Introduction to the stack Σ

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Today's goal is to define Drinfeld's stack Σ, state (and perhaps even prove) some properties about it, and say something about why we should care. Let's start with the last part: Σ is a special case of the prismatization construction $X \mapsto X^{\Delta}$ applied to the terminal object $\operatorname{Spf} \mathbb{Z}_p$ of p-adic formal schemes, i.e. $\Sigma = (\operatorname{Spf} \mathbb{Z}_p)^{\Delta}$. We'll see next time that our definition of Σ (due to Drinfeld [2]) agrees with Bhatt-Lurie's definition in [1] of WCart, where they explain deep connections to prisms and prismatic cohomology: roughly speaking, suitable sheaves on X^{Δ} correspond to sheaves on the prismatic site of X, so that the prismatic cohomology of X can be understood by looking at X^{Δ}. As classically construed, the prismatic cohomology of a point $\operatorname{Spf} \mathbb{Z}_p$ should not be very exciting, but once we allow more general sheaves it's analogous to studying the étale cohomology of a point, which should carry Galois information. If nothing else, the structure map $X \to \operatorname{Spf} \mathbb{Z}_p$ induces a map $X^{\Delta} \to \Sigma$, and so the first thing to do is to understand Σ.

First, we'd like to define it. To do so we need to talk briefly about Witt vector schemes and modules.

1. Witt vector modules and Σ

Let W denote the ring scheme of p-typical Witt vectors over $\operatorname{Spec} \mathbb{Z}$; explicitly, this can be written as $\operatorname{Spec} \mathbb{Z}\{x\} = \operatorname{Spec} \mathbb{Z}[x_0, x_1, x_2, \ldots]$, the spectrum of the free δ-ring on one generator, with $\delta(x_n) = x_{n+1}$. (Often, we'll specialize to the case over $\operatorname{Spf} \mathbb{Z}_p$, so we could think of this as $\operatorname{Spec} \mathbb{Z}_p\{x\}$ without change.)

We want to look at a modification: let Z be the locally closed subscheme of W cut out by $p = x_0 = 0$ and $x_1 \neq 0$, and let W_{prim} be the formal completion of W along Z. Thus an S-point of W_{prim} is an S-point of W such that the image of S_{red} lands in Z. Equivalently, by the description above we can think of a map $S = \operatorname{Spec} R \to W$ as a sequence of elements $x_n \in R$, and a map to W_{prim} is a sequence such that x_0 is nilpotent and x_1 is invertible. Thus we can write $W_{\text{prim}} = \operatorname{Spec} A$ where A is the (p, x_0)-adic completion of $\mathbb{Z}_p[x_0, x_1, x_2, \ldots][x_1^{-1}]$.

The Witt vectors have a Frobenius endomorphism $F : W \to W$, and one can check that it takes W_{prim} to W_{prim}. In particular we get a Cartesian diagram

$$\begin{array}{ccc} W_{\text{prim}} & \xrightarrow{F} & W_{\text{prim}} \\ \downarrow & & \downarrow \\ W & \xrightarrow{F} & W \end{array},$$

with both Frobenii representable in schemes and faithfully flat.

If $W^\times$ is the group of units of W (and so a group scheme), there is an action $W^\times \times W_{\text{prim}} \to W_{\text{prim}}$ given by $(\lambda, x) \mapsto \lambda^{-1} x$ (this is better than the action by multiplication for technical reasons, but they're mostly equivalent). We can then define

$$\Sigma = W_{\text{prim}} / W^\times.$$

Thus as a functor Σ is the (fpqc, or by a nontrivial lemma equivalently Zariski) sheafification of $S \mapsto W_{\text{prim}}(S)/W(S)^\times$.

