Application of syntomic duality: unramified p-primary class field theory for function fields

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Let X be a smooth proper curve over a finite field $k = \mathbb{F}_q$, with structure map $\pi : X \to \operatorname{Spec} k$. (We may as well assume $k = \mathbb{F}_p$.) In this setting, the core of class field theory can be thought of as an isomorphism $\operatorname{Pic}(X)^\wedge \simeq \pi_1(X)^{\mathrm{ab}}$ (satisfying certain additional properties). For $\ell \neq p$, the ℓ-adic portion of this story can by done via classical methods, typically something like this: we identify the left-hand side modulo ℓ^n with the mod ℓ^n étale cohomology of X (using a version of Tate duality), and the right-hand side mod ℓ^n with $\operatorname{Hom}(\pi_1(X), \mathbb{Z}/\ell^n) \simeq H^1_{\text{ét}}(X, \mathbb{Z}/\ell^n)$. Since the étale cohomology of X is not well-behaved at $\ell = p$, this story breaks down.

We aim to replace it using our new theory of syntomic cohomology, which should give a good theory at p. In particular, we want to show that

$$\operatorname{Pic}(X)/p^n \simeq \pi_1(X)^{\mathrm{ab}}/p^n.$$

We will deduce this from the following duality result for syntomic cohomology, originally proven by Milne.

Theorem 1. *Let X be a smooth proper scheme over k of dimension d. Then for each integer i there is a natural isomorphism*

$$\begin{aligned} R\Gamma_{\text{syn}}(X, \mathbb{Z}_p(i)) := R\Gamma(X^{\text{syn}}, \mathcal{O}_{X^{\text{syn}}}\{i\}) &\simeq R\Gamma(X^{\text{syn}}, \mathcal{O}_{X^{\text{syn}}}\{d-i\})^\vee[-2d-1] \\ &=: R\Gamma_{\text{syn}}(X, \mathbb{Z}_p(d-i))^\vee[-2d-1]. \end{aligned}$$

We can view this either as Poincaré duality for syntomic cohomology or for X^{syn}.

Before seeing how we can prove such a thing, let's see how we can deduce the above statement about class field theory. Our setting is that of the theorem with $d = 1$. In the case $i = 1$, we said last time that $\mathbb{Z}_p(1)$ identifies with $T_p\mathbb{G}_{\mathrm{m}}$; in particular $R\Gamma_{\text{syn}}(X, \mathbb{Z}_p(1))/p^n \simeq R\Gamma_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})/p^n \simeq R\Gamma(X, \mu_{p^n})$. Therefore by Theorem 1 we have

$$R\Gamma(X, \mu_{p^n}) \simeq R\Gamma(X, \mathbb{Z}/p^n)^\vee[-3].$$

In particular in degree 2 (since the isomorphism is t-exact) we get

$$H^2(X, \mu_{p^n}) \simeq H^1(X, \mathbb{Z}/p^n)^\vee =: \operatorname{Hom}(\pi_1(X), \mathbb{Z}/p^n) = \pi_1(X)^{\mathrm{ab}}/p^n.$$

On the other hand, from the Kummer sequence

$$H^1(X, \mathbb{G}_{\mathrm{m}}) \xrightarrow{p^n} H^1(X, \mathbb{G}_{\mathrm{m}}) \to H^2(X, \mu_{p^n}) \to H^2(X, \mathbb{G}_{\mathrm{m}}) \xrightarrow{p^n} H^2(X, \mathbb{G}_{\mathrm{m}})$$

we get a short exact sequence

$$0 \to H^1(X, \mathbb{G}_{\mathrm{m}})/p^n = \operatorname{Pic}(X)/p^n \to H^2(X, \mu_{p^n}) \to H^2(X, \mathbb{G}_{\mathrm{m}})[p^n] \to 0.$$

Thus if we can show that $H^2(X, \mathbb{G}_m)[p^n] = 0$ then we get an isomorphism $\mathrm{Pic}(X)/p^n \simeq H^2(X, \mu_{p^n})$, which the above identifies with $\pi_1(X)^{ab}/p^n$.

