Norm residue isomorphism theorem: introduction and overview
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1. THE CONJECTURE

Ultimately we want to study the étale cohomology of a field k, and find some relationship with its K-theory. In fact we'll use a simpler version of K-theory which is fairly explicit in order to give a description of the more mysterious étale cohomology; let's discuss this modified K-theory first.

1.1 Milnor K-theory

Let's start with the zeroth K-group, which is the Grothendieck group of the monoid of (finite-dimensional) vector bundles on a space up to isomorphism (and a splitting relation). For a field k, the vector bundles on $\operatorname{Spec} k$ are just k-vector spaces, which are classified by their dimension; the Grothendieck group adds inverses, so $K_0(k) = \mathbb{Z}$.

The first K-group is more complicated; we can define it as a quotient of matrix groups, but for our purposes it suffices to know that for a field k we have $K_1(k) = k^\times$. Matsumoto's theorem tells us that $K_2(k)$ is the tensor algebra $(k^\times \otimes_{\mathbb{Z}} k^\times)/\langle a \otimes (1-a) \rangle$ for $a \in k - \{0, 1\}$.

This suggests that we can form the full graded K-theory by taking the tensor algebra generated over $K_0(k) = \mathbb{Z}$ by $K_1(k) = k^\times$ modulo the relations $a \otimes (1-a) = 0$. Unfortunately this is not true in degree higher than 2. However, we can define *Milnor* K-theory $K_*^M(k)$ to be this algebra; we write $\{a, b\}$ for the product in K_*^M, so $\{a, 1-a\} = 0$.

1.2 The Kummer map

Now, for any field k with algebraic closure $\bar{k}$ and any prime p different from the characteristic of k we have a short exact sequence

$$1 \to \mu_p \to \bar{k}^\times \xrightarrow{p} \bar{k}^\times \to 1$$

where μ_p is the group of pth roots of unity and p denotes the pth power map $x \mapsto x^p$. Taking (continuous) Galois cohomology with respect to k (or equivalently étale cohomology of $\operatorname{Spec} k$) gives a long exact sequence

$$1 \to H^0(k, \mu_p) \to H^0(k, \bar{k}^\times) \xrightarrow{p} H^0(k, \bar{k}^\times) \to H^1(k, \mu_p) \to H^1(k, \bar{k}^\times) \to \cdots.$$

On the one hand $H^0(k, M)$ is just the $\operatorname{Gal}(\bar{k}/k)$-invariants of M, and so in particular $H^0(k, \bar{k}^\times) = (\bar{k}^\times)^{\operatorname{Gal}(\bar{k}/k)} = k^\times$; on the other hand by Hilbert's Theorem 90 $H^1(k, \bar{k}^\times) = 0$, so we have an exact sequence

$$k^\times \xrightarrow{p} k^\times \to H^1(k, \mu_p) \to 0$$

and therefore an isomorphism

$$k^\times/k^{\times p} \xrightarrow{\sim} H^1(k, \mu_p),$$

where $k^{\times p}$ denotes the image of $k^\times$ under the pth power map. This is the Kummer isomorphism.

By the definition above,

$$K_1(k) = K_1^M(k) = k^\times,$$

and so via the multiplicative action of p we have

$$K_1(k)/p = K_1^M(k)/p = k^\times/k^{\times p}.$$

Therefore the Kummer isomorphism can be viewed as an isomorphism

$$K_1^M(k)/p \to H^1(k, \mu_p).$$

On the left-hand side we have the product $K_1^M(k) \otimes K_1^M(k) \to K_2^M(k)$ sending $a \otimes b \mapsto \{a, b\}$, which we know has kernel generated by symbols of the form $a \otimes (1 - a)$. On the right-hand side we have the cup product $H^1(k, \mu_p) \otimes H^1(k, \mu_p) \xrightarrow{\cup} H^2(k, \mu_p^{\otimes 2})$.

Lemma 1 (Tate). *If $[a]$ denotes the image of $a \in k^\times$ in $H^1(k, \mu_p)$ under the Kummer map, then $[a] \cup [1 - a] = 0$ for every $a \in k - \{0, 1\}$.*

Proof. Fix $a \in k - \{0, 1\}$ and set $\alpha = \sqrt[p]{a}$ and $E = k(\alpha)$. We have

$$N_{E/k}(x - \alpha) = \prod_{\sigma \in \mathrm{Gal}(E/k)} (x - \sigma(\alpha)) = x^p - a$$

and so setting $x = 1$ gives $N_{E/k}(1 - \alpha) = 1 - a$. Let $\mathrm{res}_{E/k} : H^*(k, \mu_p^{\otimes *}) \to H^*(E, \mu_p^{\otimes *})$ be the map induced by the inclusion, and $\mathrm{cores}_{E/k} : H^*(E, \mu_p^{\otimes *}) \to H^*(k, \mu_p^{\otimes *})$ be the transfer map given at $* = 1$ by the norm map $N_{E/k}$. Since both of these are compatible with the cup product, we have

$$[a] \cup [1 - a] = [a] \cup [\mathrm{Nm}_{E/k}(1 - \alpha)] = [a] \cup [\mathrm{cores}_{E/k}(1 - \alpha)] = \mathrm{res}([\alpha^p] \cup [1 - \alpha]).$$

