Norm residue isomorphism theorem: K-theory of the integers

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1. Introduction

Our goal for today is to compute the algebraic K-theory of $\mathbf{Z}$, sketching or omitting proofs as necessary or pleasant. All our methods generalize to rings of integers of arbitrary number fields, but we stick with $\mathbf{Z}$ for simplicity.

Let's first say something about the definition of K-theory. For a commutative ring R, consider the category whose objects are finitely generated projective R-modules and whose morphisms are isomorphisms. We can build a space out of this category by letting the 0-simplices by objects, the 1-simplices morphisms, the 2-simplices commutative triangles, 3-simplices commutative tetrahedra and so on. This space has an operation given by direct sum, which makes it into a commutative monoid up to homotopy; we can then group complete it by formally adding inverses to get the desired space. We can then define $K_n(R)$ to be the nth homotopy group of this space, based at a fixed object R; we can similarly define $K_n(R, A)$ to be the nth homotopy group with coefficients in A for any abelian group A, where $\pi_n(X, A)$ is defined to be the group of homotopy classes of maps into X from a Moore space of A for n, i.e. a CW complex with nth singular homology group A and all other homology trivial. Observe that $K_0(R)$ is what we expect it to be under this definition; if R is a field, then $K_1(R)$ is the fundamental group at the base point, i.e. the group of R-linear automorphisms of R, namely $R^\times$ as expected.

Great. Let's move on to trying to compute this thing. We start by invoking a theorem of Quillen and Borel which tells us that every $K_n(\mathbf{Z})$ is finitely generated and for $n > 0$ the rank of $K_n(\mathbf{Z})$ is 1 if $n \equiv 1 \pmod 4$ and 0 otherwise. Thus the problem is reduced to computing the torsion, which we'll do one prime at a time. (Note: at $\ell = 2$ things are more complicated, so although vaguely similar arguments apply in this case we'll skip it and prove everything up to 2-torsion.)

Our primary tool for this is going to be the motivic spectral sequence

$$E_2^{p,q} = H^{p-q}(X, \mathbf{Z}/m(-q)) \implies K_{-p-q}(X, \mathbf{Z}/m),$$

which is the algebraic analogue of the Atiyah-Hirzebruch spectral sequence computing topological K-theory from singular cohomology. (Note that by taking (co)limits over prime powers $m = \ell^\nu$ we can also take coefficients in $\mathbf{Z}_\ell$ or $\mathbf{Z}/\ell^\infty$.) Here the right-hand side is K-theory with coefficients in $\mathbf{Z}/m$

Here the left-hand side is motivic cohomology, and so for $p - q \leq -q$, i.e. $p \leq 0$, it is isomorphic to $H^{p-q}_{\text{ét}}(X, \mu_m^{-q})$, and in our case of interest will be 0 otherwise. (The only possible exception is for $p = 1$, which will not occur for us; for larger p the motivic cohomology vanishes for dimension reasons, since we are interested in schemes of dimension at most 1.) Since étale cohomology vanishes in negative degrees, this means that we only need to look at the region $q \leq p \leq 0$. Applying some facts about étale cohomology of the integers and related schemes will render this spectral sequence quite manageable.

As it turns out this more or less fully computes $K_n(\mathbf{Z})$ (up to 2-torsion) for odd n, and applying the main conjecture of Iwasawa theory (now a theorem of Wiles) we can prove

Lichtenbaum's conjecture up to 2-torsion:

$$\zeta(1-2k) = (-1)^k 2^r \frac{|K_{4k-2}(\mathbf{Z})|}{|K_{4k-1}(\mathbf{Z})|}$$

for some integer r, and as a consequence (since $\zeta(1-2k)$ is well-understood) we can compute the order of $K_{4k-2}(\mathbf{Z})$ for all k. To compute $K_{4k}(\mathbf{Z})$ and determine the group structure of $K_{4k-2}(\mathbf{Z})$, we need to invoke Vandiver's conjecture.

Let's begin by studying $K_n(\mathbf{Z})$ for small n. The easiest case is of course $n = 0$: a projective $\mathbf{Z}$-module is a free abelian group, so $K_0(\mathbf{Z}) = \mathbf{Z}$. In general, for a Dedekind domain R we have $K_0(\mathbf{Z}) = \mathbf{Z} \oplus \operatorname{Pic} R$, so this can be viewed as a corollary of $\operatorname{Pic} \mathbf{Z} = 0$.