To describe the S-points of Σ more explicitly, we need to introduce W_S-modules. For a test scheme S (over $\operatorname{Spf}\mathbb{Z}_p$), we define $W_S = W \times S$. This is a ring scheme over S (i.e. the fibers of the projection to S are ring schemes, namely W); by a W_S-module we mean a commutative affine group scheme over S together with an action of W_S, and we say that a W_S-module is invertible if it is locally isomorphic to W_S (so it is essentially the same thing as a $W_S^\times$-torsor). Thus an S-point of $W/W^\times$ is an invertible W_S-module M together with a map of W_S-modules $\xi : M \to W_S$. To get S-points of Σ, we replace W by W_{prim}, which translates to enforcing a "primitiveness" condition on ξ: each fiber of ξ should have reduced part in the kernel of ξ_1 but not in the kernel of ξ_2, where ξ_n is the composition of ξ with the projection to $W_n \times S$.

The morphism $F : W_{\text{prim}} \to W_{\text{prim}}$ descends to an algebraic and faithfully flat morphism $F : \Sigma \to \Sigma$. On S-points with this description, we can view F as sending (M, ξ) to (M', ξ') defined by tensoring the map $\xi : M \to W_S$ along the map $F \times \text{id} : W_S \to W_S$.

The projection $W \to W_1 = \mathbb{A}^1$ induces, after completion, a map $W_{\text{prim}} \to \hat{\mathbb{A}}^1$ to the formal affine line, which is algebraic and flat. It follows that there is an algebraic flat map $\Sigma \to \hat{\mathbb{A}}^1/\mathbb{G}_{\text{m}}$. One can view this map on S-points as sending (M, ξ) to $(\mathscr{L}, v)$, where $\mathscr{L}$ is a line bundle on S and $v : \mathscr{L} \to \mathcal{O}_S$ is given by tensoring the map $\xi : M \to W_S$ along $W_S \to \mathbb{G}_{\text{a}} \times S$, which gives an S-point of $\hat{\mathbb{A}}^1/\mathbb{G}_{\text{m}}$.

One can do everything above replacing W everywhere with W_n; this leads to the stacks $\Sigma_n = (W_n)_{\text{prim}}/W_n^\times$. Then $\Sigma_1 = \hat{\mathbb{A}}^1/\mathbb{G}_{\text{m}}$, and $\Sigma = \varprojlim_n \Sigma_n$. The map $\Sigma \to \hat{\mathbb{A}}^1/\mathbb{G}_{\text{m}}$ from above agrees with the projection $\Sigma \to \Sigma_1$.

2. Points and divisors

We're interested in test schemes S which are locally p-nilpotent, so that they can lie over $\operatorname{Spf}\mathbb{Z}_p$. The simplest case is when S is a perfect $\mathbb{F}_p$-scheme.

Proposition. *If S is a perfect $\mathbb{F}_p$-scheme, $\Sigma(S)$ is a single point.*

Proof. By definition, Σ is the sheafification of $\operatorname{Spec} R \mapsto W_{\text{prim}}(R)/W(R)^\times$, so if we can show that the latter is a point for all perfect $\mathbb{F}_p$-algebras R the result follows. Since R is perfect, it is reduced, and so the primitiveness condition is just saying that the 0th ghost component vanishes and the 1st is invertible, i.e. $W_{\text{prim}}(R) = \{Vy | y \in W(R)^\times\}$ where V is the Verschiebung. Since R is perfect, $F : W(R)^\times \to W(R)^\times$ is an isomorphism and in particular $V(y) = V(1)F^{-1}(y)$, so the action of $W(R)^\times$ is transitive as expected and the quotient is a single point. □

One can also ask about morphisms from other p-adic formal schemes. There are two particularly natural morphisms $\operatorname{Spf}\mathbb{Z}_p \to \Sigma$ which are worth discussing more. Unlike in $W(\mathbb{F}_p) = \mathbb{Z}_p$, the points p and $V(1)$ in $W(\mathbb{Z}_p)$ are distinct; each has image 0 under the projection to $W_1(\mathbb{Z}_p)/p = \mathbb{F}_p$ and invertible image under the first ghost map and so define primitive elements, i.e. maps $\operatorname{Spf}\mathbb{Z}_p \to W_{\text{prim}}$; composing with the projection gives maps $p, V(1) : \operatorname{Spf}\mathbb{Z}_p \to \Sigma$. The Frobenius $F : \Sigma \to \Sigma$ sends $V(1)$ to p, since $F(V(1)) = p$ uniformly in the Witt vectors; in fact, it turns out that for *any* map $\operatorname{Spf}\mathbb{Z}_p \to \Sigma$, composition with Frobenius sends it to the special point $p : \operatorname{Spf} \mathbb{Z}_p \to \Sigma$. (We'll come back to this property.) Bhatt and Lurie call this point p the de Rham point of Σ: pulling back along it gives the comparison theorem between de Rham cohomology and absolute prismatic cohomology.

