Formulating the p-adic Langlands conjectures

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(Note: these notes follow [1] (mostly section 6.1) *very* closely, and should be taken as even less original than my usual notes; I am essentially copying out the results and commentary, down to wording in some cases.)

Let's first fix some notation. Let $F/\mathbb{Q}_p$ be a finite extension, $E/\mathbb{Q}_p$ another finite extension with ring of integers $\mathcal{O} = \mathcal{O}_E$; we're interested in "the Langlands program for F with coefficients in $\mathcal{O}$." The only group we're going to worry about is $G = \mathrm{GL}_d(F)$, with maximal compact open subgroup $K = \mathrm{GL}_d(\mathcal{O}_F)$ and center $Z = Z(G)$. We write $\mathrm{sm.}\, G$ for the abelian category of smooth representations of G on $\mathcal{O}$-modules, where we say a G-representation on an $\mathcal{O}$-module M is smooth if for any $m \in M$ the subgroup of G fixing m is open and for some $n \geq 0$ we have $p^n \cdot m = 0$. We write $D(\mathrm{sm.}\, G)$ for the derived stable ∞-category of $\mathrm{sm.}\, G$. This is our key object on the automorphic side, so as in Fargues–Scholze we want to spend a little time thinking about it.

We want to reframe smooth G-representations as modules over some group ring. We do this as follows: fix a compact open subgroup $H \subset G$ (e.g. K). We define

$$\mathcal{O}[[H]] = \varprojlim_{J \lhd H} \mathcal{O}[H/J]$$

where the limit ranges over normal open subgroups of H. This turns out to be a compact Noetherian linear topological ring. It can be viewed as an H-representation over $\mathcal{O}$, via roughly the regular action; therefore we can define

$$\mathcal{O}[[G]] = c\text{-}\mathrm{Ind}_H^G \mathcal{O}[[H]] = \mathcal{O}[G] \otimes_{\mathcal{O}[H]} \mathcal{O}[[H]].$$

Requiring that $\mathcal{O}[G]$ and $\mathcal{O}[[H]]$ be subrings gives this a natural ring structure as well as a G-action, and in fact it is independent of the choice of H.

If M is a smooth G-representation, then the $\mathcal{O}[G]$-action on M extends to an $\mathcal{O}[[G]]$-action. Indeed, if $D(\mathcal{O}[[G]])$ is the derived stable ∞-category of $\mathcal{O}[[G]]$-modules, then there is a fully faithful functor $D(\mathrm{sm.}\, G) \hookrightarrow D(\mathcal{O}[[G]])$, respecting the t-structures, whose essential image $D_{\mathrm{sm}}(\mathcal{O}[[G]])$ consists of the objects of $D(\mathcal{O}[[G]])$ whose cohomology groups are all smooth G-representations, so we get an equivalence $D(\mathrm{sm.}\, G) \xrightarrow{\sim} D_{\mathrm{sm}}(\mathcal{O}[[G]])$.

This equivalence is compatible with restricting to the subcategories of objects with countably generated cohomology vanishing in sufficiently high degrees:

$$D^-_{\mathrm{c.g.}}(\mathrm{sm.}\, G) \xrightarrow{\sim} D^-_{\mathrm{c.g.,sm}}(\mathcal{O}[[G]]).$$

Composing the inverse of this equivalence with the inclusion $D^-_{\mathrm{c.g.}}(\mathrm{sm.}\, G) \hookrightarrow \mathrm{Pro}\, D(\mathrm{sm.}\, G)$ gives a functor

$$D^-_{\mathrm{c.g.,sm}}(\mathcal{O}[[G]]) \to \mathrm{Pro}\, D^-(\mathrm{sm.}\, G),$$

which in fact extends to a functor

$$D^-_{\mathrm{c.g.}}(\mathcal{O}[[G]]) \to \mathrm{Pro}\, D^-(\mathrm{sm.}\, G)$$

(by a "smoothing" functor involving taking a limit over quotients by open subgroups). For example, the unit object $\mathcal{O}[[G]]$ in $D^-_{\text{c.g.}}(\mathcal{O}[[G]])$ is mapped to the pro-object $\varprojlim_{H,n} c\text{-}\operatorname{Ind}_H^G \mathcal{O}/p^n$, where the limit is over compact open subgroups H of G and positive integers n, and $\mathcal{O}/p^n$ is viewed as a trivial H-module.