It suffices to show this in the case $n = 1$, since any p^n torsion necessarily implies p-torsion (by multiplying by p^{n-1}). We can show directly via the theory of torsors that $H^2(X, \mathbb{G}_m)[p] = 0$; alternatively we can show it via Theorem 1. Taking Euler characteristics of the isomorphism $R\Gamma(X, \mu_p) \simeq R\Gamma(X, \mathbb{Z}/p)^\vee[-3]$ from above gives

$$\dim H^0(X, \mu_p) - \dim H^1(X, \mu_p) + \dim H^2(X, \mu_p) - \dim H^3(X, \mu_p) = -\chi(X, \mathbb{Z}/p).$$

By Artin–Schreier theory, we have $\chi(X, \mathbb{Z}/p) + \chi(X, \mathbb{G}_a) = \chi(X, \mathbb{G}_a)$ and so $\chi(X, \mathbb{Z}/p) = 0$, so this means that

$$\dim H^0(X, \mu_p) + \dim H^2(X, \mu_p) = \dim H^1(X, \mu_p) + \dim H^3(X, \mu_p).$$

We have $H^0(X, \mu_p) = 0$ and $H^3(X, \mu_p) \simeq H^0(X, \mathbb{Z}/p)^\vee \simeq \mathbb{Z}/p$, so this becomes

$$\dim H^2(X, \mu_p) = \dim H^1(X, \mu_p) + 1.$$

The Kummer sequence

$$H^0(X, \mathbb{G}_m) = 0 \to H^1(X, \mu_p) \to H^1(X, \mathbb{G}_m) \xrightarrow{p} H^1(X, \mathbb{G}_m)$$

shows that $H^1(X, \mu_p) \simeq H^1(X, \mathbb{G}_m)[p] = \mathrm{Pic}(X)[p]$, so we could write this as

$$\dim H^2(X, \mu_p) = \dim \mathrm{Pic}(X)[p] + 1.$$

But from the short exact sequence above (coming from the H^2 part of the Kummer sequence) we have

$$\dim H^2(X, \mu_p) = \dim \mathrm{Pic}(X)/p + \dim H^2(X, \mathbb{G}_m)[p],$$

so

$$\dim \mathrm{Pic}(X)[p] + 1 = \dim \mathrm{Pic}(X)/p + \dim H^2(X, \mathbb{G}_m)[p].$$

Since $\mathrm{Pic}(X) \simeq \mathrm{Pic}^0(X) \times \mathbb{Z}$ with $\mathrm{Pic}^0(X)$ finite, for any $(x, n) \in \mathrm{Pic}^0(X) \times \mathbb{Z} \simeq \mathrm{Pic}(X)$ if $p \cdot (x, n) = 0$ then $pn = 0$ and so $n = 0$, so $\mathrm{Pic}(X)[p] = \mathrm{Pic}^0(X)[p]$; on the other hand $\mathrm{Pic}(X)/p \simeq \mathrm{Pic}^0(X)/p \times \mathbb{Z}/p \simeq \mathrm{Pic}^0(X)[p] \times \mathbb{Z}/p$ since $\mathrm{Pic}^0(X)$ is finite. Therefore

$$\dim \mathrm{Pic}(X)/p = \dim \mathrm{Pic}(X)[p] + 1,$$

and so $H^2(X, \mathbb{G}_m)[p]$ must vanish.

We'd next like to prove Theorem 1. The key observation is that although it is pretty general in one respect (it applies to any smooth proper k-scheme X) it is specific in another (it applies only to the sheaves $\mathbb{Z}_p(i)$ on X, or equivalently $\mathcal{O}_{X^{\mathrm{syn}}}\{i\}$ on X^{syn}), and so we might hope that we can trade one level of generality for another: by allowing more general sheaves, we hope to restrict to a particular k-scheme. Indeed, we don't have to work too much harder to generalize to other sheaves, as we're already considering sheaves $\mathcal{O}_{X^{\mathrm{syn}}}\{i\}$ beyond the structure sheaf $\mathcal{O}_{X^{\mathrm{syn}}}$.