We now turn to studying certain special divisors on Σ. The most important one is the Hodge–Tate divisor Δ_0, which can be defined as the preimage of $\{0\}/\mathbb{G}_{\mathrm{m}}$ under the map $\Sigma \to \hat{\mathbb{A}}^1/\mathbb{G}_{\mathrm{m}}$. (We've skipped the discussion of effective divisors on stacks, but this is one.) Since the underlying reduced scheme of $\hat{\mathbb{A}}^1$ is just the zero point and the reduced component of Σ lies in the special fiber, note that $\Sigma_{\mathrm{red}} = \Delta_0 \otimes \mathbb{F}_p$.

We can describe the line bundle $\mathcal{O}_\Sigma(-\Delta_0)$ explicitly: for each S, if (M, ξ) is an S-point of Σ then $M/V(M')$ is a line bundle on S fitting into the exact sequence

$$0 \to M/V(M') \to \mathcal{O}_S \to \mathcal{O}_{S \times_\Sigma \Delta_0} \to 0.$$

Here $M' = M \otimes_{W_S} W_S^{(1)}$ and $W_S^{(1)}$ is W_S viewed as a W_S-module via F rather than the identity. Collecting the varying S, this gives a line bundle L on Σ, which fits into a short exact sequence

$$0 \to L \to \mathcal{O}_\Sigma \to \mathcal{O}_{\Sigma \times_\Sigma \Delta_0} = \mathcal{O}_{\Delta_0} \to 0$$

and so $L = \mathcal{O}_\Sigma(-\Delta_0)$.

We can understand Δ_0 very explicitly, this time using $V(1)$ instead of p:

Proposition. *The morphism $V(1) : \operatorname{Spf} \mathbb{Z}_p \to \Sigma$ has image in Δ_0, and induces an isomorphism $\operatorname{Spf} \mathbb{Z}_p/\mathbb{G}_m^\sharp \xrightarrow{\sim} \Delta_0$.*

Here $\mathbb{G}_{\mathrm{m}}^\sharp$ is the associated multiplicative group to the divided power additive group $\mathbb{G}_{\mathrm{a}}^\sharp = \operatorname{Spec} \mathbb{Z}_p[x_0, x_1, x_2, \ldots]/(x_{n+1} - x_n^p/p)$.

Proposition. *There is a commutative diagram*

$$\begin{array}{ccc} \Delta_0 & \longrightarrow & \Sigma \\ \downarrow & & \downarrow F \\ \operatorname{Spf} \mathbb{Z}_p & \xrightarrow{p} & \Sigma \end{array}.$$

Proof. By the previous proposition, $V(1) : \operatorname{Spf} \mathbb{Z}_p \to \Sigma$ factors through Δ_0, and the projection $\operatorname{Spf} \mathbb{Z}_p \to \Delta_0$ factors the identity on $\operatorname{Spf} \mathbb{Z}_p$, so we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Spf} \mathbb{Z}_p & & \\ \downarrow & \searrow^{V(1)} & \\ \Delta_0 & \longrightarrow & \Sigma \\ \downarrow & & \\ \operatorname{Spf} \mathbb{Z}_p & & \end{array}.$$

Since $F(V(1)) = p$, we can fill in the rest of the diagram. □

Since $F : \Sigma \to \Sigma$ is flat, one can also take the preimage of Δ_0 to get divisors $\Delta_n = (F^n)^{-1}(\Delta_0) \subset \Sigma$. Together with the special fiber $\Sigma \otimes \mathbb{F}_p$, these actually turn out to freely generate all effective divisors on Σ. They also have the following relationships:

Proposition. *(i) The intersection of any such divisors is in characteristic p, and for $m < n$ we have $\Delta_m \cap \Delta_n = \Delta_m \otimes \mathbb{F}_p$.*

(ii) After taking the special fiber, $\Delta_n \otimes \mathbb{F}_p = p^n \cdot (\Delta_0 \otimes \mathbb{F}_p)$.