We want to understand $D(\text{sm.}\, G)$ in terms of coherent sheaves, or ind-coherent complexes, on the Emerton–Gee stack $\mathcal{X}_d$. To do so we need to assume a technical condition on sm. G, which is not known:

Conjecture. *The abelian category* sm. G *is locally coherent, i.e. it is compactly generated and the compact objects form an abelian subcategory.*

The compact objects of sm. G are the finitely presented ones, and sm. G is known to be compactly generated, so this conjecture boils down to the statement that finitely presented $\mathcal{O}$-representations of G are closed under taking kernels. It is not obvious why this should be true (it is known only for $d = 1$ and partially for $d = 2$), other than that it seems to be necessary in order to give a good formulation of our main conjectures (to follow).

Now let $D^b_{\text{f.p.}}(\text{sm.}\, G)$ denote the full subcategory consisting of complexes whose cohomology groups are all finitely presented $\mathcal{O}[[G]]$-modules and vanish in sufficiently high or low degrees, i.e. the coherent objects. There is a duality on $D^b_{\text{f.p.}}(\text{sm.}\, G)$ via

$$\mathbb{D}(-) := R\operatorname{Hom}_{\mathcal{O}[[G]]}(-, \mathcal{O}[[G]])[d^2[F : \mathbb{Q}_p] + 1].$$

(Here the $d^2[F : \mathbb{Q}_p] + 1$ should be thought of as the dimension of $\mathcal{O}[[G]] = \mathcal{O}[\mathrm{GL}_d(F)]$ over $\mathbb{Q}_p$: the F-dimension of $\mathrm{GL}_d(F)$ is d^2, so the $\mathbb{Q}_p$-dimension is $d^2[F : \mathbb{Q}_p]$, and $\mathcal{O}$ should be thought of as one-dimensional.)

There is a natural isomorphism $\mathbb{D} \circ \mathbb{D} \xrightarrow{\sim} \mathrm{id}$, and for any finite length smooth representation V of a compact open subgroup H of G we have

$$\mathbb{D}(c\text{-}\operatorname{Ind}_H^G V) = c\text{-}\operatorname{Ind}_H^G V^\vee,$$

where $V^\vee = \operatorname{Hom}_{\mathcal{O}}(V, E/\mathcal{O})$ is the Pontryagin dual.

Let $\underline{\lambda}$ be a regular Hodge type, i.e. a tuple of integers $\lambda_{\sigma,i}$ for every embedding $\sigma : F \hookrightarrow E$ and integer $1 \le i \le d$ such that $\lambda_{\sigma,i} > \lambda_{\sigma,i+1}$ for every σ, i. In particular, given a suitable representation $\rho : W_F \to \mathrm{GL}(V)$ for some E-vector space V (where W_F is the Weil group of F), we can assign it a Hodge type by looking at the eigenvalues of the action of W_F, all of the form χ^i for some i where χ is the cyclotomic character; for each σ and i, we define $\lambda_{\sigma,i}$ to be the E-dimension of the Galois invariants $(V \otimes_{\sigma,F} \widehat{\overline{F}}(i))^{\mathrm{Gal}_F}$ where the tensor product is along the map $E \to \overline{F}$ induced by σ. For example, if ρ is the cyclotomic character, the invariants are only nonzero if $i = -1$.