But $[\alpha^p]$ is the image of α^p under the map $E^\times \to H^1(k, \mu_p)$, which we know has kernel containing all pth powers in $E^\times$; so $[\alpha^p] = 0$ and therefore $\mathrm{res}([\alpha^p] \cup [1 - \alpha]) = 0$. $\qquad\square$

Therefore the tensor product of Kummer maps $k^{\otimes n} \to H^1(k, \mu_p) \otimes \cdots \otimes H^1(k, \mu_p) \xrightarrow{\cup} H^n(k, \mu_p^{\otimes n})$ factors through $K_n^M(k)$, and in fact through $K_n^M(k)/p$. Thus we have the desired map

$$K_n^M(k)/p \to H^n(k, \mu_p^{\otimes n}),$$

called the norm residue homomorphism. The main theorem of this seminar is the following.

Theorem 2 (Norm residue isomorphism theorem). *For every field k with characteristic different from p and every positive integer n, the norm residue homomorphism is an isomorphism. Equivalently for every such field k*

$$K_*^M(k)/p \to H^*(k, \mu_p^{\otimes n})$$

is a ring isomorphism.

For local fields k, the Hilbert symbol (a, b) (or norm residue symbol) which is 1 if $ax^2 + by^2 = z^2$ has a nonzero solution over k and -1 otherwise satisfies the relation $(a, 1 - a) = 1$ for all $a \in k - \{0, 1\}$, so it factors through $K_2^M(k)$. Its image can be interpreted as the 2-torsion in the Brauer group of F, or equivalently $H^2(k, \mu_2) = H^2(k, \mu_2^{\otimes 2})$; thus the Hilbert symbol gives the norm residue homomorphism in this case.

The Hilbert symbol can also be viewed as detecting whether b is a norm of an element of $k[\sqrt{a}]$, thus the term "norm residue symbol"; this lends its name to the more general homomorphism.

2. FIRST REDUCTIONS

The first thing to do is to reduce to certain fields with convenient properties. The central tool here is called transfer.

2.1   Transfer

Fix a base field k_0, and let F be a functor from the category of algebraic field extensions of k_0 to $\mathbb{F}_p$-modules. In addition to F being a covariant functor, i.e. a field extension $k \hookrightarrow k'$ induces a map $F(k) \to F(k')$, we also assume that F is contravariant for *finite* extensions, i.e. if k'/k is a finite extension then we also have a map $F(k') \to F(k)$, and that the resulting composite $F(k) \to F(k') \to F(k)$ is given by multiplication by $[k' : k]$. The contravariant maps are called transfer maps.

If $[k' : k]$ is prime to p, then $F(k) \to F(k') \to F(k)$ is invertible, and so $F(k) \to F(k')$ must be injective. More generally if k' is algebraic over k and for every $\alpha \in k'$ the degree of α over k is prime to p, then $k \hookrightarrow k(\alpha)$ induces an injection $F(k) \to F(k(\alpha))$ and so adding all of the α gives an injection $F(k) \hookrightarrow F(k')$. Thus if we want to prove that $F(k)$ vanishes it suffices to prove that $F(k')$ vanishes for an algebraic extension k' whose elements have degree prime to p.

Now consider the functors $k \mapsto K_n^M(k)/p$ and $k \mapsto H^n(k, \mu_p^{\otimes n})$. Both of these are covariant functors on algebraic extensions with values in $\mathbb{F}_p$-modules; the transfer maps for K_n^M are induced by the norm map $\mathrm{Nm}_{k'/k} : k'^\times \to k^\times$ in degree 1, and for Galois cohomology are given by corestriction. The condition that the composition is multiplication by the degree is clear in degree 1 for Milnor K-theory, since the norm map has degree $[k' : k]$, and thus holds in all degrees; for Galois cohomology it is the restriction-corestriction composition. Therefore the kernel and cokernel of the norm residue homomorphism are functorial in k and also satisfy the transfer hypotheses, and so to show that the norm residue homomorphism is an isomorphism for k it suffices to prove it for k'/k with elements of degree prime to p. More broadly we can assume by this argument that k is p-special, i.e. it has no finite extensions of degree prime to p, since if there exists such an extension we can replace k with it.

2.2   Characteristic 0

Next, we want to reduce to characteristic 0 fields. We do this using Witt vectors: if k is a field of characteristic $q \neq p$, then we can adjoin all qth roots by the transfer argument, so we can assume k is perfect and thus consider its Witt vectors W. We have an induced map

$K_n^M(W) \to K_n^M(k)$ and a transfer map $K_n^M(\text{Frac } W) \to K_n^M(W)$ induced by the degree 1 map $(\text{Frac } W)^\times \simeq \mathbb{Z} \times T \times U \to U \hookrightarrow W^\times$, where $\mathbb{Z}$ comes from the degree of the uniformizer, T is a finite torsion group, and $U \subset W^\times$ is the preimage of 1 under the reduction map $W \to k$. These compose to a specialization map $K_n^M(\text{Frac } W) \to K_n^M(k)$.