In particular, for any (derived) sheaf $\mathcal{F}$ on X^{syn}, we can consider the (derived) push-forward $\pi_*^{\mathrm{syn}}\mathcal{F}$ on $(\mathrm{Spec}\, k)^{\mathrm{syn}}$. Thus we might hope that Theorem 1 should follow from a sufficiently strong version of Poincaré duality for $(\mathrm{Spec}\, k)^{\mathrm{syn}}$, which in the following we abbreviate as $*^{\mathrm{syn}}$.

Indeed, we have the following theorem.

Theorem 2 (Serre duality on $(\operatorname{Spec}\mathbb{F}_p)^{\mathrm{syn}}$)**.**

(1) There is a natural isomorphism $R\Gamma(*^{\mathrm{syn}}, \mathcal{O}_{*^{\mathrm{syn}}}) \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p[-1]$ *in* $\operatorname{Perf}(\mathbb{Z}_p)$*.*

(2) For any $E \in \operatorname{Perf}(*^{\mathrm{syn}})$*, the pairing*

$$R\Gamma(*^{\mathrm{syn}}, E) \otimes R\Gamma(*^{\mathrm{syn}}, E^\vee) \to R\Gamma(*^{\mathrm{syn}}, \mathcal{O}_{*^{\mathrm{syn}}}) \to \mathbb{Z}_p[-1]$$

induced by $E \otimes E^\vee \xrightarrow{\sim} \mathcal{O}_{*^{\mathrm{syn}}}$ *and the isomorphism from (1) induces an isomorphism*

$$R\Gamma(*^{\mathrm{syn}}, E) \simeq R\Gamma(*^{\mathrm{syn}}, E^\vee)^\vee[-1]$$

in $\operatorname{Perf}(\mathbb{Z}_p)$*.*

Letting $E = \mathscr{H}_{\mathrm{syn}}(X)\{i\} := \pi_*^{\mathrm{syn}}\mathcal{O}_{X^{\mathrm{syn}}}\{i\} \in \operatorname{Perf}(*^{\mathrm{syn}})$, this gives

$$R\Gamma(X, \mathcal{O}_{X^{\mathrm{syn}}}\{i\}) = R\Gamma(*^{\mathrm{syn}}, \mathscr{H}_{\mathrm{syn}}(X)\{i\}) \simeq R\Gamma(*^{\mathrm{syn}}, \mathscr{H}_{\mathrm{syn}}(X)\{i\}^\vee)^\vee[-1].$$

To proceed further, we need a description of $\mathscr{H}_{\mathrm{syn}}(X)\{i\}^\vee$. This comes from the work of Longke Tang on Poincaré duality for prismatic cohomology. In this case, since we're in characteristic p this is really a version of Poincaré duality for crystalline cohomology with coefficients in a prismatic F-gauge, and can by summarized by the following theorem:

Theorem 3 (Poincaré duality for $\mathscr{H}_{\mathrm{syn}}(X)$)**.** *Suppose* X *is smooth and proper over* k *of dimension* d*.*

(1) There is a natural isomorphism $\mathscr{H}^{2d}_{\mathrm{syn}}(X)\{d\} \simeq \mathcal{O}_{*^{\mathrm{syn}}}$*.*