(iii) $\Sigma \otimes \mathbb{F}_p = \bigcup_{n\geq 0} \Delta_n \otimes \mathbb{F}_p$.

(iv) For any morphism $f : S \to \Sigma$ from a quasi-compact scheme S, for all n sufficiently large we have $f^{-1}(\Delta_n) = S \otimes \mathbb{F}_p$.

Thus in a certain sense $\Delta_n \to \Sigma \otimes \mathbb{F}_p$ as $n \to \infty$.

3. The contracting property of Frobenius

We want to come back to the claim that F sends all $\mathbb{Z}_p$-points of Σ to the de Rham point p. This turns out to be because F is *contracting*: let's first say what this means.

Let $\mathcal{C}$ be any category and $F : \mathcal{C} \to \mathcal{C}$ a functor. We write $\mathcal{C}^F$ for the category of pairs (c, α) where c is an object of $\mathcal{C}$ and $\alpha : c \xrightarrow{\sim} F(c)$ is an isomorphism; this is called the category of fixed points of F. There is a canonical faithful functor $\mathcal{C}^F \to \mathcal{C}$ sending $(c, \alpha) \mapsto c$, but it is not in general fully faithful.

On the other hand, one can also study the localization

$$\mathcal{C}[F^{-1}] = \varinjlim \left(\mathcal{C} \xrightarrow{F} \mathcal{C} \xrightarrow{F} \mathcal{C} \xrightarrow{F} \cdots \right).$$

We say that F is contracting if $\mathcal{C}[F^{-1}]$ is a point, i.e. the trivial category with one object and one morphism.

The above two notions are closely related:

Proposition. *If $F : \mathcal{C} \to \mathcal{C}$ is contracting, then $\mathcal{C}^F$ is a point.*

Proof. First, we show that $\mathcal{C}^F$ is nonempty. Since $\mathcal{C}[F^{-1}]$ is a point, it in particular has an object and so $\mathcal{C}$ is nonempty; let c be an object. Then c and $F(c)$ have the same image in $\mathcal{C}[F^{-1}]$, so there is some n such that $F^n(c) \simeq F^n(F(c)) = F^{n+1}(c) = F(F^n(c))$, so $F^n(c)$ is in the essential image of the functor $\mathcal{C}^F \to \mathcal{C}$: there is some isomorphism $F^n(c) \xrightarrow{\sim} F(F^n(c))$.

Next, let c' be any object together with an isomorphism $\alpha : c' \xrightarrow{\sim} F(c')$. Since c' and $F^n(c)$ necessarily agree in $\mathcal{C}[F^{-1}]$, there is some m such that $F^m(c') \simeq F^m(F^n(c))$; since F is an isomorphism on both c' and $F^n(c)$, in fact it follows that $c' \simeq F^n(c)$. Therefore there is only one isomorphism class in $\mathcal{C}^F$. It remains to show only that there is only one morphism.

Let c, c' be in the essential image of $\mathcal{C}^F$ in $\mathcal{C}$. Then F gives a map

$$f : \mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{C}}(F(c), F(c')) \simeq \mathrm{Hom}_{\mathcal{C}}(c, c').$$

We can view $\mathrm{Hom}_{\mathcal{C}}(c, c')$ as a discrete category with f giving an endofunctor, and then $\mathrm{Hom}_{\mathcal{C}}(c, c')[f^{-1}]$ is a point because $\mathcal{C}[F^{-1}]$ is, so f is contracting; on the other hand by the above this implies that there is only one object, and so $\mathrm{Hom}_{\mathcal{C}}(c, c')$ is a point as desired. □

Now, for each p-nilpotent scheme S, we get a category $\Sigma(S)$ and a functor $F(S) : \Sigma(S) \to \Sigma(S)$.