Thus for our fixed regular Hodge type $\underline{\lambda}$, we can associate to it an $\mathcal{O}$-representation $V_{\underline{\lambda}}$ of K as follows: for each $\sigma : F \hookrightarrow E$, set $\xi_{\sigma,i} = i - 1 - \lambda_{\sigma,d+1-i}$, so that $\xi_{\sigma,1} \ge \cdots \ge \xi_{\sigma,d}$. Then $\xi_\sigma = (\xi_{\sigma,i})$ is a dominant weight of the algebraic group GL_d (with respect to the upper triangular Borel subgroup), so it corresponds to an algebraic $\mathcal{O}_F$-representation M_{ξ_σ} of K of highest weight ξ_σ. We then define

$$V_{\underline{\lambda}} = \bigotimes_\sigma M_{\xi_\sigma} \otimes_{\mathcal{O}_F,\sigma} \mathcal{O}.$$

We now briefly turn back to the Galois side. First, the objects corresponding to derived smooth G-representations should be derived coherent sheaves on $\mathcal{X}_d$ from last time; more simply we can think of $\mathcal{X}_d$ as parametrizing suitable Galois representations. Since we have a duality on the automorphic side, given by $\mathbb{D}$, we hope that there is a similar duality on the Galois side. There is a natural involution ι of $\mathcal{X}_d$ given by $\rho \mapsto \rho^\vee$; composing with Grothendieck–Serre duality $R\mathscr{H}\mathrm{om}_{\mathcal{O}_{\mathcal{X}_d}}(-, \omega_{\mathcal{X}_d})$, where $\omega_{\mathcal{X}_d}$ is the dualizing sheaf (we expect that $\mathcal{X}_d$ will be lci, so this should be well-behaved), we get our duality

$$\mathcal{D}_{\mathcal{X}_d} = \iota^* \circ R\mathscr{H}\mathrm{om}_{\mathcal{O}_{\mathcal{X}_d}}(-, \omega_{\mathcal{X}_d}) : D_{\mathrm{coh}}(\mathcal{X}_d) \to D_{\mathrm{coh}}(\mathcal{X}_d),$$

which is an antiequivalence with $\mathcal{D}_{\mathcal{X}_d} \circ \mathcal{D}_{\mathcal{X}_d} \simeq \mathrm{id}$.

An inertial type τ is a representation $\tau : I_F \to \mathrm{GL}_d(E)$, where I_F is the inertia subgroup of Gal_F, which extends to a representation of W_F with open kernel. In particular, it should have finite image. By the "inertial local Langlands correspondence," due to Schneider and Zink, we can associate to any inertial type τ a finite-dimensional smooth irreducible representation $\sigma^{\mathrm{crys}}(\tau)$ of K. We choose a K-stable $\mathcal{O}$-lattice $\sigma^{\mathrm{crys},\circ}(\tau) \subset \sigma^{\mathrm{crys}}(\tau)$.

For any regular Hodge type $\underline{\lambda}$ and inertial type τ, we could restrict to only looking at (generalized) Galois representations of those types. In practice we often want to restrict to crystalline representations (or various other conditions); we write $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\tau}$ for the substack of $\mathcal{X}_d$ parametrizing crystalline representations (equivalently (φ, Γ)-modules) of those types.

Conjecture (p-adic local Langlands conjectures for GL_d in the Banach setting)**.** *There is an $\mathcal{O}$-linear cocontinuous fully faithful functor*

$$\mathfrak{A} : D^b_{\mathrm{f.p.}}(\mathrm{sm}.\, G) \to D^b_{\mathrm{coh}}(\mathcal{X}_d),$$

extending to a fully faithful functor

$$\mathfrak{A} : \mathrm{Ind}\, D^b_{\mathrm{f.p.}}(\mathrm{sm}.\, G) \to \mathrm{Ind}\,\mathrm{Coh}(\mathcal{X}_d),$$

such that:

(1) $L_\infty = \mathfrak{A}(\mathcal{O}[[G]])$ is a pro-coherent sheaf on $\mathcal{X}_d$, concentrated in degree 0, and is flat over $\mathcal{O}[[K]]$.

(2) There is a natural equivalence

$$\mathfrak{A} \circ \mathbb{D} \xrightarrow{\sim} (\mathcal{D}_{\mathcal{X}_d} \circ \mathfrak{A}) \left[\frac{d(d+1)}{2} [F : \mathbb{Q}_p] + 1 \right]$$

of contravariant functors $D^b_{\mathrm{f.p.}}(\mathrm{sm}.\, G) \to D_{\mathrm{coh}}(\mathcal{X}_d)$.

