Overview of Fargues–Scholze

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1. Geometrizing the local Langlands correspondence

For today, and indeed for much of the seminar, we're going to focus in on the local Langlands correspondence. As Kevin told us last time, this is a correspondence of the following shape: if G is a reductive group (say split for simplicity) over a local field E (such as $\mathbb{Q}_p$, $\mathbb{R}$, or $\mathbb{F}_q((t))$), we're interested in the set of smooth irreducible representations π of $G(E)$. In the case of archimedean fields, i.e. $E = \mathbb{R}$ or $\mathbb{C}$, these are understood by classical work of Langlands and Harish-Chandra: to each π we associate a map $\varphi_\pi : W_E \to \widehat{G}(\mathbb{C})$, called the L-parameter of π, where W_E is the Weil group of E (given by $\mathbb{C}^\times$ for $E = \mathbb{C}$ and a nonsplit extension of $\mathbb{C}^\times$ by $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\pm 1\}$ for $E = \mathbb{R}$) and $\widehat{G}$ is the Langlands dual group Kevin told us about. This association $\pi \mapsto \varphi_\pi$ has finite fibers, which are the objects of a large amount of study; in the case $G = \mathrm{GL}_n$, the fibers are singletons and so $\pi \mapsto \varphi_\pi$ is a bijection.

The idea is that the same thing should hold true for arbitrary local fields E; in particular we'll focus on E nonarchimedean, so not $\mathbb{R}$ or $\mathbb{C}$, and the case of most interest is E a p-adic field, i.e. a finite extension of $\mathbb{Q}_p$ (though we'll generally try to handle nonarchimedean E uniformly). The only thing we have to change in the above is the definition of the Weil group W_E: for E nonarchimedean, with residue field $k = \mathbb{F}_q$, we have a map $\mathrm{Gal}(\overline{E}/E) \to \mathrm{Gal}(\overline{k}/k) \simeq \widehat{\mathbb{Z}}$, and W_E is the preimage under this map of $\mathbb{Z} \subset \widehat{\mathbb{Z}}$ generated by the Frobenius $x \mapsto x^q$.

As Kevin told us, this is known for $G = \mathrm{GL}_n$, by the work of Michael Harris among others, and these days for certain other classical groups, but not in general. The goal of Fargues–Scholze is to propose a uniform local Langlands correspondence, constructing a map $\pi \mapsto \varphi_\pi$, which comes out of some more powerful structure and which we hope explains the relationship between the representation theory of $G(E)$ and W_E, and the appearance of the Langlands dual group $\widehat{G}$. The upshot is a categorical local Langlands correspondence, which was previously exclusively the realm of geometric Langlands.

The basic idea of Fargues is to geometrize the local Langlands program using the Fargues–Fontaine curve. This is something roughly like a curve which can be associated to a given local field E. We know a Langlands program for arbitrary curves: the geometric Langlands program, which additionally can give not only a finite-to-one map with various properties but a full equivalence of categories (in particular the Fargues–Fontaine curve does not live over a finite field, and so is not in the realm of classical function field Langlands). Thus the idea is to try to do geometric Langlands for the Fargues–Fontaine curve, and then the hope is that this a priori geometric program will shed light on, or even be equivalent to, the arithmetic local Langlands program.

There are a number of issues here. One is that the Fargues–Fontaine curve is not literally a curve, and is best understood as not even a scheme but an adic space, or better a diamond. Another is that even if a geometric Langlands correspondence for the Fargues–Fontaine curve can be written down, it's not obvious it should have any connection to the problem for the original local field. Nevertheless, this turns out to be possible, after developing a remarkable amount of machinery to deal with the technical difficulties, including the geometry of diamonds and v-stacks and condensed and solid mathematics. We will mostly skip lightly over these ideas, and say only what the constructions allow us to do (with at most hints of what goes into them).

2. Bun_G AND SHEAVES ON IT

In the usual geometric Langlands program, the main object of study on the geometric side is Bun_G, the stack of G-bundles on our curve X. Specifically, if the base field is k, for any test k-scheme S the S-points of Bun_G are G-bundles on $S \times_k X$. As Kevin discussed, functions on $\mathrm{Bun}_G(k)$ can be thought of as automorphic forms, and correspond (at least when k is finite) via trace of Frobenius to sheaves on Bun_G.

Much of the relevance of Bun_G comes from the Hecke action on it. This works as follows: we have some stack Hk_G, which roughly speaking parametrizes a pair of G-bundles $\mathcal{E}_1$, $\mathcal{E}_2$, together with a "modification," i.e. an isomorphism between $\mathcal{E}_1$ and $\mathcal{E}_2$ away from some point x. Remembering only $\mathcal{E}_1$, $\mathcal{E}_2$, or x gives two natural projections from Hk_G to Bun_G and one to X, which we usually think of as a diagram

$$\begin{tikzcd} & \mathrm{Hk}_G \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times X \end{tikzcd}.$$

If we start with a sheaf on Bun_G (which is a geometric incarnation of an automorphic form) we can pull it back to Hk_G and then push forward to the product to get a sheaf on $\mathrm{Bun}_G \times X$, which can be thought of as a $\pi_1(X)$-equivariant object on Bun_G. (This is because in a certain sense the Hecke operator given by this push-pull is locally constant in x, and so varying x gives an action of $\pi_1(X)$.) Since $\pi_1(X)$ is the geometric incarnation of the Galois group (and for function field Langlands is literally the same thing), this shows where the Galois action is coming from.