Similarly, we get an induced map $H_{\text{ét}}^n(\text{Spec } W, \mu_p^{\otimes n}) \to H_{\text{ét}}^n(\text{Spec } k, \mu_p^{\otimes n}) = H^n(k, \mu_p^{\otimes n})$ and a transfer map $H^n(\text{Frac } W, \mu_p^{\otimes n}) = H_{\text{ét}}^n(\text{Spec}(\text{Frac } W), \mu_p^{\otimes n}) \to H_{\text{ét}}^n(\text{Spec } W, \mu_p^{\otimes n})$ by pushforward. Thus we have a specialization map $H^n(\text{Frac } W, \mu_p^{\otimes n}) \to H^n(k, \mu_p^{\otimes n})$.

These specialization maps turn out to be surjective and compatible with the norm residue homomorphism in the sense that the diagram

$$
\begin{array}{ccc}
K_n^M(\text{Frac } W)/p & \longrightarrow & H^n(\text{Frac } W, \mu_p^{\otimes n}) \\
\downarrow & & \downarrow \\
K_n^M(k)/p & \longrightarrow & H^n(k, \mu_p^{\otimes n})
\end{array}
$$

commutes. If the top arrow is an isomorphism, since the composition with the right arrow is surjective it follows that the bottom arrow must be surjective. We can choose splittings s_0 and s_1 of the left and right surjections respectively such that the corresponding diagram again commutes, so that for a in the kernel of the bottom map $s_0(a)$ must correspond so s_1 of the image of a, i.e. $s_1(0) = 0$; and since the top map is an isomorphism it follows that $s_0(a) = 0$ and so $a = 0$. Therefore it suffices to prove the theorem for characteristic 0 fields.

3. THE HILBERT 90 CONDITION

3.1   Motivic cohomology

To proceed further, we need the notion of motivic cohomology on a smooth variety X. We won't make any rigorous definitions for the moment, but just claim the existence of complexes with certain properties. These are the motivic complexes $\mathbb{Z}(i)$, which are cochain complexes of étale sheaves on X. We define the motivic cohomology $H_{\text{Zar}}^n(X, \mathbb{Z}(i))$ to be the hypercohomology of $\mathbb{Z}(i)$ in the Zariski topology. The complexes satisfy $\mathbb{Z}(i) = 0$ for $i < 0$ and $\mathbb{Z}(0) = \mathbb{Z}$, so $H_{\text{Zar}}^n(X, \mathbb{Z}(i)) = 0$ for $i < 0$ or $i = 0$ and $n > 0$; there are pairings $\mathbb{Z}(i) \otimes \mathbb{Z}(j) \to \mathbb{Z}(i+j)$, so $H_{\text{Zar}}^*(X, \mathbb{Z}(*))$ is a bigraded ring. For k a field we'll often write $H_{\text{Zar}}^n(k, \mathbb{Z}(i))$ for $H_{\text{Zar}}^n(\text{Spec } k, \mathbb{Z}(i))$.

Motivic cohomology recovers Milnor K-theory: there is an isomorphism $H_{\text{Zar}}^n(k, \mathbb{Z}(n)) \simeq K_n^M(k)$. It follows that $\mathbb{Z}(1)$ is quasi-isomorphic to $\mathcal{O}^\times[1]$. It also recovers the Chow group as $\text{Ch}^n(X) \simeq H_{\text{Zar}}^{2n}(X, \mathbb{Z}(n))$. We can also take motivic cohomology with coefficients in any abelian group A by setting $A(i) = A \otimes_{\mathbb{Z}} \mathbb{Z}(i)$, after which we recover a map to étale cohomology $H_{\text{Zar}}^n(X, (\mathbb{Z}/p)(i)) \to H_{\text{ét}}^n(X, \mu_p^{\otimes i})$, generalizing the cycle map.

Since Zariski cohomology commutes with direct limits, we can take $A = \mathbb{Z}_{(p)}$ to get

$$
H_{\text{Zar}}^n(k, \mathbb{Z}_{(p)}(i)) \simeq H_{\text{Zar}}^n(k, \mathbb{Z}(i)) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = H_{\text{Zar}}^n(k, \mathbb{Z}(i))_{(p)}.
$$

We have a short exact sequence

$$
0 \to \mathbb{Z}_{(p)}(n) \xrightarrow{p} \mathbb{Z}_{(p)}(n) \to (\mathbb{Z}/p\mathbb{Z})(n) \to 0,
$$

and taking the nth cohomology over k gives

$$H^n_{\text{Zar}}(k, \mathbb{Z}_{(p)}(n)) \xrightarrow{p} H^n_{\text{Zar}}(k, \mathbb{Z}_{(p)}(n)) \to H^n_{\text{Zar}}(k, (\mathbb{Z}/p\mathbb{Z})(n)) \to H^{n+1}_{\text{Zar}}(k, \mathbb{Z}_{(p)}(n)).$$

This rightmost module turns out to be 0 (essentially for dimension reasons), so using the above we get that

$$H^n_{\text{Zar}}(k, (\mathbb{Z}/p\mathbb{Z})(n)) \simeq H^n_{\text{Zar}}(k, \mathbb{Z}(n))_{(p)}/p = H^n_{\text{Zar}}(k, \mathbb{Z}(n))/p \simeq K^M_n(k)/p.$$