(3) For any regular Hodge type $\underline{\lambda}$ and inertial type τ, the scheme-theoretic support of $\mathfrak{A}(c\text{-}\mathrm{Ind}_K^G V_{\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{crys,\circ}(\tau))$ is $\mathcal{X}_d^{crys,\underline{\lambda},\tau}$.

There is also a statement about the action of the Bernstein center which I omit because I don't understand it; maybe we'll come back to it in future talks.

The rest of today's talk is devoted to a series of remarks about this conjecture.

The first thing to note is that this gives a fully faithful functor, not an equivalence of categories as in Fargues–Scholze. To fix this, we should study instead of $\operatorname{Ind} D^b_{\text{f.p.}}(\text{sm.}\, G)$ a similar (ind-) derived category of sheaves on Bun_G, in Fargues–Scholze's sense or a similar one. The main obstruction is that there is no such good category of sheaves with coefficients in $\mathcal{O}$; it seems likely that this can be solved using the work of Lucas Mann, by combining his thesis work (which uses the machinery of condensed and solid sheaves to give a good notion of sheaves of $\mathcal{O}_X^+/\pi$-modules on diamonds and v-sheaves) and his recent work on nuclear sheaves of $\mathbb{Z}_\ell$-modules on diamonds and v-sheaves (also using the condensed formalism), and it's even possible that by the end of this seminar such a category will be available. (Or maybe this optimism will come to look foolish.) If so, we expect that $\mathfrak{A}$ should extend to an equivalence of categories, as in Fargues–Scholze.

Since $\mathfrak{A}$ is a fully faithful functor of ∞-categories, we should get an identification of certain endomorphism algebras as E_1-rings. In particular, we should get an isomorphism of E_1-rings between p-adic derived Hecke algebras and certain endomorphism rings of coherent sheaves on $\mathcal{X}_d$; in the ℓ-adic case these correspond to Feng's spectral Hecke algebras, and in that case there is a precise conjecture due to Zhu.

In the conjecture, we restrict to crystalline representations; we could replace this condition with potentially semistable, with more exposition.

From the perspective that we want to describe smooth representations of G via coherent sheaves on $\mathcal{X}_d$, we want to know what happens to π under this functor; for formal reasons it must be of the form

$$\pi \mapsto L_\infty \otimes^{\mathbb{L}}_{\mathcal{O}[[G]]} \pi,$$

where we view $L_\infty = \mathfrak{A}(\mathcal{O}[[G]])$ as a G-module by functoriality, since $\operatorname{End} \mathcal{O}[[G]] = \mathcal{O}[[G]]$. Indeed, by our first, technical conjecture, $\text{sm.}\, G$ is generated by finitely presented objects and so we can resolve a smooth representation π by a (possibly infinite) complex

$$\cdots \to \mathcal{O}[[G]]^{\oplus m_2} \to \mathcal{O}[[G]]^{\oplus m_1} \to \mathcal{O}[[G]]^{\oplus m_0} \to \pi \to 0.$$

Therefore $\mathfrak{A}(\pi)$ is computed by the complex

$$\cdots \to \mathfrak{A}(\mathcal{O}[[G]])^{\oplus m_2} \to \mathfrak{A}(\mathcal{O}[[G]])^{\oplus m_1} \to \mathfrak{A}(\mathcal{O}[[G]])^{\oplus m_0},$$

which via $L = \mathfrak{A}(\mathcal{O}[[G]])$ is the same thing as

$$L_\infty \otimes_{\mathcal{O}[[G]]} \left(\cdots \to \mathcal{O}[[G]]^{\oplus m_2} \to \mathcal{O}[[G]]^{\oplus m_1} \to \mathcal{O}[[G]]^{\oplus m_0}\right),$$

which in the derived category by the above is the same thing as

$$L_\infty \otimes^{\mathbb{L}}_{\mathcal{O}[[G]]} \pi.$$

In the case $d = 1$, we have $L_\infty = \mathcal{O}_{\mathcal{X}_1}$, and there is a description in the case $d = 2$ and $F = \mathbb{Q}_p$, but in general there is no explicit description.