For any commutative ring R, the group $K_1(R)$ is the product of the group of units $R^\times$ with the abelianization of the infinite special linear group $\mathrm{SL}(R)$; for the integers this latter factor turns out to be trivial, and so $K_1(\mathbf{Z}) = \mathbf{Z}^\times \simeq \mathbf{Z}/2$.

For any number field F, the group of units and the Picard (or class) group are related by the exact sequence

$$0 \to \mathcal{O}_F^\times \to F^\times \xrightarrow{\oplus_{\mathfrak{p}} \nu_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}} \mathbf{Z} \to \operatorname{Pic} \mathcal{O}_F \to 0,$$

which in terms of K-groups is

$$0 \to K_1(\mathcal{O}_F) \to F^\times \to \bigoplus_{\mathfrak{p}} \mathbf{Z} \to K_0(\mathbf{Z})/\mathbf{Z} \to 0,$$

or equivalently upon expanding and using the fact that $K_0(L) = \mathbf{Z}$ and $K_1(L) = L^\times$ for any field L

$$0 \to K_1(\mathcal{O}_F) \to K_1(F) \to \bigoplus_{\mathfrak{p}} K_0(\mathcal{O}_F/\mathfrak{p}) \to K_0(\mathcal{O}_F) \to K_0(F) \to 0.$$

This extends to a long exact sequence

$$\cdots \to \bigoplus_{\mathfrak{p}} K_n(\mathcal{O}_F/\mathfrak{p}) \to K_n(\mathcal{O}_F) \to K_n(F) \to \bigoplus_{\mathfrak{p}} K_{n-1}(\mathcal{O}_F/\mathfrak{p}) \to \cdots .$$

The K-theory of finite fields is fully known, thanks to Quillen: $K_{2n}(\mathbf{F}_q) = 0$ and $K_{2n-1}(\mathbf{F}_q) = \mathbf{Z}/(q^n - 1)$ for $n > 0$. In particular if n is odd then $K_n(\mathbf{Z}) \to K_n(\mathbf{Q})$ is surjective, and by a result of Soulé it is always injective; therefore for odd n we have $K_n(\mathbf{Z}) \simeq K_n(\mathbf{Q})$, and for even n we have a short exact sequence

$$0 \to K_n(\mathbf{Z}) \to K_n(\mathbf{Q}) \to \bigoplus_{p} K_{n-1}(\mathbf{F}_p) \to 0.$$

In particular taking $n = 2$ we can compute $K_2(\mathbf{Q}) = K_2^M(\mathbf{Q})$ modulo every integer m via the norm residue isomorphism theorem: $K_2(\mathbf{Q})/m \simeq H^2_{\text{ét}}(\mathbf{Q}, \mu_m^{\otimes 2})$, which is more or less the Brauer group; doing some class field theory gives $K_2(\mathbf{Z}) \simeq \mathbf{Z}/2$.

Somewhat mysteriously, though, the next few groups are $K_3(\mathbf{Z}) \simeq \mathbf{Z}/48$, $K_4(\mathbf{Z}) = 0$, and $K_5(\mathbf{Z}) = \mathbf{Z}$. We want to explain these, as well as give a general formula.

2. Applying the spectral sequence

Our goal is to apply the motivic spectral sequence, with the input of Bloch-Kato, to compute the ℓ-primary part of $K_n(\mathbf{Z})$. To do so we need the following lemma.

Lemma 1. *Let R be a ring such that $K_n(R)$ is finite and $K_{n-1}(R)$ is finitely generated. Then $K_n(R)[\ell^\infty] \simeq K_n(R, \mathbf{Z}_\ell)$ and $K_{n-1}(R)[\ell^\infty] \simeq K_n(R, \mathbf{Z}/\ell^\infty)$ for each prime ℓ.*

Proof. The short exact sequence $0 \to \mathbf{Z} \xrightarrow{\ell^\nu} \mathbf{Z} \to \mathbf{Z}/\ell^\nu \to 0$ induces a long exact sequence

$$\cdots \to K_n(R) \xrightarrow{\ell^\nu} K_n(R) \to K_n(R, \mathbf{Z}/\ell^\nu) \to K_{n-1}(R) \xrightarrow{\ell^\nu} K_{n-1}(R) \to K_{n-1}(R, \mathbf{Z}/\ell^\nu) \to \cdots .$$