(2) The pairing

$$\mathscr{H}_{\mathrm{syn}}(X) \otimes \mathscr{H}_{\mathrm{syn}}(X) \to \mathscr{H}_{\mathrm{syn}}(X) \to \mathcal{O}_{*^{\mathrm{syn}}}\{-d\}[-2d]$$

coming from (1) is perfect, giving an isomorphism

$$\mathscr{H}_{\mathrm{syn}}(X)^\vee \simeq \mathscr{H}_{\mathrm{syn}}(X)\{d\}[2d]$$

in $\operatorname{Perf}(*^{\mathrm{syn}})$*.*

Now we can complete the proof of Theorem 1: we get

$$R\Gamma(X, \mathcal{O}_{X^{\mathrm{syn}}}\{i\}) \simeq R\Gamma(*^{\mathrm{syn}}, \mathscr{H}_{\mathrm{syn}}(X)\{d-i\}[-2d-1] = R\Gamma(X^{\mathrm{syn}}, \mathcal{O}_{X^{\mathrm{syn}}}\{d-i\})^\vee[-2d-1]$$

as claimed.

We won't prove Theorem 3, as it's essentially just about crystalline cohomology in this setting and unfolding definitions (it also holds in mixed characteristic, where it is much more interesting but also more difficult, but we avoid this for now). However it does remain to prove Theorem 2; essentially what we've done is reduce Theorem 1 to a duality on $*^{\mathrm{syn}}$, which we can understand reasonably explicitly.

The first statement of Theorem 2, the calculation of the cohomology of the structure sheaf, is relatively straightforward: by definition, $R\Gamma(*^{\mathrm{syn}}, \mathcal{O}_{*^{\mathrm{syn}}}) =: R\Gamma_{\mathrm{syn}}(\operatorname{Spec}\mathbb{F}_p, \mathbb{Z}_p(0))$, and we observed last time that $\mathbb{Z}_p(0)$ just corresponds to the constant sheaf $\mathbb{Z}_p$ and so this is

$R\Gamma(\operatorname{Spec}\mathbb{F}_p, \mathbb{Z}_p) = R\Gamma(\operatorname{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p), \mathbb{Z}_p) = R\Gamma(\widehat{\mathbb{Z}}, \mathbb{Z}_p)$. As a complex, we can view $R\Gamma(\widehat{\mathbb{Z}}, M)$ as $(M \xrightarrow{\gamma-1} M)$ for any $\widehat{\mathbb{Z}}$-module M, where γ is a topological generator; on $M = \mathbb{Z}_p$ with the trivial action, $\gamma \mapsto 1$ and so is this the trivial complex $\mathbb{Z}_p \oplus \mathbb{Z}_p[-1]$.

The second statement is more subtle, and proceeds via several steps. The idea is this: for any $E \in D_{\mathrm{qc}}(*^{\mathrm{syn}})$, via the first statement there is a map

$$R\Gamma(*^{\mathrm{syn}}, E) \otimes R\Gamma(*^{\mathrm{syn}}, E^\vee[1]) \to R\Gamma(*^{\mathrm{syn}}, \mathcal{O}_{*^{\mathrm{syn}}}[1]) \simeq \mathbb{Z}_p[1] \oplus \mathbb{Z}_p \to \mathbb{Z}_p,$$

yielding by adjunction a map

$$\eta_E : R\Gamma(*^{\mathrm{syn}}, E) \to R\Gamma(*^{\mathrm{syn}}, E^\vee[1])^\vee.$$

We let $\mathcal{C}$ denote the full subcategory of $D_{\mathrm{qc}}(*^{\mathrm{syn}})$ spanned by E such that η_E is an isomorphism. The idea is that step by step, we will show that $\operatorname{Perf}(*^{\mathrm{syn}}) \subseteq \mathcal{C}$.

Step 1. Observe that $\mathcal{O}_{*^{\mathrm{syn}}}$ is in $\mathcal{C}$, since by the computation above the right-hand side is $(\mathbb{Z}_p[1] \oplus \mathbb{Z}_p)^\vee \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p[-1]$, agreeing with the left-hand side, and the map $\eta_{\mathcal{O}}$ is self-dual and so self-inverse.