In particular, if V is any smooth $\mathcal{O}[[K]]$-module, then

$$\mathfrak{A}(c\text{-}\operatorname{Ind}_K^G V) = L_\infty \otimes^{\mathbb{L}}_{\mathcal{O}[[G]]} c\text{-}\operatorname{Ind}_K^G V = L_\infty \otimes^{\mathbb{L}}_{\mathcal{O}[[G]]} \mathcal{O}[[G]] \otimes^{\mathbb{L}}_{\mathcal{O}[[K]]} V = L_\infty \otimes^{\mathbb{L}}_{\mathcal{O}[[K]]} V,$$

so since L_∞ is (conjecturally) flat and concentrated in degree 0, we conclude that $\mathfrak{A}(c\text{-}\mathrm{Ind}_K^G V)$ is concentrated in degree 0. In particular, $\mathfrak{A}(c\text{-}\mathrm{Ind}_K^G V_{\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{\mathrm{crys},\circ}(\tau))$, which appears in part (3) of the conjecture, is concentrated in degree 0.

By comparing compact objects, we deduce that an ind-object $\pi \in \mathrm{Ind}\, D^b_{\mathrm{f.p.}}(\mathrm{sm.}\, G)$ is a genuine object of $D^b_{\mathrm{f.p.}}(\mathrm{sm.}\, G)$ if and only if $\mathfrak{A}(\pi) \in \mathrm{Ind}\,\mathrm{Coh}(\mathcal{X}_d)$ is in fact in $D^b_{\mathrm{coh}}(\mathcal{X}_d)$. In particular, if π is concentrated in degree 0, then it is of finite presentation if and only if $\mathfrak{A}(\pi)$ is bounded with coherent cohomology sheaves. (In fact we expect that $\mathfrak{A}(\pi)$ is automatically bounded, and that $\mathfrak{A}$ in general has amplitude $[1-d, 0]$.)

If π is concentrated in degree 0 and is finitely generated but not of finite presentation, then $\mathfrak{A}(\pi)$ is not coherent. However we claim that $H^0(\mathfrak{A}(\pi))$ is coherent. Since π is finitely generated, there is a surjection $c\text{-}\mathrm{Ind}_K^G U \to \pi$ for some finite length K-representation U; by the right t-exactness of $\mathfrak{A}$, we get a surjection $H^0(\mathfrak{A}(c\text{-}\mathrm{Ind}_K^G(U)) \to H^0(\mathfrak{A}(\pi))$, and since the left-hand side is coherent (by construction and the above) so is the right-hand side.

It is not clear whether the properties given in the conjecture (even including the Bernstein action) uniquely characterize the conjecture, or how close they come; they seem to constrain it to be "close to unique" in some heuristic sense.

For any $x \in \mathcal{X}_d(\overline{\mathbb{F}_p})$, assuming $p \nmid 2d$, there is a versal morphism $f : \mathrm{Spf}\, R_\infty \to \mathcal{X}_d$ at x, together with an R_∞-module M_∞ with a commuting action of G. Here R_∞ is a power series over the universal deformation ring for x (as a Galois representation); the variables for R_∞ over this ring correspond to the "patching variables" for Taylor–Wiles, with M_∞ a patched version of completed cohomology for certain unitary groups. These constructions are global and depend on various choices, but we expect that we can construct the R_∞-module M_∞ as $f^* L_\infty$. This gives a purely local construction of M_∞, and justifies some of our expectations for L_∞: for example, M_∞ is always flat over $\mathcal{O}[[K]]$, which explains why we conjecture that L_∞ is flat over $\mathcal{O}[[K]]$. Similarly the expectation that L_∞ is concentrated in degree 0, analogous to similar conjectures for the coherent Springer sheaf in geometric Langlands, is motivated by the fact that its pullback to a versal ring should be concentrated in degree 0.

