Introduction to Σ'

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Recall: $\Sigma(S)$ is the groupoid of pairs (M,ξ) for M an invertible W_S-module and $\xi : M \to W_S$ a primitive morphism. (In light of our comparison to WCart from last time, we could think of (M,ξ) as giving a "generalized ideal" making W_S a "generalized prism.") Here an invertible module is one which is locally isomorphic to W_S (in either the Zariski or fpqc topologies).

We want to define a larger stack, and so we take a generalization: instead of requiring M to be invertible, we require it only to be *admissible*. This means that there is a short exact sequence

$$0 \to M_0 \to M \to M' \to 0$$

of W_S-modules where M_0 is locally isomorphic to $W_S^{(F)}$ and M' is locally isomorphic to $W_S^{(1)}$ as W_S-modules, where $W_S^{(1)}$ is W_S viewed as a W_S-module via Frobenius F rather than via the identity, and $W_S^{(F)}$ is the kernel of the resulting map $F : W_S \to W_S^{(1)}$ of W_S-modules. If M is locally isomorphic to W_S, then it is admissible via the short exact sequence

$$0 \to W_S^{(F)} \to W_S \to W_S^{(1)} \to 0$$

of W_S-modules, so this is indeed a generalization.

If $S = \operatorname{Spec} k$ for k a perfect field of characteristic p, then $\Sigma'(S)$ has exactly three isomorphism classes, represented by

$$(W_S, p), \qquad (W_S^{(F)} \oplus W_S^{(1)}, 0 + V), \qquad (W_S^{(F)} \oplus W_S^{(1)}, 1 + V).$$

Since invertible modules are admissible, we get a natural fully faithful functor $j_+(S) : \Sigma(S) \to \Sigma'(S)$, functorial in S, and so an embedding $j_+ : \Sigma \hookrightarrow \Sigma'$. We call its image Σ_+. It is an open substack affine over Σ'.

As for Σ, there is a natural Frobenius F' on Σ' given by twisting (M,ξ) by Frobenius. However, now its image actually lies in Σ, since Frobenius kills $W_S^{(F)}$, and so we get a morphism $F' : \Sigma' \to \Sigma$. One can check that $F' \circ j$ recovers the original Frobenius F on Σ. It turns out that F' is algebraic, and composing with the map $\Sigma \to \hat{\mathbb{A}}^1/\mathbb{G}_{\mathrm{m}}$ we get that Σ' is algebraic over $\hat{\mathbb{A}}^1/\mathbb{G}_{\mathrm{m}}$.

We can define a second map $j_- : \Sigma \to \Sigma'$, by

$$M \mapsto \tilde{M} = M^{(1)} \times_{W_S^{(1)}} W_S$$

via the maps $\xi : M \to W_S$ and $F : W_S \to W_S^{(1)}$, and $\xi \mapsto \tilde{\xi} : \tilde{M} \to W_S$ given by the projection onto the second factor. Again j_- is an embedding, with image Σ_- an open substack of Σ'; restricting $F' : \Sigma' \to \Sigma$ to Σ_- gives an isomorphism $F' : \Sigma_- \xrightarrow{\sim} \Sigma$. Thus we have

$$F' \circ j_+ = F, \qquad F' \circ j_- = \mathrm{id}_\Sigma .$$

One can work out that Σ_+ and Σ_- are disjoint in Σ'.

One can also interpret $j_- : \Sigma \to \Sigma'$ as right adjoint to $F' : \Sigma' \to \Sigma$.

Recall on Σ, we defined a line bundle $\mathscr{L}_\Sigma = \mathcal{O}_\Sigma(-\Delta_0)$ and used it to define $\mathcal{O}_\Sigma\{1\} = (1-F^*)^{-1}\mathscr{L}_\Sigma$. We want to do something similar for Σ'. In an analogous way (omitted since we haven't introduced Δ_0' yet), we can define a line bundle $\mathscr{L}_{\Sigma'}$ on Σ' which again comes with a map $v_- : \mathscr{L}_{\Sigma'} \to \mathcal{O}_{\Sigma'}$, and the locus on which v_- is an isomorphism is precisely Σ_-. Thus $j_-^*\mathscr{L}_{\Sigma'} \xrightarrow{\sim} \mathcal{O}_{\Sigma_-} \simeq \mathcal{O}_\Sigma$, while $j_+^*\mathscr{L}_{\Sigma'} = \mathscr{L}_\Sigma$.