Localizing everything at ℓ, since everything is finitely generated we can assume that everything is a direct sum of some number of copies of $\mathbf{Z}_{(\ell)}$ and a finite ℓ-primary abelian group. In particular for ν sufficiently large the map $K_n(R) \xrightarrow{\ell^\nu} K_n(R)$ is the zero map since $K_n(R)$ is finite, and similarly $K_{n-1}(R) \xrightarrow{\ell^\nu} K_{n-1}(R)$ kills the torsion component for ν sufficiently large. Since $K_n(R, \mathbf{Z}/\ell^\nu)$ is an ℓ-primary finite group we get a short exact sequence

$$0 \to K_n(R)[\ell^\infty] \to K_n(R, \mathbf{Z}/\ell^\nu) \to K_{n-1}(R)[\ell^\infty] \to 0$$

for ν sufficiently large. Changing coefficients from $\mathbf{Z}/\ell^\nu$ to $\mathbf{Z}/\ell^{\nu-1}$ corresponds to multiplying by ℓ on the right-hand group; doing this repeatedly eventually kills it, so taking the inverse limit gives an isomorphism $K_n(R)[\ell^\infty] \simeq K_n(R, \mathbf{Z}_\ell)$. Similarly, changing coefficients from $\mathbf{Z}/\ell^\nu$ to $\mathbf{Z}/\ell^{\nu+1}$ corresponds to multiplying by ℓ on the left-hand group, and so taking the direct limit gives an isomorphism $K_n(R, \mathbf{Z}/\ell^\infty) \simeq K_{n-1}(R)[\ell^\infty]$. □

We're now ready to apply our spectral sequence. We need one additional piece of notation: the étale cohomology group $H^0_{\text{ét}}(\operatorname{Spec} \mathbf{Z}[1/\ell], \mathbf{Z}/\ell^\infty(i))$ is finite and cyclic, and we will call its order $w_i^{(\ell)}$, which must be some power of ℓ. It is possible to compute these numbers directly, but we'll get a more direct result from the following result together with the main conjecture of Iwasawa theory.

Proposition 2. *Fix an odd prime ℓ. For every $n \geq 2$ we have*

$$K_n(\mathbf{Z})_{(\ell)} \cong \begin{cases} H^2_{\text{ét}}(\operatorname{Spec} \mathbf{Z}[1/\ell], \mathbf{Z}_\ell(i+1)) & n = 2i \\ \mathbf{Z}/w_i^{(\ell)} & n = 2i-1,\ i \text{ even} \\ \mathbf{Z}_{(\ell)} \oplus \mathbf{Z}/w_i^{(\ell)} & n = 2i-1,\ i \text{ odd} \end{cases}$$

Proof. Taking the ℓ-part of the localization sequence

$$0 \to K_n(\mathbf{Z}) \to K_n(\mathbf{Z}[1/\ell]) \to K_{n-1}(\mathbf{Z}/\ell) \to 0,$$

since $K_{n-1}(\mathbf{Z}/\ell) \simeq \mathbf{Z}/(\ell^{n-1} - 1)$ has no ℓ-primary component after localizing we have $K_n(\mathbf{Z})_{(\ell)} \simeq K_n(\mathbf{Z}[1/\ell])_{(\ell)}$, so it suffices to work with the latter. Since we know the rank part of the claim, it suffices to show that $K_{2i}(\mathbf{Z}[1/\ell])_{(\ell)} = K_{2i}(\mathbf{Z}[1/\ell], \mathbf{Z}_\ell) \simeq H^2_{\text{ét}}(\operatorname{Spec} \mathbf{Z}[1/\ell], \mathbf{Z}_\ell(i+1))$ and $K_{2i-1}(\mathbf{Z}[1/\ell])[\ell^\infty] \simeq \mathbf{Z}/w_i^{(\ell)}$. By Lemma 1 with $n = 2i$ (which applies since every

K-group of $\mathbf{Z}$ is finitely generated and the even-indexed groups are finite), the latter claim is equivalent to $K_{2i}(R, \mathbf{Z}/\ell^\infty) \simeq \mathbf{Z}/w_i^{(\ell)}$.

