WCart: Bhatt–Lurie's perspective and connections to prisms

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Last time, we talked about Drinfeld's stack $\Sigma = W_{\text{prim}}/W^\times$, together with one interpretation of its functor of points and various properties. Today, we want to give another interpretation which will let us more directly tie it back to prismatic cohomology and give the desired equivalence between quasi-coherent sheaves on Σ and prismatic crystals on $\operatorname{Spf} \mathbb{Z}_p$. We'll do this by defining a new stack WCart via a functor of points, and then show that in fact WCart $= \Sigma$; from there we can use this new interpretation to develop the equivalence.

1. The Cartier–Witt stack

Recall our definition of Σ via the scheme of primitive Witt vectors

$$W_{\text{prim}} = \operatorname{Spf} \mathbb{Z}_p[x_0, x_1, x_2, \ldots][x_1^{-1}]^\wedge_{(p,x_0)}$$

together with the action of the group scheme $W^\times$ by division, so

$$\Sigma = W_{\text{prim}}/W^\times.$$

We're going to introduce a new stack WCart, and will eventually see that the two agree.

We begin with the notion of Cartier–Witt divisors. First, recall that a Cartier divisor of a scheme X is a closed subscheme D such that the ideal sheaf $\mathcal{O}_X(-D)$ is an invertible $\mathcal{O}_X$-module. This is poorly behaved with respect to base change, i.e. the inverse image of a Cartier divisor need not be a Cartier divisor. Therefore we'll replace it with a more general notion:

Definition. A generalized Cartier divisor of a scheme X is a pair $(\mathcal{I}, \alpha)$ where $\mathcal{I}$ is an invertible $\mathcal{O}_X$-module and $\alpha : \mathcal{I} \to \mathcal{O}_X$ is a morphism.

The classical Cartier divisors correspond, by the morphism $\mathcal{O}_X(-D) \to \mathcal{O}_X$, to the generalized Cartier divisors for which α is a monomorphism. We write $\operatorname{Cart}(X)$ for the groupoid of generalized Cartier divisors on X, where morphisms are isomorphisms of $\mathcal{O}_X$-modules commuting with the maps to $\mathcal{O}_X$.

This is now (contravariantly) functorial in X, i.e. there is a good notion of pullback to a generalized Cartier divisor $(f^*\mathcal{I}, f^*\alpha)$ along $f : Y \to X$ (given by pullback as usual on $\mathcal{I}$, and $f^*\alpha : f^*\mathcal{I} \to f^*\mathcal{O}_X = \mathcal{O}_Y$). However $f^*\alpha$ is not in general a monomorphism even if α is, so pullback does not preserve the genuine Cartier divisors.

The map $X \mapsto \operatorname{Cart}(X)$ thus gives a contravariant functor from schemes to groupoids, and is a stack for the fpqc topology, which we call Cart. A similar argument to what we discussed last time with Drinfeld's approach shows that Cart $= [\mathbb{A}^1/\mathbb{G}_{\text{m}}]$, which we won't use and so won't go into detail on. We'll sometimes write $\operatorname{Cart}(R)$ for $\operatorname{Cart}(\operatorname{Spec} R)$.

We now want to incorporate Witt vectors. The naive thing (turning to the affine perspective for now) is to consider the stack sending a p-nilpotent ring R to the groupoid of generalized Cartier divisors on $\operatorname{Spec} W(R)$. We want to require something additional, analogous to the "primitive" condition from last time: given such a generalized Cartier divisor

(I, α) of $\operatorname{Spec} W(R)$, the image of $I \xrightarrow{\alpha} W(R) \twoheadrightarrow R$ should be a nilpotent ideal of R; and the image of $I \xrightarrow{\alpha} W(R) \xrightarrow{\delta} W(R)$ should generate the unit ideal, where δ is the δ-structure automatically attached to $W(R)$. If both of these conditions are satisfied, we say that (I, α) is a Cartier–Witt divisor of R. These form a full subcategory of $\mathrm{Cart}(W(R))$, which we call $\mathrm{WCart}(R)$.