Each $A(i)$ is a complex of étale sheaves, and so we can also look at its étale cohomology rather than Zariski, which we write as $H^n_{\text{ét}}(X, A(i))$. Since the étale topology is finer than the Zariski, we have a change-of-topology map $H^*_{\text{Zar}} \to H^*_{\text{ét}}$; we have an isomorphism $H^n_{\text{ét}}(X, (\mathbb{Z}/p\mathbb{Z})(i)) \to H^n_{\text{ét}}(X, \mu_p^{\otimes i})$, and so the Zariski-to-étale map recovers the map to étale cohomology mentioned above. The same relation with localization hold as for the Zariski topology. However, the condition $H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(p)}(n)) = 0$ is (at least a priori) no longer necessarily true.

3.2 The condition

We say that the Hilbert 90 condition holds for n, written H90(n), if $H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(p)}(n)) = 0$ for every field k with characteristic different from p.

For $n = 1$, this is $H^2_{\text{ét}}(k, \mathbb{Z}_{(p)}(1)) = H^2_{\text{ét}}(k, \mathbb{Z}(1))_{(p)} = H^2_{\text{ét}}(k, \mathcal{O}^\times[1])_{(p)} = H^1_{\text{ét}}(k, \mathcal{O}^\times)_{(p)} = 0$ by Hilbert's theorem 90; thus the name.

Why is this condition important? Let's connect it to Milnor K-theory to see:

Lemma 3. *For $n > i$, $H^n_{\text{ét}}(k, \mathbb{Z}(i))$ is torsion, its p-torsion subgroup is $H^n_{\text{ét}}(k, \mathbb{Z}_{(p)}(i))$, and if the characteristic of k is different from p, then $H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(p)}(i)) \simeq H^n_{\text{ét}}(k, (\mathbb{Q}/\mathbb{Z}_{(p)})(i))$. For $n = i$, there is an exact sequence*

$$K^M_n(k)/p \otimes \mathbb{Q}/\mathbb{Z}_{(p)} \to H^n_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}_{(p)}(n)) \to H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(p)}(n)) \to 0.$$

This lemma will allow us to show that our main theorem implies H90(n).

Theorem 4. *Fix $n \geq 1$ and a prime p. If $K^M_n(k)/p \to H^n(k, \mu_p^{\otimes n})$ is an isomorphism for every field k of characteristic different from p, then H90(n) holds.*

Proof of Lemma 3. For $\mathbb{Q}(i)$, there is no difference between the étale and Zariski cohomology, and $H^n(k, \mathbb{Q}(i))$ vanishes for all $n > i$. Since $H^n(k, \mathbb{Q}(i)) = H^n(k, \mathbb{Z}(i)) \otimes_{\mathbb{Z}} \mathbb{Q}$, it follows that $H^n(k, \mathbb{Z}(i))$ is torsion for all $n > i$. Tensoring with $\mathbb{Z}_{(p)}$ kills all torsion other than the p-torsion, so we've proven the first two statements. Taking the étale cohomology sequence for

$$0 \to \mathbb{Z}_{(p)}(i) \to \mathbb{Q}(i) \to (\mathbb{Q}/\mathbb{Z}_{(p)})(i) \to 0$$

gives

$$H^n_{\text{ét}}(k, \mathbb{Q}(i)) \to H^n_{\text{ét}}(k, (\mathbb{Q}/\mathbb{Z}_{(p)})(i)) \to H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(p)}(i)) \to H^{n+1}_{\text{ét}}(k, \mathbb{Q}(i)),$$

and the first and last groups vanish for $n > i$, giving the claimed isomorphism. Finally, if $n = i$ we taking the cohomology sequence for Zariski and étale cohomology, together with the map between them, gives a commutative diagram

$$
\begin{array}{ccccccc}
H^n_{\mathrm{Zar}}(k, \mathbb{Z}_{(p)}(n)) & \longrightarrow & H^n_{\mathrm{Zar}}(k, \mathbb{Q}(n)) & \longrightarrow & H^n_{\mathrm{Zar}}(k, (\mathbb{Q}/\mathbb{Z}_{(p)})(n)) & \longrightarrow & H^{n+1}_{\mathrm{Zar}}(k, \mathbb{Z}_{(p)}(n)) \\
\downarrow & & \downarrow{\wr} & & \downarrow & & \downarrow \\
H^n_{\mathrm{\acute{e}t}}(k, \mathbb{Z}_{(p)}(n)) & \longrightarrow & H^n_{\mathrm{\acute{e}t}}(k, \mathbb{Q}(n)) & \longrightarrow & H^n_{\mathrm{\acute{e}t}}(k, (\mathbb{Q}/\mathbb{Z}_{(p)})(n)) & \longrightarrow & H^{n+1}_{\mathrm{\acute{e}t}}(k, \mathbb{Z}_{(p)}(n))
\end{array}
$$