The étale ℓ-cohomological dimension of $\mathbf{Z}[1/\ell]$ is 2, and so the only nonzero terms of the motivic spectral sequence are as follows: (sketch). Using the fact that $H^2_{\text{ét}}(\operatorname{Spec}\mathbf{Z}[1/\ell], \mathbf{Z}/\ell^\infty(i)) = 0$ for all i (which is proved by some nonsense with étale Chern classes), taking the direct limit we see that there are two diagonal lines of nonzero entries, and so the spectral sequence degenerates and

$$K_{2i}(\mathbf{Z}[1/\ell], \mathbf{Z}/\ell^\infty) \simeq H^0_{\text{ét}}(\operatorname{Spec}\mathbf{Z}[1/\ell], \mathbf{Z}/\ell^\infty(i)) \simeq \mathbf{Z}/w_i^{(\ell)}$$

and

$$K_{2i-1}(\mathbf{Z}[1/\ell], \mathbf{Z}/\ell^\infty) \simeq H^1_{\text{ét}}(\operatorname{Spec}\mathbf{Z}[1/\ell], \mathbf{Z}/\ell^\infty).$$

The first of these is what we wanted to prove for odd n by the above.

To prove the statement for even n, we do the same thing but take the inverse limit of the spectral sequence: we have $H^n_{\text{ét}}(\operatorname{Spec}\mathbf{Z}[1/\ell], \mathbf{Z}_\ell(i)) = 0$ for $i > 0$ and $n \neq 1, 2$, since $\operatorname{Spec}\mathbf{Z}[1/\ell]$ has cohomological dimension 0 and $\mathbf{Z}_\ell(i)$ has trivial invariants for $i > 0$, so the spectral sequence degenerates with

$$K_{2i}(\mathbf{Z}[1/\ell], \mathbf{Z}_\ell) = H^2_{\text{ét}}(\operatorname{Spec}\mathbf{Z}[1/\ell], \mathbf{Z}_\ell)$$

and

$$K_{2i-1}(\mathbf{Z}[1/\ell], \mathbf{Z}_\ell) = H^1_{\text{ét}}(\operatorname{Spec}\mathbf{Z}[1/\ell], \mathbf{Z}_\ell(i)).$$

The first of these gives the desired result in the even case. □

For even i we have $2i - 1 \equiv -1 \pmod 4$ so that K_{2i-1} is finite, so by Lemma 1 we have $K_{2i-1}(\mathbf{Z})[\ell^\infty] \simeq K_{2i-1}(\mathbf{Z}, \mathbf{Z}_\ell) \simeq K_{2i}(\mathbf{Z}, \mathbf{Z}/\ell^\infty)$, so by the proof of Proposition 2 since replacing $\mathbf{Z}$ by $\mathbf{Z}[1/\ell]$ does not change the ℓ-primary subgroup we get an isomorphism

$$H^1_{\text{ét}}(\operatorname{Spec}\mathbf{Z}[1/\ell], \mathbf{Z}_\ell(i)) \simeq K_{2i-1}(\mathbf{Z}, \mathbf{Z}_\ell) \simeq K_{2i}(\mathbf{Z}, \mathbf{Z}/\ell^\infty) \simeq H^0_{\text{ét}}(\operatorname{Spec}\mathbf{Z}[1/\ell], \mathbf{Z}/\ell^\infty(i)) \simeq \mathbf{Z}/w_i^{(\ell)}.$$

We now need the main conjecture of Iwasawa theory, proven by Wiles, in the special case for $\mathbf{Q}$ in the following form.

Theorem 3. *Let ℓ be an odd prime. Then for every integer k there exists a rational number u_i prime to ℓ such that*

$$\zeta(1-2k) = \frac{|H^2_{\text{ét}}(\operatorname{Spec}\mathbf{Z}[1/\ell], \mathbf{Z}_\ell(2k))|}{|H^1_{\text{ét}}(\operatorname{Spec}\mathbf{Z}[1/\ell], \mathbf{Z}_\ell(2k))|} u_k.$$

Corollary 4. *For every integer k, there exists some integer r_k such that*

$$\zeta(1-2k) = (-1)^k 2^{r_k} \frac{|K_{4k-2}(\mathbf{Z})|}{|K_{4k-1}(\mathbf{Z})|}.$$

Proof. The sign comes from the functional equation; for the rest of the statement it suffices to show that each side has the same valuation for each odd prime ℓ. By Proposition 2, the ℓ-primary part of $K_{4k-2}(\mathbf{Z})$ is $H^2_{\text{ét}}(\operatorname{Spec}\mathbf{Z}[1/\ell], \mathbf{Z}_\ell(2k))$, and by the above observation $H^1_{\text{ét}}(\operatorname{Spec}\mathbf{Z}[1/\ell], \mathbf{Z}_\ell(2k)) \simeq H^0_{\text{ét}}(\operatorname{Spec}\mathbf{Z}[1/\ell], \mathbf{Z}_\ell(2k)) \simeq \mathbf{Z}/w_k^{(\ell)}$ is the ℓ-primary part of $K_{4k-1}(\mathbf{Z})$, so the ℓ-primary part of the right hand side is, by Theorem 3, the ℓ-primary part of $\zeta(1-2k)$ as desired. □

In fact it turns out that r_k is always 1.