In our case, the group we are hoping will show up is not the whole Galois group but only the Weil group W_E. In the function field case, this is analogous to replacing our curve X/k by first base changing to the algebraic closure $X_{\bar{k}}$ and then quotienting by Frobenius: the map $X_{\bar{k}}/\varphi \to \bar{k}/\varphi \to k$ at least heuristically corresponds to taking only the piece of $\pi_1(\operatorname{Spec} k)$ generated by Frobenius. We don't have a curve X but instead a local field E; base changing to $\bar{k}$ can be thought of as replacing E by $\breve{E} = E \otimes_{W(k)} W(\bar{k})$, the completion of the maximal unramified extension of E. Thus the object X we want to appear in the above diagram should be something like $(\operatorname{Spec} \breve{E})/\varphi$.

However, we run into trouble using this X to define Bun_G, since it isn't a curve in any usual sense. Instead, we think about the functor of points: Bun_G is supposed to send a test scheme S (over the base field, which for now we take to be $\overline{\mathbb{F}_p}$) to the set, or groupoid, of G-torsors on $S \times_k (\operatorname{Spec} \breve{E})/\varphi$. If $E = \mathbb{F}_q((t))$ is a local function field, this more or less literally works: if $S = \operatorname{Spec} R$, then this should be $\operatorname{Spec} R((t))/\varphi$. (This quotient is still tricky, but if we work in the category of adic spaces instead of schemes there is no difficulty.)

If E is a p-adic field, however, we have more substantial problems: this product no longer makes any sense. The key idea is that $R \otimes_k \breve{E}$ should be something like $W(R) \otimes_{W(k)} \breve{E}$: this carries a canonical map from $\breve{E}$, as expected, and although there can be no map of rings from R, since R is characteristic p and $W(R)$ is characteristic 0, there is a map of multiplicative monoids, sometimes called the Teichmüller map, $R \to W(R)$, so this is some analogue of the tensor product (or dually the fiber product). To make sure this is well-behaved, we replace the category of all $\overline{\mathbb{F}_p}$-schemes by the category **Perf** of (adic) perfectoid spaces over $\overline{\mathbb{F}_p}$, which is generally the test category for diamonds and v-sheaves.

Now we can form the Fargues–Fontaine curve: given a test space S in **Perf** and a local field E, we define $X_{S,E} = (S \dot{\times} \operatorname{Spa} \breve{E})/\varphi$, where Spa means that we're taking adic spaces rather than schemes, the symbol $\dot{\times}$ means this weird product defined via Witt vectors (in the mixed characteristic case), and φ comes from the Frobenius on S. This is *not* a single curve associated to E: instead, it's like the family of spaces $S \times_k X$ for a given base curve X, in that these are supposed to be analogous to products. Therefore we define the S-points of Bun_G to be G-torsors on $X_{S,E}$.

We can recover something like our original target of $(\operatorname{Spec} \breve{E})/\varphi$ by looking at "degree 1 divisors" on the Fargues–Fontaine curve: unlike with a usual curve, these are classified not by the same thing but by a space Div^1 which can be seen to be isomorphic to $(\operatorname{Spa} \breve{E})/\varphi$. Unlike the Fargues–Fontaine curve, which is an honest adic space and can even be viewed as a scheme, Div^1 is only a diamond. This does mean though that modifications of vector bundles, classified by the Hecke stack Hk_G, are projected to Bun_G and $\mathrm{Bun}_G \times \mathrm{Div}^1$, so $\pi_1(\mathrm{Div}^1) = W_E$ arises as desired.

It is not at all obvious that this functor Bun_G on **Perf** should have any kind of reasonable geometry admitting categories of sheaves, but it does in fact turn out to be true and there is a particular derived category $D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ with good properties (which takes quite a lot of work to define; recent work, as in Tuesday, of Lucas Mann provides a slightly different but in some ways more natural category, and it will be interesting to see how his version behaves in the Langlands world). It should be emphasized that this is not the naive notion (it is instead what Fargues–Scholze call $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$, but we haven't said what this would mean).