The top-right entry is 0, and the next entry in the bottom row is $H^{n+1}_{\mathrm{\acute{e}t}}(k, \mathbb{Q}(n)) = 0$ so the rightmost map in the bottom row is surjective. Observing that $H^n_{\mathrm{Zar}}(k, (\mathbb{Q}/\mathbb{Z}_{(p)})(n)) = H^n_{\mathrm{Zar}}(k, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}_{(p)} = K^M_n(k) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$, we have a series of maps

$$
K^M_n(k) \otimes \mathbb{Q}/\mathbb{Z}_{(p)} \to H^n_{\mathrm{\acute{e}t}}(k, (\mathbb{Q}/\mathbb{Z}_{(p)})(n)) \to H^{n+1}_{\mathrm{\acute{e}t}}(k, \mathbb{Z}_{(p)}(n)) \to 0
$$

which we know is an exact sequence at $H^{n+1}_{\mathrm{\acute{e}t}}(k, \mathbb{Z}_{(p)}(n))$. It is a sequence because the image of $K^M_n(k) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$ in $H^{n+1}_{\mathrm{\acute{e}t}}(k, \mathbb{Z}_{(p)}(n))$ is 0 by the commutativity of the above diagram; and it is exact everywhere because the kernel of the map $H^n_{\mathrm{\acute{e}t}}(k, (\mathbb{Q}/\mathbb{Z}_{(p)})(n)) \to H^{n+1}_{\mathrm{\acute{e}t}}(k, \mathbb{Z}_{(p)}(n))$ is the image of $H^n_{\mathrm{\acute{e}t}}(k, \mathbb{Q}(n)) = H^n_{\mathrm{Zar}}(k, \mathbb{Q}(n))$, which by commutativity is the image of $H^n_{\mathrm{Zar}}(k, (\mathbb{Q}/\mathbb{Z}_{(p)})(n))$ since the top middle map is surjective. $\square$

Proof of Theorem 4. Writing $K^M_n(k) = H^n_{\mathrm{Zar}}(k, \mathbb{Z}(n))$, the Zariski-to-étale map on the obvious short exact sequence gives a commutative diagram

$$
\begin{array}{ccccccc}
H^n_{\mathrm{Zar}}(k, \mathbb{Z}(n)) & \xrightarrow{p} & H^n_{\mathrm{Zar}}(k, \mathbb{Z}(n)) & \longrightarrow & H^n_{\mathrm{Zar}}(k, (\mathbb{Z}/p)(n)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^n_{\mathrm{\acute{e}t}}(k, \mathbb{Z}(n)) & \xrightarrow{p} & H^n_{\mathrm{\acute{e}t}}(k, \mathbb{Z}(n)) & \longrightarrow & H^n(k, \mu_p^{\otimes n}) & \longrightarrow & H^{n+1}_{\mathrm{\acute{e}t}}(k, \mathbb{Z}(n))
\end{array}
$$

The rightmost vertical map is the norm residue homomorphism. If we assume that this is an isomorphism, then by the commutativity of this diagram the map $H^n_{\mathrm{\acute{e}t}}(k, \mathbb{Z}(n)) \to H^n(k, \mu_p^{\otimes n})$ must be surjective, and so the image of $H^n(k, \mu_p^{\otimes n})$ in $H^{n+1}_{\mathrm{\acute{e}t}}(k, \mathbb{Z}(n))$ is trivial. Since the next map to the right is given by multiplication by p again, it follows that the p-torsion of $H^{n+1}_{\mathrm{\acute{e}t}}(k, \mathbb{Z}(n))$ is trivial. But by Lemma 3 this p-torsion is precisely $H^{n+1}_{\mathrm{\acute{e}t}}(k, \mathbb{Z}_{(p)}(n))$, so this must be zero for all k, i.e. H90(n) holds. $\square$

The converse of Theorem 4 is also true: H90(n) is equivalent to our main theorem for n, which we'll denote by BK(n). This is harder to prove, however; I think this'll be in Haodong's talk in a few weeks.

4. PROOF, UP TO MAIN THEOREMS

We're now (almost) ready to prove the main theorem! Recall that a p-special field is one that has no finite extensions of degree prime to p, and that we can assume that k is p-special for the purposes of proving our theorem.

Theorem 5. *Suppose that $H90(n-1)$ holds, and that k is a p-special field. If $K_n^M(k)/p = 0$, then $H^n(k, \mu_p^{\otimes n}) = 0$, and therefore $H_{\text{ét}}^{n+1}(k, \mathbb{Z}_{(p)}(n)) = 0$.*

The latter implication follows since if $H^n(k, \mu_p^{\otimes n}) = 0$ then extending the lower exact sequence in the proof of Theorem 4 we get

$$0 \to H_{\text{ét}}^{n+1}(k, \mathbb{Z}(n)) \xrightarrow{p} H_{\text{ét}}^{n+1}(k, \mathbb{Z}(n)),$$

i.e. the p-torsion of $H_{\text{ét}}^{n+1}(k, \mathbb{Z}(n))$ is trivial, and since this is precisely $H_{\text{ét}}^{n+1}(k, \mathbb{Z}_{(p)}(n)) = 0$ the conclusion follows.

This theorem is the upshot of the first part of the seminar, and is currently scheduled to be proven in Baiqing's talk in a few weeks.

Since we can reduce to p-special fields, what we still need to show in order to apply this theorem to induct is that we can choose k such that $K_n^M(k)/p = 0$. This is done via the following theorem.

