DM^{eff,-}_{Nis}}(M(X), \mathbb{Z}(i)[n])$.

3. Mayer-Vietoris: $X = U \cup V$ opens

$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \xrightarrow{+1}$$

4. Vector bundles $E/X \rightsquigarrow M(E) \cong M(X)$.

5. Projective bundles $\mathbb{P}(\mathcal{E})/X$ of rank $n+1$

$\rightsquigarrow M(\mathbb{P}(\mathcal{E})) \cong \bigoplus_{i=0}^{n} M(X)(i)[2i]$.

6. If $Z \subset X$ smooth, codim c, $X' \to X$ blow-up w.r.t Z

$M(X') \cong M(X) \oplus \left(\bigoplus_{i=1}^{c-1} M(Z)(i)[2i] \right)$

Also $\exists M(X) \longrightarrow M(Z)(c)[2c]$, inducing Gysin triangle

$M(X-Z) \longrightarrow M(X) \longrightarrow M(Z)(c)[2c] \xrightarrow{+1}$

7. $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(A(1), B(1))$

8. About Chow motive....

Contraction: If F a homotopic invariant presheaf, $F_{-1}(X) := F(X \times \mathbb{G}_m)/F(X)$. l_1 gives a canonical splitting $F(X) \oplus F_{-1}(X) = F(X \times \mathbb{G}_m)$.

$H^p(U, F_{-1}) = H^p(\cdot, F)_{-1}(U)$.

Suppose F is a bounded above complex of $Sh_{Nis}(Cor(k))$

$F_{-1} := \underline{R\mathrm{Hom}}(\mathbb{L}, F)[1]$

where $\mathbb{L} = \mathbb{Z}(1)[2] \cong \dfrac{\mathbb{Z}_{tr}(\mathbb{G}_m)}{\mathbb{Z}_{tr}}[1] \cong \dfrac{\mathbb{Z}_{tr}(\mathbb{P}^1)}{\mathbb{Z}}$.

Lemma. $H^p(X, F_{-1}) = H^p(\cdot, F)_{-1}(X)$.

Proof. $H^p(X \times \mathbb{G}_m, F) = H^p(X, F) \oplus H^p(X \otimes \mathbb{L}, F[1])$
$= H^p(X, F) \oplus H^p(X, F_{-1})$.

Consider $\mathrm{id} \in \mathrm{Hom}(F \otimes \mathbb{L}, F \otimes \mathbb{L})$

$\rightsquigarrow \delta \in \mathrm{Hom}(F, \underline{R\mathrm{Hom}}(\mathbb{L}, F \otimes \mathbb{L}))$
$= \mathrm{Hom}(F[-1], \underline{R\mathrm{Hom}}(\mathbb{L}, F(1))[1])$
$= \mathrm{Hom}(F[-1], F(1)[-1])$.

Example. $A_{-1} = 0$. $\delta: \mathbb{Z}(n-1)[-1] \cong \mathbb{Z}(n)_{-1}$

$\mathbb{Z}/\ell\mathbb{Z}(n-1)[-1] \cong \mathbb{Z}/\ell\mathbb{Z}(n)_{-1}$

$R\pi_*(\mu_\ell^{\otimes n-1})[-1] \cong R\pi_*(\mu_\ell^{\otimes n})_{-1}$.

Lemma. Consider cohomology sheaf $\mathcal{H}^p$.

$$\mathcal{H}^p(F_{-1})(X) = \frac{\mathcal{H}^p(F)(X \times \mathbb{G}_m)}{\mathcal{H}^p(F)(X)}.$$

Proof. $\mathcal{H}^p(F_{-1})$ is the Nis sheafification of
$X \longmapsto H^p_{Nis}(X, F_{-1}) = H^p(\cdot\,, F)_{-1}(X)$.
Contraction commute with Nis sheafification
$\Rightarrow \mathcal{H}^p(F_{-1})(X) = \mathcal{H}^p(F)_{-1}(X)$.

Consider $(\tau^{\leq n} F)_{-1} \longrightarrow F_{-1}$. This has image in $\tau^{\leq n} F_{-1}$ and is quasi-isom as both has cohomology sheaf

$$\begin{cases} \mathcal{H}^p(F_{-1}) & p \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Coro. $\mathcal{Y}/\ell(n-1)[-1] \overset{\delta}{\simeq} \mathcal{L}/\ell(n)_{-1}$.

Consider $\mathbb{Z}/\ell(n) \to \mathcal{Y}/\ell(n) \to R\pi_*(\mu_\ell^{\otimes n})$

$$\rightsquigarrow \begin{tikzcd} H^{p-1}(X, \mathbb{Z}/\ell(n-1)) \arrow[r, "\simeq"] \arrow[d, "\alpha_{n-1}"] & H^p(X, \mathbb{Z}/\ell(n)_{-1}) \arrow[d, "\alpha_n"] \\ H^{p-1}(X, \mathcal{Y}/\ell(n)) \arrow[r, "\simeq"] \arrow[d] & H^p(X, \mathcal{Y}/\ell(n)_{-1}) \\ H^{p-1}_{et}(X, \mu_\ell^{\otimes n-1}) \arrow[r, "\simeq"] & H^p_{et}(X, \mu_\ell^{\otimes n}{}_{-1}) \end{tikzcd}$$

$BL(n) \Rightarrow \alpha_{n-1}$ is isom $\forall$ p, X. Apply this on all local X $\Rightarrow$ isom on stalks $\Rightarrow$
$\mathbb{Z}/\ell(n-1) \longrightarrow \mathbb{Y}/\ell(n-1)$ is quasi-isim $= BL(n-1)$.

$\forall$ $U \subset \mathbb{G}_m$ dense open, $\exists$ Gysin triangle

$$\bigoplus_{x \in \mathbb{A}^1 - U} M(x)(1)[1] \longrightarrow M(\mathbb{G}_m) \longrightarrow M(\mathbb{A}^1) = M(k)$$

Split exact by $i_1 : \operatorname{Spec}(k) \to \mathbb{G}_m$.

Take colimit over U

$$\rightsquigarrow 0 \longrightarrow H^p(k, C) \longrightarrow H^p(k(t), C) \longrightarrow \bigoplus_{x \in |\mathbb{A}^1|} H^{p-1}(k(x), C(-1)) \longrightarrow 0.$$

split exact.

$p = n$, $C = \mathbb{Z}(n)$, $C' = R\pi_* \mathbb{Z}(n)_{et}$ yields

$$\bigoplus_{x \in |\mathbb{A}^1|} H^{n-1}(k(x), \mathbb{Z}(n-1))/\ell \longrightarrow H^{n-1}_{et}(k(x), \mu_\ell^{\otimes n-1})$$

is a summand of $H^n(k(t), \mathbb{Z}(n))/\ell \to H^n(k(t), \mu_\ell^{\otimes n})$.

$\Rightarrow BK(n) \Rightarrow BK(n-1)$.

$p = n+1$, $C = \mathbb{Z}_{(\ell)}(n)$:

$\bigoplus_{x \in |\mathbb{A}^1|} H^n(k(x), \mathbb{Z}_{(\ell)}(n-1))$ is a summand of

$H^{n+1}(k(t), \mathbb{Z}_{(\ell)}(n))$. $H90(n) \Rightarrow H90(n-1)$.