Proposition. *Let R be a p-nilpotent ring and $\alpha : I \hookrightarrow W(R)$ an invertible ideal. Then the following are equivalent:*

- *The pair (I, α) is a Cartier–Witt divisor of R.*
- *The pair $(W(R), I)$ is a prism.*

Thus we can think of $\mathrm{WCart}(R)$ as the groupoid of "generalized" prism structures on $W(R)$, with genuine prism structures corresponding to (I, α) with α injective. Note that again, Cartier–Witt divisors and thus generalized prism structures are functorial, but genuine prism structures are not: the pullback of a genuine prism structure need not be a genuine prism structure, since the pullback of α need not be injective. (The proof is essentially by inspection: the first condition on Cartier–Witt divisors corresponds to $W(R)$ having to be (p, I)-complete, and the second translates to the prismatic condition.)

We can also go the other way: given an arbitrary prism (A, I) and a ring homomorphism $f : A \to R$, since A is a δ-ring there is a canonical lift $\tilde{f} : A \to W(R)$. Let $\tilde{f}^* I = I \otimes_A W(R)$, viewed as a $W(R)$-module. The inclusion $I \hookrightarrow A$ induces a map $\alpha : \tilde{f}^* I \to W(R)$ via tensoring with $W(R)$ over A, so that $(W(R), \alpha)$ is a generalized Cartier divisor of $W(R)$. It is a Cartier–Witt divisor if and only if the image of (p, I) in R is nilpotent, and so as R varies we get a morphism

$$\rho_A : \operatorname{Spf} A \to \mathrm{WCart}$$

for the (p, I)-adic topology on A.

There is a canonical projection $W(R) \to R$, and correspondingly a map $\operatorname{Spec} R \to \operatorname{Spec} W(R)$; therefore there is a pullback $\mathrm{Cart}(W(R)) \to \mathrm{Cart}(R)$, which restricts to a functor $\mathrm{WCart}(R) \to \mathrm{Cart}(R)$. This is functorial in R and so gives a morphism of stacks $\mathrm{WCart} \to \mathrm{Cart} = [\mathbb{A}^1/\mathbb{G}_\mathrm{m}]$. One can check that in fact it factors through $[\hat{\mathbb{A}}^1/\mathbb{G}_\mathrm{m}]$, and so keeping in mind that WCart will turn out to be the same as Σ this corresponds to our projection from last time $\Sigma \to [\hat{\mathbb{A}}^1/\mathbb{G}_\mathrm{m}]$.

We now turn to our first main claim.

Theorem. *The stacks Σ and* WCart *agree.*

Proof. First, we give a morphism $W_{\mathrm{prim}} \to \mathrm{WCart}$; we'll then show that this presents WCart as $W_{\mathrm{prim}}/W^\times = \Sigma$.

For any p-nilpotent ring R, let $v \in W_{\mathrm{prim}}(R)$ be a primitive Witt vector. Via the inclusion $W_{\mathrm{prim}}(R) \hookrightarrow W(R)$, it defines an ideal $\iota_v : (v) \hookrightarrow W(R)$, and the fact that v is primitive implies that $(W(R), \iota_v)$ is a Cartier–Witt divisor: the image of v under the projection to R is nilpotent and $\delta(v)$ is a unit, so v is distinguished and therefore its image in $W_2(R)$ is a unit. Since the construction $v \mapsto (W(R), \iota_v)$ is functorial in R, this gives a morphism $W_{\mathrm{prim}} \to \mathrm{WCart}$.

First, we want to show that locally the induced functor is essentially surjective, i.e. every point of $\mathrm{WCart}(R)$ arises from some $v \in W_{\mathrm{prim}}(R)$ in this fashion. A generalized Cartier divisor of $W(R)$ is a principle ideal if and only if it is abstractly isomorphic to $W(R)$, which is true if and only if it's true after tensoring down to R since p is nilpotent in R, i.e. if $I \otimes_{W(R)} R \simeq R$ as an R-module. This is Zariski-locally true on $\operatorname{Spec} R$ since it is a Cartier divisor, so every Cartier divisor on $W(R)$ comes from some $v \in W(R)$; those which are Cartier–Witt divisors are exactly those which come from $v \in W_{\mathrm{prim}}(R)$.