Theorem 6. *Suppose that $H90(n - 1)$ holds, and k is a field of characteristic 0. Then for every $a \in K_n^M(k)/p$ there exists a smooth projective variety X_a such that the induced map $K_n^M(k)/p \to K_n^M(k(X_a))/p$ sends a to 0 and the induced map $H_{\text{ét}}^{n+1}(k, \mathbb{Z}_{(p)}(n)) \to H_{\text{ét}}^{n+1}(k(X_a), \mathbb{Z}_{(p)}(n))$ is an injection.*

These are Rost varieties. We'll define them later today; the first condition is part of the definition, but the latter is not and has to be proved.

Assuming these two theorems, we can now prove the norm residue isomorphism theorem.

Proof of Theorem 2. First, we can restrict to the case where k has characteristic 0. For each $a \in K_n^M(k)/p$, we apply Theorem 6 to find a field $k_a = k(X_a)$ in which a vanishes and such that $H_{\text{ét}}^{n+1}(k, \mathbb{Z}_{(p)}(n))$ embeds into $H_{\text{ét}}^{n+1}(k_a, \mathbb{Z}_{(p)}(n))$. Repeating the construction for k_a, by transfinite induction we end up with a field k' over k such that $K_n^M(k)/p \to K_n^M(k')/p$ is zero and $H_{\text{ét}}^{n+1}(k, \mathbb{Z}_{(p)}(n))$ embeds in $H_{\text{ét}}^{n+1}(k', \mathbb{Z}_{(p)}(n))$. We can then choose a p-special algebraic extension k'' of k' by transfer such that $H_{\text{ét}}^{n+1}(k', \mathbb{Z}_{(p)}(n))$ embeds in $H_{\text{ét}}^{n+1}(k'', \mathbb{Z}_{(p)}(n))$ and the composition $K_n^M(k)/p \to K_n^M(k')/p \to K_n^M(k'')/p$ is zero.

Set $k^{(1)} = k''$, and repeat the construction to produce a field $k^{(2)}$ with the same properties with respect to $k^{(1)}$. Iterating, we get an increasing tower of field extensions $k^{(m)}$; let L be their union. Then every symbol $a \in K_n^M(L)/p$ comes from some symbol in $K_n^M(k^{(m)})/p$ for some m; but each such symbol goes to zero in all higher extensions, and therefore is zero in L. Therefore $K_n^M(L)/p = 0$. If $H90(n-1)$ holds, then since L is a union of p-special fields it is also p-special and therefore it follows that $H_{\text{ét}}^{n+1}(k, \mathbb{Z}_{(p)}(n)) = 0$ from Theorem 5. By construction each $H_{\text{ét}}^{n+1}(k^{(m)}, \mathbb{Z}_{(p)}(n))$ embeds in $H_{\text{ét}}^{n+1}(L, \mathbb{Z}_{(p)}(n)) = 0$ and so each is 0, and since $H_{\text{ét}}^{n+1}(k, \mathbb{Z}_{(p)}(n)) \hookrightarrow H_{\text{ét}}^{n+1}(k^{(1)}, \mathbb{Z}_{(p)}(n))$ it follows that $H_{\text{ét}}^{n+1}(k, \mathbb{Z}_{(p)}(n)) = 0$, i.e. $H90(n)$ holds for k. Since k is arbitrary up to the permitted reductions, it follows by induction that $H90(n)$ holds for all n, and by the equivalence to the main theorem we get the result. $\qquad\square$

Next we want to say something about the mysterious varieties of the key Theorem 6; we'll see that in the case $n = 2$ they are not too mysterious. Finally we'll say something about how Theorem 5 will be proved, which will occupy the rest of this part of the seminar.

5. ROST VARIETIES

5.1   Severi-Brauer varieties

Let's start with an example, with $k = \mathbb{Q}$ and $n = 2$. Consider the projective curve C defined by $X^2 + Y^2 = 3Z^2$. This is smooth and of degree 2, and so it has genus 0 and therefore is isomorphic to the projective line and so should have many points.

Except it doesn't: this curve actually has no rational points over $\mathbb{Q}$. (Exercise.) The problem is solved by extension of scalars: if E is a number field such that $C(E)$ is nonempty, then the base change C_E is isomorphic to the projective line over E, just as we'd like. In this case we say that E splits C. For example, $\mathbb{Q}(i)$ splits C, since $1^2 + i^2 = 3 \cdot 0^2$.

This curve turns out to be closely related to the algebra $A_{-1}(-1, 3)$, i.e. the algebra on two generators x and y by $x^2 = -1$, $y^2 = 3$, and $xy = -yx$. We say that a field E splits this algebra if over E it is isomorphic to the matrix algebra; over $\mathbb{Q}(i)$, we can take generators

$$x = \begin{pmatrix} i & \\ & -i \end{pmatrix}, \qquad y = \begin{pmatrix} & 3 \\ 1 & \end{pmatrix},$$

so $\mathbb{Q}(i)$ splits $A_{-1}(-1, 3)$.