Further, we have a stratification of Bun_G by locally closed substacks Bun_G^b for $b \in B(G)$ classifying inner forms G_b of G, and further $\mathrm{Bun}_G^b \simeq [*/\widetilde{G}_b(E)]$ for some extension $\tilde{G}_b$. For b basic, this extension is trivial, and the semistable locus $\mathrm{Bun}_G^{\mathrm{ss}}$ decomposes as

$$\coprod_b [*/G_b(E)]$$

over b basic in $B(G)$. In particular, $D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ decomposes as a sum of the $D(\mathrm{Bun}_G^b, \overline{\mathbb{Q}}_\ell)$, and for b basic $D(\mathrm{Bun}_G^b, \overline{\mathbb{Q}}_\ell) \simeq D([*/G_b(E), \overline{\mathbb{Q}}_\ell) \simeq D(G_b(E), \overline{\mathbb{Q}}_\ell)$, the derived category of smooth $G_b(E)$-representations. In particular $D(G(E), \overline{\mathbb{Q}}_\ell)$ embeds into $D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$, so studying sheaves on Bun_G gives a direct geometrization of studying smooth representations of $G(E)$.

3. Geometric Satake

We have sort of seen where the Galois action comes in to the picture when we're studying representations of $G(E)$ (we'll be more explicit about this soon), but we still haven't said anything to explain the presence of the Langlands dual group $\widehat{G}$. The answer is again through the Hecke action. In particular, we have the following version of geometric Satake: there is a natural symmetric monoidal functor $\mathrm{Rep}\,\widehat{G} \to D(\mathrm{Hk}_G, \overline{\mathbb{Q}}_\ell)$. Therefore given a representation V of $\widehat{G}$, we get a derived sheaf of $\overline{\mathbb{Q}}_\ell$-modules on Hk_G, which by the push-pull above and tensoring with this sheaf in the middle (essentially integrating against this kernel) we get an operator $T_V : D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \to D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^{BW_E}$, i.e. landing in W_E-equivariant objects.

Before we talk about how we can assemble this structure to produce L-parameters for a given representation π of $G(E)$, let's first say something about how this functor is built. Roughly, this will be as follows: first, we look at a certain subcategory of sheaves Sat_G^I on a Hecke stack with multiple legs indexed by a finite set I, and define a fusion product $\mathrm{Sat}_G^I \times \mathrm{Sat}_G^I \to \mathrm{Sat}_G^I$. Just as Hk_G lives over Div^1, once we have legs indexed by I this lives over $(\mathrm{Div}^1)^I$, so we can push forward to I to get local systems on $(\mathrm{Div}^1)^I$, which by a version of Drinfeld's lemma are the same thing as W_E^I-representations, so we get a symmetric monoidal functor $\mathrm{Sat}_G^I \to \mathrm{Rep}\,W_E^I$. By Tannakian theory, there should be some Hopf algebra H^I (ind-)internal to $\mathrm{Rep}\,W_E^I$ such that Sat_G^I is the representation category, internal to $\mathrm{Rep}\,W_E^I$, of H^I. Since everything is functorial in I, this boils down to the case $I = \{*\}$, where $H^{\{*\}}$ is some affine group scheme over the coefficients with a W_E-action. This turns out to be exactly $\widehat{G}$, with $\mathrm{Sat}_G^I \simeq \mathrm{Rep}(\widehat{G} \rtimes W_E)^I$, and so we get a natural map from representations of $\widehat{G}$ into W_E-equivariant sheaves on Hk_G.

4. L-parameters

We are now ready to construct the L-parameters. I don't want to write this out again so I'll refer to notes here.

5. The categorical conjecture

We now have two geometric objects: Bun_G and this stack of Langlands parameters up to conjugacy $Z^1(W_E, \widehat{G})/\widehat{G}$. Following the form of the geometric Langlands correspondence suggests that the conjecture should be the following:

$$D(\mathrm{Bun}_G, \mathbb{Z}_\ell)^\omega \simeq D^{b,\mathrm{qc}}_{\mathrm{coh,nilp}}(Z_1(W_E, \widehat{G})/\widehat{G})$$

as stable infinity-categories, where ω denotes compact objects and Nilp denotes nilpotent singular support, which we will not get into. In fact we can say slightly more: we discussed a W_E-action on each side through geometric Satake, which can be upgraded to an action of the category $\mathbf{Perf}(Z^1(W_E, \widehat{G})/\widehat{G})$ of perfect complexes on the stack of Langlands parameters up to conjugacy on each side, and this equivalence should respect this action. This correspondence should send the structure sheaf on the right to the Whittaker sheaf on Bun_G corresponding to the Whittaker representation, which depends on a choice of Whittaker datum on which the equivalence depends.

In particular, the left-hand side canonically includes $D(G(E), \mathbb{Z}_\ell)^\omega$, and so gives a categorical description (with no decategorification necessary) of smooth representations of $G(E)$. This is roughly what we hope to accomplish with coefficients in $\mathbb{Z}_p$-algebras.

References

[1] Laurent Fargues and Peter Scholze. Geometrization of the local Langlands correspondence. *arXiv preprint arXiv:2102.13459*, 2021.