Step 2. For every i, the Breuil–Kisin twist $\mathcal{O}_{*^{\mathrm{syn}}}\{i\}$ is also in $\mathcal{C}$: the case $i = 0$ is the previous step, and for $i \neq 0$ since syntomic cohomology vanishes for i negative or greater than the dimension of the scheme (here 0) both sides vanish for $i \neq 0$.

Step 3. Recall that there is a canonical map $j_{\Delta} : *^{\Delta} \to *^{\mathrm{syn}}$, given by gluing the maps $j_{\pm} : *^{\Delta} \to *^{\mathcal{N}}$. We claim that $E = j_{\Delta *}\mathcal{O}_{*^{\Delta}}$ is also in $\mathcal{C}$. First, observe that it is sufficient to check that η_E is an isomorphism modulo p, as a map of $\mathbb{Z}_p$-modules is an isomorphism if and only if it induces an isomorphism modulo p. Now we need to know something about $*^{\mathrm{syn}} = (\operatorname{Spec}\mathbb{F}_p)^{\mathrm{syn}}$. First, $*^{\Delta} = (\operatorname{Spec}\mathbb{F}_p)^{\Delta}$ is just $\operatorname{Spf}\mathbb{Z}_p$. Passing to $*^{\mathcal{N}}$ is given by taking the Rees algebra $(\operatorname{Spf}\mathbb{Z}_p[u,t]/(ut-p))/\mathbb{G}_{\mathrm{m}}$ for t in degree 1 and u in degree -1. This is greatly simplified modulo p: here it is just $(\operatorname{Spec}\mathbb{F}_p[u,t]/(ut))/\mathbb{G}_{\mathrm{m}}$, which we can think of as a pair of intersecting axes, one labeled by t and one by u, modulo $\mathbb{G}_{\mathrm{m}}$. The images of j_+ and j_- are given by the loci $t \neq 0$ and $u \neq 0$ respectively: e.g. if $t \neq 0$ then $u = 0$ and so this is just $(\operatorname{Spec}\mathbb{F}_p[t^{\pm 1}])/\mathbb{G}_{\mathrm{m}} = \mathbb{G}_{\mathrm{m}}/\mathbb{G}_{\mathrm{m}} \bmod p = \operatorname{Spec}\mathbb{F}_p$, which is the same as $*^{\Delta}$ modulo p. Finally we can understand $*^{\mathrm{syn}}$ as gluing the loci $t \neq 0$ and $u \neq 0$ (and quotienting by $\mathbb{G}_{\mathrm{m}}$). In particular this means there are two points of $*^{\mathrm{syn}}$: an open point $j_{\Delta} : \operatorname{Spf}\mathbb{Z}_p \to *^{\mathrm{syn}}$, and a closed point $i_H : (\operatorname{Spec}\mathbb{F}_p)/\mathbb{G}_{\mathrm{m}} \to *^{\mathrm{syn}}$ coming from the locus $t = u = 0$ of $*^{\mathcal{N}}$.

If we write $j_{\mathcal{N}} : *^{\mathcal{N}} \to *^{\mathrm{syn}}$ for the projection, the above description gives an exact triangle

$$\mathcal{O}_{*^{\mathrm{syn}}}/p \to j_{\mathcal{N}*}\mathcal{O}_{*^{\mathcal{N}}}/u \oplus j_{\mathcal{N}*}\mathcal{O}_{*^{\mathcal{N}}}/t \to j_{\Delta *}\mathcal{O}_{*^{\Delta}}/p \oplus i_{H*}\mathcal{O}_{*/\mathbb{G}_{\mathrm{m}}}.$$