On the other hand, v and v' give rise to isomorphic Cartier–Witt divisors if and only if $(v) = (v')$, which is true if and only if they differ by a unit of $W(R)$. Thus the map $W_{\mathrm{prim}} \to \mathrm{WCart}$ induces an isomorphism $W_{\mathrm{prim}}/W^\times = \Sigma \xrightarrow{\sim} \mathrm{WCart}$. □

In fact, we can even identify the map $W_{\mathrm{prim}} \to \mathrm{WCart}$ with a prism. Let $A^0 = \mathbb{Z}_p[x_0, x_1, x_2, \ldots][x^{-1}]^\wedge_{(p,x_0)}$, so that $W_{\mathrm{prim}} = \operatorname{Spf} A^0$. Then $(A^0, (x_0))$ is a prism, and the corresponding morphism $\rho_{A_0} : \operatorname{Spf} A^0 \to \mathrm{WCart}$ agrees with the map $W_{\mathrm{prim}} \to \mathrm{WCart}$ inducing the identification above.

Another important example is the de Rham point, which last time was denoted as $p : \operatorname{Spf} \mathbb{Z}_p \to \Sigma = \mathrm{WCart}$. Here, this corresponds to the prism $(\mathbb{Z}_p, (p))$, and so we can naturally call it $\rho_{\mathbb{Z}_p}$ as in the notation above; we'll also write ρ_{dR} to denote the de Rham point.

2. Complexes on WCart $= \Sigma$

For any ring R, write $\mathcal{D}(R)$ for the derived ∞-category of R-modules. We define

$$\mathcal{D}(\mathrm{WCart}) = \varprojlim_{\operatorname{Spec} R \to \mathrm{WCart}} \mathcal{D}(R),$$

a symmetric monoidal stable ∞-category which we call the ∞-category of quasi-coherent complexes on WCart. We can think of its objects as assigning to each ring R and Cartier–Witt divisor $\alpha : I \to W(R)$ a complex of R-modules, functorially in R. One object to note is the unit object $\mathcal{O}_{\mathrm{WCart}}$, which for each R sends $\alpha : I \to W(R)$ to the underlying ring R (as a complex of R-modules, concentrated in degree 0). This is called the structure sheaf of WCart.

Another natural object in this category we could consider is the Hodge–Tate ideal sheaf $\mathcal{I}$, which for each R sends $\alpha : I \to W(R)$ to the R-module $I \otimes_{W(R)} R$, again viewed as a complex concentrated in degree 0.

For any prism (A, I), we defined above an induced morphism $\rho_A : \operatorname{Spf} A \to \mathrm{WCart}$. (Note that this does depend on I via the topology on A, though the notation does not include the dependence.) Then pullback along A gives a functor

$$\rho_A^* : \mathcal{D}(\mathrm{WCart}) \to \mathcal{D}(\operatorname{Spf} A)$$

for every prism (A, I). For technical reasons we restrict to bounded prisms (i.e. $A[p^\infty] = A[p^n]$ for some integer n): this allows us to write $\mathcal{D}(\operatorname{Spf} A) = \hat{\mathcal{D}}(A)$, where $\hat{\mathcal{D}}(A)$ denotes the full subcategory of $\mathcal{D}(A)$ spanned by (p, I)-complete A-modules (again implicitly depending on I).

Theorem. *The above construction induces an equivalence of categories*

$$\mathcal{D}(\mathrm{WCart}) \to \varprojlim_{(A,I)} \hat{\mathcal{D}}(A)$$

where the limit is over all bounded prisms (A, I).

We need the boundedness condition to get $\mathcal{D}(\mathrm{Spf}\, A) = \hat{\mathcal{D}}(A)$, but in fact the limit over all prisms is the same as the limit over bounded prisms, so we could drop "bounded" from the theorem statement without harm.