Proposition. *If S is quasi-compact, $F(S)$ is contracting.*

Let Σ^F denote the stack sending $S \mapsto \Sigma(S)^F$. By the above two propositions, it follows that Σ^F is a point:

Corollary. *We have $\Sigma^F = \operatorname{Spf} \mathbb{Z}_p$, and the natural morphism $\Sigma^F = \operatorname{Spf} \mathbb{Z}_p \to \Sigma$ is given by the de Rham point p.*

Proof. The first part follows from the above, and the second part follows from the first part together with the fact that p is fixed by F. □

In particular, we recover the statement that F sends every $\operatorname{Spf} \mathbb{Z}_p$-point to p.

4. Global sections

In fact, we can say more about p: its image is dense in the following sense.

Proposition. *If $Y \subset \Sigma$ is a closed substack such that $p : \operatorname{Spf} \mathbb{Z}_p \to \Sigma$ factors through Y, then $Y = \Sigma$.*

(In particular, p is *not* a monomorphism.)

Corollary. *The canonical homomorphism $\mathbb{Z}_p \to H^0(\Sigma, \mathcal{O}_\Sigma)$ is an isomorphism.*

Proof. Let $\varphi : H^0(\Sigma, \mathcal{O}_\Sigma) \to \mathbb{Z}_p$ denote the pullback along $p : \operatorname{Spf} \mathbb{Z}_p \to \Sigma$. Then $\mathbb{Z}_p \to H^0(\Sigma, \mathcal{O}_\Sigma) \xrightarrow{\varphi} \mathbb{Z}_p$ is the identity on $\mathbb{Z}_p$, so the canonical homomorphism is an injection and φ is a surjection. On the other hand, $\ker \varphi$ consists of functions on Σ which vanish on the pullback along p, which would have to be supported on the complement of the image of p; by the density result above this is impossible, so $\ker \varphi = 0$ and therefore both maps $\mathbb{Z}_p \to H^0(\Sigma, \mathcal{O}_\Sigma) \to \mathbb{Z}_p$ are isomorphisms. □

5. Line bundles

Recall that we constructed (and described) a line bundle $\mathcal{O}_\Sigma(-\Delta_0)$, the kernel of the map $\mathcal{O}_\Sigma \to \mathcal{O}_{\Delta_0}$. This gives an element of $\operatorname{Pic} \Sigma$ and further of $\operatorname{Pic}' \Sigma$, which classifies line bundles on Σ together with a trivialization at p: we can interpret the map p as a $W_{\operatorname{Spf} \mathbb{Z}_p}$-module M together with a map $\xi : M \to W_{\operatorname{Spf} \mathbb{Z}_p}$, which (as this is just a point) is given explicitly by $M = W_{\operatorname{Spf} \mathbb{Z}_p}$ and ξ is multiplication by p. Therefore pulling back $\mathcal{O}_\Sigma(-\Delta_0)$ automatically gives it a trivialization.

The morphism F induces maps $F^* : \operatorname{Pic} \Sigma \to \operatorname{Pic} \Sigma$ and $F^* : \operatorname{Pic}' \Sigma \to \operatorname{Pic}' \Sigma$. One can check that in particular $1 - F^* : \operatorname{Pic}' \Sigma \to \operatorname{Pic}' \Sigma$ is invertible. We then define $\mathcal{O}_\Sigma\{1\} = (1 - F^*)^{-1}(\mathcal{O}_\Sigma(-\Delta_0))$. This is the first Breuil-Kisin twist; we can define $\mathcal{O}_\Sigma\{n\} = \mathcal{O}_\Sigma\{1\}^{\otimes n}$. These are supposed to be analogous to Tate twists, and form the most basic example of F-crystals on Σ (and thus prismatic F-crystals on $\operatorname{Spf} \mathbb{Z}_p$): for $n \leq 0$ we get canonical morphisms $F^*\mathcal{O}_\Sigma\{n\} \to \mathcal{O}_\Sigma\{n\}$.

References

[1] Bhargav Bhatt and Jacob Lurie. Absolute prismatic cohomology. *arXiv preprint arXiv:2201.06120*, 2022.

[2] Vladimir Drinfeld. Prismatization. *arXiv preprint arXiv:2005.04746*, 2020.