Applying the functor $R\operatorname{Hom}_{*^{\mathrm{syn}}}(E, -)$ corepresented by E kills both middle terms as well as $i_{H*}\mathcal{O}_{*/\mathbb{G}_{\mathrm{m}}}$. The latter claim is clear since j_{Δ} and i_H have distinct images; for the former, by adjunction it suffices to show that on the loci cut out by $u = 0$

and $t = 0$, both of which are isomorphic to $\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}$, the inclusion j of the open point $\mathbb{G}_{\mathrm{m}}/\mathbb{G}_{\mathrm{m}}$ gives $R\operatorname{Hom}(j_*\mathcal{O}, \mathcal{O}) = 0$, which follows by a formal completeness argument. Thus we have an exact triangle

$$R\operatorname{Hom}(E, \mathcal{O}_{*^{\mathrm{syn}}}/p) \simeq E^\vee/p \to 0 \to R\operatorname{Hom}(E, E/p) \simeq E/p,$$

and so an isomorphism

$$E^\vee/p \simeq E/p[1],$$

which is just $\eta_E^\vee/p$.

Step 4. It follows from the previous step that in fact for any $E \in \operatorname{Perf}(*^{\mathbb{\Delta}})$ we have $j_{\mathbb{\Delta}*}E$ in $\mathcal{C}$: indeed, any perfect complex on $*^{\mathbb{\Delta}} = \operatorname{Spf}\mathbb{Z}_p$ is a perfect complex of complete $\mathbb{Z}_p$-modules and so is built from copies of $\mathcal{O}_{\operatorname{Spf}\mathbb{Z}_p} = \mathcal{O}_{*^{\mathbb{\Delta}}}$, so we are reduced to the claim of Step 3.

Step 5. We claim that further for any $E \in \operatorname{Perf}(*^{\mathcal{N}})$, we have $j_{\mathcal{N}*}E$ in $\mathcal{C}$. We mentioned in a previous talk that $\operatorname{Perf}(*^{\mathcal{N}})$ is generated (under finite colimits, shifts, and retracts) by $\mathcal{O}_{*^{\mathcal{N}}}\{i\}$ for integers i. By the projection formula, $(j_{\mathcal{N}*}\mathcal{O}_{*^{\mathcal{N}}}\{i\}) \otimes \mathcal{O}_{*^{\mathrm{syn}}}\{i\} = j_{\mathcal{N}*}(\mathcal{O}_{*^{\mathcal{N}}}\{i\} \otimes j_{\mathcal{N}}^*\mathcal{O}_{*^{\mathrm{syn}}}\{i\}) = j_{\mathcal{N}*}\mathcal{O}_{*^{\mathcal{N}}}$ since $j_{\mathcal{N}}^*\mathcal{O}_{*^{\mathrm{syn}}}\{i\} = \mathcal{O}_{*^{\mathcal{N}}}\{-i\}$ from a previous talk, so it suffices to show the claim for $j_{\mathcal{N}*}\mathcal{O}_{*^{\mathcal{N}}}$ since we already know it for $\mathcal{O}_{*^{\mathrm{syn}}}\{i\}$ (by Step 2 above). By the gluing description of $*^{\mathrm{syn}}$, we have an exact triangle

$$\mathcal{O}_{*^{\mathrm{syn}}} \to j_{\mathcal{N}*}\mathcal{O}_{*^{\mathcal{N}}} \to j_{\mathbb{\Delta}*}\mathcal{O}_{*^{\mathbb{\Delta}}},$$

so since the outer two terms lie in $\mathcal{C}$ so must the inner one.

Step 6. Finally, for any $E \in \operatorname{Perf}(*^{\mathrm{syn}})$, tensoring the above triangle with E gives

$$E \to E \otimes j_{\mathcal{N}*}\mathcal{O}_{*^{\mathcal{N}}} = j_{\mathcal{N}*}j_{\mathcal{N}}^*E \to E \otimes j_{\mathbb{\Delta}*}\mathcal{O}_{*^{\mathbb{\Delta}}} = j_{\mathbb{\Delta}*}j_{\mathbb{\Delta}}^*E,$$

and so the second and third terms lie in $\mathcal{C}$ by steps 4 and 5 and so so does the first.