With the view that L_∞ is a "universal" patched module in this way, the conjectural property (1) can be viewed as explaining the fact that the patched modules M_N at finite level are maximal Cohen–Macaulay over their supports. In particular, we should be able to strengthen the conjectural property (3) as follows: as remarked above, $\mathfrak{A}(c\text{-}\mathrm{Ind}_K^G V_{\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{\mathrm{crys},\circ}(\tau))$ should be concentrated in degree 0, and we further conjecture that it should be maximal Cohen–Macaulay over the support $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\tau}$, and its fiber $\mathfrak{A}(c\text{-}\mathrm{Ind}_K^G V_{\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{\mathrm{crys},\circ}(\tau)) \otimes_{\mathcal{O}} E$ should be locally free of rank 1 over $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\tau} \otimes_{\mathcal{O}} E$. This is via the following heuristic:

Let $i : \mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\tau} \hookrightarrow \mathcal{X}_d$ be the inclusion. We expect this to be pure of codimension $[F:\mathbb{Q}_p] \cdot \frac{d(d+1)}{2}$, though the dimension theory doesn't really exist yet. Thus we expect that $i^! \omega_{\mathcal{X}_d} = \omega_{\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\tau}}[-[F:\mathbb{Q}_p] \cdot \frac{d(d+1)}{2}]$, so for any $\mathcal{F} \in D_{\mathrm{coh}}(\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\tau})$ we have

$$i_* R\mathscr{H}om_{\mathcal{O}_{\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\tau}}}(\mathcal{F}, \omega_{\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\tau}}) = R\mathscr{H}om_{\mathcal{O}_{\mathcal{X}_d}}(i_*\mathcal{F}, \omega_{\mathcal{X}_d}) \left[[F:\mathbb{Q}_p] \frac{d(d+1)}{2} \right].$$

If $\mathcal{D}_{\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\tau}}$ is the corresponding duality (given by the composition of Grothendieck–Serre duality with pullback by the natural involution $\rho \mapsto \rho^\vee$ as on $\mathcal{X}_d$), which replaces $\underline{\lambda}$ with $-\underline{\lambda}$, then taking $\mathcal{F} = \mathfrak{A}(c\text{-}\mathrm{Ind}_K^G V_{\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{\mathrm{crys},\circ}(\tau))$, by our previous calculation of duality applied to compact induction we have

$$\mathbb{D}(c\text{-}\operatorname{Ind}_K^G V_{\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{\text{crys},\circ}(\tau)) = (c\text{-}\operatorname{Ind}_K^G V_{-\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{\text{crys},\circ}(\tau^\vee))[1]$$

and so (by the compatibility with duality of the conjecture)

$$\mathfrak{A}(c\text{-}\operatorname{Ind}_K^G V_{-\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{\text{crys},\circ}(\tau^\vee)) \xrightarrow{\sim} \mathcal{D}_{\mathcal{X}_d^{\text{crys},\underline{\lambda},\tau}}(\mathfrak{A}(c\text{-}\operatorname{Ind}_K^G V_{\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{\text{crys},\circ}(\tau))).$$

Since $\mathfrak{A}(c\text{-}\operatorname{Ind}_K^G V_{\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{\text{crys},\circ}(\tau))$ is supposed by be concentrated in degree 0 and supported on $\mathcal{X}_d^{\text{crys},\underline{\lambda},\tau}$ and similarly for $\mathfrak{A}(c\text{-}\operatorname{Ind}_K^G V_{-\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{\text{crys},\circ}(\tau^\vee))$, by commutative algebra it follows that $\mathfrak{A}(c\text{-}\operatorname{Ind}_K^G V_{\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{\text{crys},\circ}(\tau))$ is maximal Cohen–Macaulay over $\mathcal{X}_d^{\text{crys},\underline{\lambda},\tau}$. The same thing is true after tensoring with E, and by regularity it follows that $\mathfrak{A}(c\text{-}\operatorname{Ind}_K^G V_{\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{\text{crys},\circ}(\tau)) \otimes_{\mathcal{O}} E$ is locally free. The full faithfulness of $\mathfrak{A}$ should then (with some more work) imply that the rank is 1.

Next time, we'll move towards examples, consequences, and connections: the geometric Breuil–Mézard conjecture, known cases, perhaps comparisons with the $\ell \neq p$ case.

References

[1] Matthew Emerton, Toby Gee, and Eugen Hellmann. An introduction to the categorical p-adic Langlands program. *arXiv preprint arXiv:2210.01404*, 2022.