Corollary 5. *Let B_k be the kth Bernoulli number, and let c_k be the numerator of $\frac{B_k}{4k}$. Then $|K_{4k-2}(\mathbf{Z})| = c_k$ if k is even and $2c_k$ if k is odd, and $w_k = \prod_\ell w_k^{(\ell)}$ is the denominator of $\frac{B_k}{4k}$ up to a possible factor of 2, i.e. $\frac{B_k}{4k}$ is equal to $\frac{c_k}{w_k}$ for k even and $\frac{c_k}{w_k/2}$ for k odd.*

Proof. We have $\zeta(1-2k) = (-1)^k \frac{B_k}{2k}$, so the formula follows (up to powers of 2) from Corollary 4. □

We now know everything except the group structure of $K_{4k-2}(\mathbf{Z})$, though we do know its order, and anything about the groups $K_{4k}(\mathbf{Z})$. To understand these we need to introduce a further ingredient: Vandiver's conjecture.

3. Vandiver's conjecture

We consider the cyclotomic extension $\mathbf{Z}[\zeta]$ of $\mathbf{Z}$ where ζ is a primitive ℓth root of unity. Since $G = \mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$ has order $\ell - 1$ prime to ℓ, we can apply transfer arguments after tensoring with $\mathbf{Z}_\ell$ to get an isomorphism $K_n(\mathbf{Z}) \otimes \mathbf{Z}_\ell \simeq K_n(\mathbf{Z}[\zeta])^G \otimes \mathbf{Z}_\ell$. Again it suffices to work with $\mathbf{Z}[1/\ell]$.

Proposition 6. *If ℓ is a regular prime, i.e. $\mathrm{Pic}\,\mathbf{Z}[\zeta]$ has no ℓ-torsion, then $K_{2i}(\mathbf{Z})[\ell] = 0$.*

Proof. Let $R = \mathbf{Z}[\zeta, 1/\ell]$. We have isomorphisms $H^2_{\text{ét}}(R, \mathbf{Z}_\ell(i))/\ell \simeq H^2_{\text{ét}}(R, \mu_\ell^i) \simeq H^2_{\text{ét}}(R, \mu_\ell) \otimes \mu_\ell^{i-1}$ has components coming from the Brauer group, which has trivial ℓ-torsion since R contains all ℓth roots of unity, and from the Picard group, which has trivial ℓ-torsion since ℓ is regular, so $H^2_{\text{ét}}(R, \mathbf{Z}_\ell(i)) = 0$. As in Proposition 2 this is the ℓ-primary subgroup of $K_{2i}(R) = K_{2i}(\mathbf{Z}[\zeta])$. Therefore $K_{2i}(\mathbf{Z}[\zeta])$ has no ℓ-primary torsion, so neither does its subgroup of G-invariants $K_{2i}(\mathbf{Z})$. □

To check the ℓ-primary torsion of $K_{2i}(\mathbf{Z})$, it remains to look at irregular primes ℓ. We now need Vandiver's conjecture.

Conjecture 7 (Vandiver's conjecture)**.** *Let ℓ be an irregular prime, and let ζ be a primitive ℓth root of unity. Then $\mathrm{Pic}\,\mathbf{Z}[\zeta + \zeta^{-1}]$ has no ℓ-torsion.*

Proposition 8. *Assuming Vandiver's conjecture, for ℓ an irregular prime $K_{4i}(\mathbf{Z})$ has no ℓ-torsion.*

This is roughly analogous to Proposition 6.

Proposition 9. *Assuming Vandiver's conjecture for all ℓ, the groups $K_{4i-2}(\mathbf{Z})$ are cyclic for all i.*

Proof. Vandiver's conjecture implies that the ith summand of $\mathrm{Pic}(R)/\ell$ is cyclic, and taking Galois invariants of the tensor product with powers of μ_ℓ identifies this subgroup with $H^2(\mathrm{Spec}\,\mathbf{Z}[1/\ell], \mu_\ell^{2i})$; by Proposition 2 this is the ℓ-primary part of $K_{4i-2}(\mathbf{Z})$. □

References