Thus we can equivalently think of $\mathcal{D}(\mathrm{WCart})$ as the ∞-category of (p, I_{Δ})-complete crystals of complexes of $\mathcal{O}_{\Delta}$-modules on the prismatic site of $\mathrm{Spf}\, \mathbb{Z}_p$.

The proof proceeds by constructing the coproduct of n copies of the prism $(A^0, (x_0))$ from the previous section, and combining these to form a simplicial prism $(A^\bullet, I^\bullet)$. The result then follows from the presentation of WCart as a quotient of $\mathrm{Spf}\, A^0$ and formal properties of the simplicial prism (in particular its cofinality). In fact, this description of WCart via the simplicial prism $(A^\bullet, I^\bullet)$ also allows us to form a "global sections" functor via totalization:

Corollary. *The functor* $\mathcal{D}(\mathbb{Z}) \to \mathcal{D}(\mathrm{WCart})$ *sending* $M \mapsto M \otimes \mathcal{O}_{\mathrm{WCart}}$ *has a right adjoint* $R\Gamma(\mathrm{WCart}, -) : \mathcal{D}(\mathrm{WCart}) \to \mathcal{D}(\mathbb{Z})$, *sending* $\mathcal{F}$ *to the totalization of* $\rho_{A^\bullet}(\mathcal{F})$.

3. The Hodge–Tate divisor

The map WCart $\to$ Cart from above can be thought of as defining a generalized Cartier divisor on WCart. We can also describe this divisor explicitly: the above map is given by pulling back along $W(R) \to R$, so the divisor (evaluated at R) consists of Cartier–Witt divisors $\alpha : I \to W(R)$ such that the composition $I \xrightarrow{\alpha} W(R) \to R$ is zero. We call this the Hodge–Tate divisor $\mathrm{WCart}^{\mathrm{HT}}$. It forms a closed substack of WCart.

For each prism (A, I), we have a morphism $\rho_A : \mathrm{Spf}\, A \to \mathrm{WCart}$. The formal subscheme $\mathrm{Spf}\, A/I$ is carried under this map into $\mathrm{WCart}^{\mathrm{HT}}$; we call the restriction $\rho_A^{\mathrm{HT}} : \mathrm{Spf}\, A/I \to \mathrm{WCart}^{\mathrm{HT}}$. It forms a pullback diagram

$$\begin{array}{ccc} \mathrm{Spf}\, A/I & \xrightarrow{\rho_A^{\mathrm{HT}}} & \mathrm{WCart}^{\mathrm{HT}} \\ \downarrow & & \downarrow \\ \mathrm{Spf}\, A & \xrightarrow{\rho_A} & \mathrm{WCart} \end{array} .$$

For example, for any perfectoid ring R, there is a canonical R-point of $\mathrm{WCart}^{\mathrm{HT}}$: we can write R uniquely as A/I for a perfect prism (A, I), and so we get a map $\rho_A^{\mathrm{HT}} : \mathrm{Spf}\, A/I = \mathrm{Spf}\, R \to \mathrm{WCart}$.

In addition to the de Rham point $\rho_{\mathrm{dR}} = \rho_{\mathbb{Z}_p} = p : \mathrm{Spf}\, \mathbb{Z}_p \to \mathrm{WCart}$, last time we also had the point $V(1) : \mathrm{Spf}\, \mathbb{Z}_p \to \mathrm{WCart}$. We have this here as well: for any p-nilpotent ring R (i.e. affine scheme over $\mathrm{Spf}\, \mathbb{Z}_p$), there is a Cartier–Witt divisor $(W(R), V(1))$, giving an R-point $V(1) : \mathrm{Spec}\, R \to \mathrm{WCart}$ functorially in R. Letting R vary gives a point $V(1) : \mathrm{Spf}\, \mathbb{Z}_p \to \mathrm{WCart}$. In fact, for every R the composition $W(R) \xrightarrow{V(1)} W(R) \to R$ is 0, since

$V(1)$ is 0 in R, so $V(1) : \operatorname{Spec} R \to \mathrm{WCart}$ factors through $\mathrm{WCart}^{\mathrm{HT}}$; therefore so does $V(1) : \operatorname{Spf} \mathbb{Z}_p \to \mathrm{WCart}$.