To form $\mathcal{O}_\Sigma\{1\}$, it was important to fix a trivialization of $\mathscr{L}_\Sigma$ via pullback along $p : \operatorname{Spf}\mathbb{Z}_p \to \Sigma$. Via $j_\pm$ this induces trivializations along $\operatorname{Spf}\mathbb{Z}_p \xrightarrow{p} \Sigma \xrightarrow{j_\pm} \Sigma'$. We define

$$\mathcal{O}_{\Sigma'}\{1\} = \mathscr{L}_{\Sigma'} \otimes F'^*\mathcal{O}_\Sigma\{1\}.$$

Proposition. *We have isomorphisms*

$$j_+^*\mathcal{O}_{\Sigma'}\{1\} \simeq \mathcal{O}_\Sigma\{1\}, \qquad j_-^*\mathcal{O}_{\Sigma'}\{1\} \simeq \mathcal{O}_\Sigma\{1\}.$$

Proof. By the identities above (and the fact that pullback is symmetric monoidal), we have

$$j_+^*\mathcal{O}_{\Sigma'} = j_+^*\mathscr{L}_{\Sigma'} \otimes j_+^*F'^*\mathcal{O}_\Sigma\{1\} = \mathscr{L}_\Sigma \otimes (F' \circ j_+)^*\mathcal{O}_\Sigma\{1\}$$

and

$$j_-^*\mathcal{O}_{\Sigma'} = j_-^*\mathscr{L}_{\Sigma'} \otimes j_-^*F'^*\mathcal{O}_\Sigma\{1\} = \mathcal{O}_\Sigma \otimes (F' \circ j_-)^*\mathcal{O}_\Sigma\{1\}.$$

Since $F' \circ j_+ = F$ and $F' \circ j_- = \mathrm{id}_\Sigma$, this is

$$j_+^*\mathcal{O}_{\Sigma'} = \mathscr{L}_\Sigma \otimes F^*\mathcal{O}_\Sigma\{1\}, \qquad j_-^*\mathcal{O}_{\Sigma'} = \mathcal{O}_\Sigma\{1\}.$$

To finish, we write $\mathcal{O}_\Sigma\{1\} = (1-F^*)^{-1}\mathscr{L}_\Sigma$, so $F^*\mathcal{O}_\Sigma\{1\} = F^*(1-F^*)^{-1}\mathscr{L}_\Sigma = ((1-F^*)^{-1} - 1)\mathscr{L}_\Sigma$, and so

$$j_+^*\mathcal{O}_{\Sigma'} = (1 + (1-F^*)^{-1} - 1)\mathscr{L}_\Sigma = (1-F^*)^{-1}\mathscr{L}_\Sigma = \mathcal{O}_\Sigma\{1\}.$$

□

Thus this is the "right" definition in that it extends $\mathcal{O}_\Sigma\{1\}$ from both copies of Σ in Σ' to the whole thing. One can then define $\mathcal{O}_{\Sigma'}\{n\}$ by tensoring as usual.

Like for Σ, we can define a Hodge–Tate divisor $\Delta_0' \subset \Sigma'$, as the zero locus of $v_- : \mathscr{L}_{\Sigma'} \to \mathcal{O}_{\Sigma'}$. This must be disjoint from Σ_-; its intersection with Σ_+ is exactly $j_+(\Delta_0)$. Again, we can give an explicit description of Δ_0'. Recall that on Σ, we found $\Delta_0 \simeq \operatorname{Spf}\mathbb{Z}_p/\mathbb{G}_m^\sharp$. In this setting, it turns out that $\Delta_0' \simeq (\mathbb{A}^1\hat{\otimes}\mathbb{Z}_p)^{\mathrm{dR}}/\mathbb{G}_m$.

For any p-nilpotent scheme S, we can describe $\Delta_0'(S)$ explicitly: it is the category of line bundles $\mathscr{L}$ on S together with an extension of $W_S^{(1)}$ by $\mathscr{L} \otimes W_S^{(F)}$. Analogous to the diagram of last time, we have a commutative diagram

$$\begin{array}{ccc} \Delta_0' & \longrightarrow & \Sigma' \\ \big\downarrow & & \big\downarrow{\scriptstyle F'} \\ \operatorname{Spf}\mathbb{Z}_p & \xrightarrow{p} & \Sigma \end{array}.$$

Indeed, if we restrict the upper right corner to Σ_+, we obtain the analogous diagram with Δ_0. As a consequence, we obtain that the restrictions of $\mathcal{O}_{\Sigma'}\{1\}$ and $\mathscr{L}_{\Sigma'}$ to Δ_0' agree: the pullback of $\mathcal{O}_\Sigma\{1\}$ along p is canonically trivial, so $\mathcal{O}_{\Sigma'}\{1\} = \mathscr{L}_{\Sigma'} \otimes F'^*\mathcal{O}_\Sigma\{1\}$ restricted to Δ_0' is just the restriction of $\mathscr{L}_{\Sigma'}$.

References

[1] Vladimir Drinfeld. Prismatization. *arXiv preprint arXiv:2005.04746*, 2020.