For $X, Y, Z, W \in \overline{\mathbb{Q}}$, we can define the norm $N(X + Yx + Zy + Wxy)$ to be the determinant of the corresponding matrix, since $A_{-1}(-1, 3)$ certainly splits over $\overline{\mathbb{Q}}$. We can compute this explicitly: it is $X^2 + Y^2 - 3Z^2 + 3W^2$. The elements of $A_{-1}(-1, 3)$ with norm 0 are precisely the zero divisors. Thus if E splits $A_{-1}(-1, 3)$, then we can consider the two-dimensional space of matrices $\begin{pmatrix} X & \\ Y & \end{pmatrix}$; this must have nontrivial intersection with the three-dimensional subspace of $A_{-1}(-1, 3)$ cut out by $W = 0$, i.e. generated by $\{1, x, y\}$. Therefore there exist (X, Y, Z) not all zero such that the norm of the corresponding matrix is 0, i.e. $X^2 + Y^2 = 3Z^2$. We conclude that E splits C if it splits $A_{-1}(-1, 3)$.

Conversely, if E splits C there exists such a point, and therefore a nontrivial subspace of $A_{-1}(-1, 3)$ on which both the norm and W vanish. If $A_{-1}(-1, 3)$ does not split, it is a division algebra, which is impossible since it has a nonzero space of zero divisors; therefore we conclude that E splits C if and only if it splits $A_{-1}(-1, 3)$. We say that C is a Severi-Brauer variety for $A_{-1}(-1, 3)$.

This generalizes as follows. Let A be a central simple algebra over k of dimension m^2, and let $\mathrm{Gr}(m, A)$ be the scheme classifying subspaces of A of dimension m. Let X be the subscheme of $\mathrm{Gr}(m, A)$ classifying subspaces which are invariant under right multiplication by elements of A; this is a smooth scheme over k of dimension $p - 1$, and its base change to a field E/k is isomorphic to $\mathbb{P}_E^{p-1}$ if and only if it has an E-point. Essentially the same argument shows that a field extension E/k splits A if and only if it splits X, i.e. $X_E \simeq \mathbb{P}_E^{p-1}$ if and only if X has an E-point if and only if E splits A. In particular, the function field of X splits A, since X certainly has a $k(X)$-rational point, namely its generic point.

For $\{a_1, a_2\} \in K_2^M(k)/p$ and ζ a primitive pth root of unity, we can define the algebra $A_\zeta(a_1, a_2)$ to be the algebra generated by x and y with the relations $x^p = a_1$, $y^p = a_2$, and $xy = \zeta yx$. We define the Severi-Brauer variety X_a of $a = \{a_1, a_2\}$ to be the Severi-Brauer variety of $A_\zeta(a_1, a_2)$ as defined above.

Since $k(X_a)$ splits X_a, it splits $A_\zeta(a_1, a_2)$, i.e. the image of the class of $A_\zeta(a_1, a_2)$ in the Brauer group $H^2(k(X_a), \mathcal{O}^\times)$ is trivial, and so it is also trivial in $H^2(k(X_a), \mu_p^{\otimes 2})$. Therefore $\{a_1, a_2\}$ is in the kernel of the norm residue map for $k(X_a)$. However, it's a (fairly explicit) result due to Milnor that this is possible only if a_2 is a norm for the extension $k(X_a)(\sqrt[p]{a_2})$ over $k(X_a)$, in which case $\{a_1, a_2\}$ is zero in $K_2^M(k(X_a))/p$. (This doesn't prove the theorem for $n = 2$ since the kernel could contain nonzero linear combinations of symbols, just not individual symbols.) Therefore we see that Severi-Brauer varieties give a candidate for the varieties X_a of Theorem 6 in the case $n = 2$, and satisfy at least the first desired condition.

5.2 ν_i-varieties

The next part of the definition of Rost varieties involves the notion of a ν_i-variety, for which we need the characteristic number s_d.

For a smooth projective variety X of dimension d, there is a characteristic class $s_d : K_0(X) \to \mathrm{Ch}^d(X)$ corresponding to the sum of dth powers of Chern roots; we define $s_d(X) = s_d(TX)$. A ν_i-variety over a field k, for a fixed prime p, is a smooth projective variety of dimension $p^i - 1$ with $s_{p^i - 1}(X) \not\equiv 0 \pmod{p^2}$.

In particular, since over a sufficiently large field extension any Severi-Brauer variety is isomorphic to $\mathbb{P}^{p-1}$ it satisfies $s_d(X) = s_d(\mathbb{P}^{p-1}) = p \not\equiv 0 \pmod{p^2}$.

5.3   Rost varieties

There is a final condition for Rost varieties which we will not go into more than superficially today, due to time constraints. The outline is this: to a scheme X we can associate the Borel-Moore homology $H_{-1,-1}(X)$ and its quotient $\overline{H}_{-1,-1}(X)$. It carries a norm map $N : H_{-1,-1}(X) \to H_{-1,-1}(k) = k^\times$, given by the pushforward along the structure morphism $X \to k$, which descends to a map on the reduced homology $\overline{H}_{-1,-1}(X) \to \overline{H}_{-1,-1}(k) = k^\times$. It turns out that $\overline{H}_{-1,-1}(X)$ for the Severi-Brauer variety X of an algebra A is given by $K_1(A)$, which is the image of the norm map $A^\times \to k^\times$.