This can be used to describe $\mathrm{WCart}^{\mathrm{HT}}$ quite explicitly.

Proposition. *The map* $V(1) : \operatorname{Spf} \mathbb{Z}_p \to \mathrm{WCart}^{\mathrm{HT}}$ *extends to an isomorphism*

$$\operatorname{Spf} \mathbb{Z}_p \times B\mathbb{G}_m^\sharp \to \mathrm{WCart}^{\mathrm{HT}}.$$

As for all of WCart, we can look at the category of complexes on $\mathrm{WCart}^{\mathrm{HT}}$: we similarly define

$$\mathcal{D}(\mathrm{WCart}^{\mathrm{HT}}) = \varprojlim_{\operatorname{Spec} R \to \mathrm{WCart}^{\mathrm{HT}}} \mathcal{D}(R),$$

and by a similar argument, now pulling back along $\rho_A^{\mathrm{HT}} : \operatorname{Spf} A/I \to \mathrm{WCart}^{\mathrm{HT}}$ instead of ρ_A, we get

$$\mathcal{D}(\mathrm{WCart}^{\mathrm{HT}}) \xrightarrow{\sim} \varprojlim_{(A,I)} \hat{\mathcal{D}}(A/I)$$

where the completion is now just with respect to the p-adic topology.

Just as for Σ, we have a Frobenius endomorphism F of WCart, which can be viewed as twisting the Cartier–Witt divisors by Frobenius. This preserves the maps ρ_A: the diagram

$$\begin{array}{ccc} \operatorname{Spf} A & \xrightarrow{\rho_A} & \mathrm{WCart} \\ \big\downarrow{\scriptstyle \varphi_A} & & \big\downarrow{\scriptstyle F} \\ \operatorname{Spf} A & \xrightarrow{\rho_A} & \mathrm{WCart} \end{array}$$

commutes, where φ_A is the Frobenius lift on A coming from its δ-structure. However, F does *not* preserve $\mathrm{WCart}^{\mathrm{HT}}$: instead it takes it to a point, i.e. the diagram

$$\begin{array}{ccc} \mathrm{WCart}^{\mathrm{HT}} & \longrightarrow & \mathrm{WCart} \\ \big\downarrow & & \big\downarrow{\scriptstyle F} \\ \operatorname{Spf} \mathbb{Z}_p & \xrightarrow{\rho_{\mathrm{dR}}} & \mathrm{WCart} \end{array}$$

commutes. This is Bhatt–Lurie's version of Drinfeld's statement that the Frobenius is contracting.

4. Examples of crystals

We have our equivalence between prismatic crystals and quasi-coherent complexes on WCart; let's look at some examples of such things on both sides. The first example is prismatic cohomology itself: let X be a smooth and proper $\mathbb{Z}_p$-scheme. Then $(A, I) \mapsto R\Gamma_{\Delta}(X_{A/I}, A)$, the prismatic cohomology of $X_{A/I} = X \times_{\operatorname{Spf} \mathbb{Z}_p} \operatorname{Spf} A/I$ over A, is a prismatic crystal: to every prism (A, I) it associates a perfect complex of (p, I)-complete A-modules. By our equivalence, this means it corresponds to a complex of sheaves on WCart, which we call $\mathscr{H}_{\Delta}(X)$, the prismatic cohomology sheaf of X.

On the other side, we have the simplest sheaf on WCart: the structure sheaf $\mathcal{O}_{\mathrm{WCart}}$, which sends a Cartier–Witt divisor on R $\alpha : I \to W(R)$ simply to R. Given a prism (A, I), the corresponding prismatic crystal to $\mathcal{O}_{\mathrm{WCart}}$ evaluated on (A, I) is given by $\rho_A^* \mathcal{O}_{\mathrm{WCart}}$, which is just the structure sheaf of A, i.e. A as an A-module; thus the prismatic crystal corresponding to $\mathcal{O}_{\mathrm{WCart}}$ is $(A, I) \mapsto A$.