In general, we define a Rost variety for $a \in K_n^M(k)/p$ to be a ν_{n-1}-variety X such that X splits a, i.e. the image of a in $K_n^M(k(X))/p$ is 0; for each $1 \le i < n$, there exists a ν_i-variety with a map to X; and the norm map $N : \overline{H}_{-1,-1}(X) \to k^\times$ is an injection. From the above, we see that Severi-Brauer varieties are Rost varieties for $n = 2$.

It is not yet clear that Rost varieties satisfy the second property of Theorem 6; this will follow from the existence of Rost motives, which we will see we can associate to any Rost variety.

6. BEILINSON-LICHTENBAUM

Consider the morphism of sites $\pi : (\mathbf{Sm}/k)_{\text{ét}} \to (\mathbf{Sm}/k)_{\text{Zar}}$. The total direct image functor $\mathbf{R}\pi_*$ sends an étale sheaf (complex) $\mathcal{F}$ to a Zariski complex whose cohomology satisfies $H^*_{\text{Zar}}(X, \mathbf{R}\pi_*\mathcal{F}) = H^*_{\text{ét}}(X, \mathcal{F})$. In particular we have $H^*_{\text{Zar}}(X, \mathbf{R}\pi_*\mu_p^{\otimes n}) = H^*_{\text{ét}}(X, \mu_p^{\otimes n}) = H^*_{\text{ét}}(X, (\mathbb{Z}/p\mathbb{Z})(n)) = H^*_{\text{Zar}}(X, \mathbf{R}\pi_*(\mathbb{Z}/p\mathbb{Z})(n))$.

The truncation $\tau^{\le n}C$ of a complex C is the universal subcomplex which has the same cohomology in degrees up to n and trivial cohomology in higher degrees. The above equalities

suggest truncating $R\pi_*(\mathbb{Z}/p\mathbb{Z})(n)$, and indeed we define

$$(L/p)(n) = \tau^{\leq n}\mathbf{R}\pi_*(\mathbb{Z}/p\mathbb{Z})(n).$$

As the notation suggests, as well as the previous sections, we also define

$$L(n) = \tau^{\leq n}\mathbf{R}\pi_*\mathbb{Z}_{(p)}(n).$$

Both complexes inherit transfers from $\mathbb{Z}_{(p)}(n)$ and $(\mathbb{Z}/p\mathbb{Z})(n)$.

The derived pushforward $\mathbf{R}\pi_*$ has an adjoint π^*, and the adjunction sends the identity $\pi^*\mathcal{F} \to \pi^*\mathcal{F}$ to a canonical morphism $\mathcal{F} \to \mathbf{R}\pi_*\pi^*\mathcal{F}$. Taking $\mathcal{F}$ equal to $\mathbb{Z}_{(p)}(n)$ or $(\mathbb{Z}/p\mathbb{Z})(n)$, observe that as Zariski sheaves both have vanishing cohomology in degrees greater than n, and so $\tau^{\leq n}$ is the identity on both. Therefore applying $\tau^{\leq n}$ to the adjunction morphism gives canonical morphisms

$$\tilde{\alpha} : \mathbb{Z}_{(p)}(n) \to L(n), \qquad \alpha : (\mathbb{Z}/p\mathbb{Z})(n) \to (L/p)(n).$$

The Beilinson-Lichtenbaum condition for a fixed integer n states that α is a quasi-isomorphism for every k of characteristic different from p; we will write this as $\mathrm{BL}(n)$.

Theorem 7. *If $BL(n)$ holds then $BK(n)$ holds, and therefore so does $H90(n)$.*

Proof. Taking H^n_{Zar} of α gives a map

$$
\begin{aligned}
H^n_{\mathrm{Zar}}(k, (\mathbb{Z}/p\mathbb{Z})(n)) = K^M_n(k)/p \to H^n_{\mathrm{Zar}}(k, \tau^{\leq n}\mathbf{R}\pi_*(\mathbb{Z}/p\mathbb{Z})(n)) &= H^n_{\mathrm{Zar}}(k, \mathbf{R}\pi_*(\mathbb{Z}/p\mathbb{Z})(n)) \\
&= H^n_{\text{ét}}(k, (\mathbb{Z}/p\mathbb{Z})(n)) \\
&= H^n(k, \mu_p^{\otimes n}).
\end{aligned}
$$

By the assumption that α is a quasi-isomorphism, this is an isomorphism. Since k is arbitrary up to characteristic this implies $\mathrm{BK}(n)$, which implies $\mathrm{H90}(n)$ by Theorem 4. $\qquad\square$

In Haodong's talk, we'll prove that not only is $\mathrm{H90}(n)$ equivalent to $\mathrm{BK}(n)$, it's also equivalent to $\mathrm{BL}(n)$; in other words we'll prove converses to Theorems 4 and 7. To do this we'll need the reverse induction steps $\mathrm{H90}(n) \implies \mathrm{H90}(n-1)$ and $\mathrm{BL}(n) \implies \mathrm{BL}(n-1)$, as well as a good understanding of the cohomology of derived Nisnevich sheaves.

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