A generalization of this example is given by the Breuil–Kisin twists. There are a few ways we could go about defining these; one simple one, parallel to the definition of Tate twists, is to define, given a prism (A, I),

$$A\{-1\} = H^2_{\mathbb{\Delta}}(\mathbb{P}^1_{A/I}, A).$$

This is an invertible A-module; we call its inverse $A\{1\}$, and define the higher twists $A\{n\}$ via tensor powers. Each of these gives a prismatic crystal: $(A, I) \mapsto A\{n\}$. Since each $A\{n\}$ is an invertible A-module, the corresponding complex on WCart should be an invertible $\mathcal{O}_{\mathrm{WCart}}$-module; since the case $n = 0$ corresponds to $\mathcal{O}_{\mathrm{WCart}}$, we call the complex corresponding to $(A, I) \mapsto A\{n\}$ $\mathcal{O}_{\mathrm{WCart}}\{n\}$.

Instead of $(A, I) \mapsto A$, as for usual prismatic cohomology, we could also look at $(A, I) \mapsto I$ or $(A, I) \mapsto A/I$, as for Hodge–Tate cohomology. These correspond to the Hodge–Tate ideal sheaf $\mathcal{I}$ and $\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}$ respectively, by the same argument as for $\mathcal{O}_{\mathrm{WCart}}$.

With this framing of prismatic cohomology, we can now come back to the prismatic-de Rham comparison theorem: the proof is not too long but is somewhat technical, so we omit it, but we can at least say precisely what we mean. On the WCart side, the prismatic cohomology of a scheme (or suitable p-adic formal scheme) X corresponds to the sheaf $\mathscr{H}_{\mathbb{\Delta}}(X)$. The map $\rho_{\mathrm{dR}} : \mathrm{Spf}\, \mathbb{Z}_p \to \mathrm{WCart}$ allows us to pull back $\mathscr{H}_{\mathbb{\Delta}}(X)$ to a sheaf $\mathscr{H}_{\mathbb{\Delta}}(X)_{\mathrm{dR}} := \rho_{\mathrm{dR}}^* \mathscr{H}_{\mathbb{\Delta}}(X)$, which is an object of $\mathcal{D}(\mathrm{Spf}\, \mathbb{Z}_p) = \hat{\mathcal{D}}(\mathbb{Z}_p)$. The case we're particularly interested in is when X is affine but animated: $X = \mathrm{Spec}\, R$ for an animated ring R, for which we write $\mathscr{H}_{\mathbb{\Delta}}(R)_{\mathrm{dR}}$. On the other hand we have the p-completed derived de Rham complex $\widehat{\mathrm{dR}}_R$ of R, which also lives in $\mathcal{D}(\mathbb{Z}_p)$. The comparison theorem is that this pullback gives an isomorphism in this category:

$$\mathscr{H}_{\mathbb{\Delta}}(R)_{\mathrm{dR}} \simeq \widehat{\mathrm{dR}}_R$$

functorially in R.

This also gives a sense of how exactly even absolute prismatic cohomology is supposed to specialize to various other cohomology theories: each cohomology theory should correspond to a choice of prism (A, I) (e.g. $(\mathbb{Z}_p, (p))$ for de Rham/crystalline cohomology, $(\mathbb{Z}_p[[q-1]], (1 + q + \cdots + q^{p-1}))$ for q-de Rham cohomology, a suitable perfect prism for étale comparison, $(W(C), \ker \theta)$ for C a perfectoid field of characteristic p for A_{inf}-cohomology, etc. Once an appropriate prism (A, I) is chosen, the specialization from prismatic cohomology can be viewed as pulling back $\mathscr{H}_{\mathbb{\Delta}}(X)$ along $\rho_A : \mathrm{Spf}\, A \to \mathrm{WCart}$ (or ρ_A^{HT} in the case of Hodge–Tate cohomology).

References

[1] Bhargav Bhatt and Jacob Lurie. Absolute prismatic cohomology. *arXiv preprint arXiv:2201.06120*, 2022.