Integral models*

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For purposes such as point-counting and p-adic uniformization, we need to have an integral model of Shimura varieties over something like $\mathbb{Z}_p$. Rapoport-Zink spaces are built using integral structure and so are already naturally integral, and to get the comparison to the moduli space of shtukas we had to take the generic fiber; one might hope for a moduli-theoretic construction on the same footing as shtukas which would give an integral structure for our general theory of local Shimura varieties. In the cases where we've defined Rapoport-Zink spaces, these should agree with (the diamondization of) the corresponding Rapoport-Zink spaces, not just the generic fiber.

In fact, our construction of shtukas naturally extends to an integral structure. Recall that the moduli space $\mathrm{Sht}_{\mathcal{G},b,\{\mu\}}$ parametrized shtukas on a test space S, i.e. tuples $(\mathcal{P}, S^\sharp, \varphi_\mathcal{P}, \iota_r)$ where $\mathcal{P}$ is a $\mathcal{G}$-torsor on $S \dot{\times} \operatorname{Spa} \mathbb{Z}_p$, $S^\sharp$ is an untilt of S to $\breve{E} = \breve{\mathbb{Q}}_p \cdot E$ for the field of definition E of μ, $\varphi_\mathcal{P}$ is an isomorphism away from $S^\sharp$ of $\varphi_\mathcal{P}$ with its pullback by the Frobenius of S giving $\mathcal{P}$ and its pullback of relative position bounded by μ, and ι_r is a trivialization $\mathcal{P} \cong G \times \mathcal{Y}_{[r,\infty)}(S)$ for r sufficiently large, carrying $\varphi_\mathcal{P}$ to (b, σ).

This lives over $\operatorname{Spd} \breve{E}$ by sending the tuple above to only the data of $S^\sharp$ over $\breve{E}$. To get something living over $\operatorname{Spd} \breve{\mathcal{O}}_E$, we only need to slightly broaden this data: allow $S^\sharp$ to be an untilt to $\breve{\mathcal{O}}_E$ rather than $\breve{E}$, so that the natural map is to $\operatorname{Spd} \breve{\mathcal{O}}_E$. We call the moduli space of such data $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$.

The space of shtukas only depends on the choice of a model $\mathcal{G}$ through the compact open subgroup $\mathcal{G}(\mathbb{Z}_p)$, which we can replace by any compact open subgroup $K \subset G(\mathbb{Q}_p)$. We can do the same thing in this setting to get an integral model $\mathcal{M}^{\mathrm{int}}_{G,b,\mu,K}$ for $\mathcal{M}_{G,b,\mu,K}$, which if we choose a model $\mathcal{G}$ such that $\mathcal{G}(\mathbb{Z}_p) = K$ (imposing the technical restriction that $\mathcal{G}$ be parahoric) then $\mathcal{M}^{\mathrm{int}}_{G,b,\mu,K} = \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ classifies the data above, i.e. an S-point is a tuple $(\mathcal{P}, S^\sharp, \varphi_\mathcal{P}, \iota_r)$ exactly as for shtukas but with $S^\sharp$ an untilt to $\breve{\mathcal{O}}_E$ rather than E. (Technically there is also some complication on the boundedness condition: taking the relative position bounded by μ in the case of shtukas is about taking the preimage under the period map of a particular subspace $\mathrm{Gr}_{\mathcal{G},\operatorname{Spd} E,\mu}$ of $\mathrm{Gr}_{\mathcal{G},\operatorname{Spd} E}$ bounded by μ, and although we can define an integral version $\mathrm{Gr}_{\mathcal{G},\operatorname{Spd} \mathcal{O}_E}$ in a natural way extending the subspace bounded by μ over $\mathcal{O}_E$ can only conjecturally be done canonically; for now we have to make an arbitrary choice of extension, which we fix once and for all.)

The generic fiber over $\operatorname{Spd} \breve{\mathcal{O}}_E$ consists of S-points with image in the generic point $\operatorname{Spd} \breve{E}$, i.e. the same data but where $S^\sharp$ is restricted to be over $\breve{E}$ rather than just $\breve{\mathcal{O}}_E$. Thus $\mathcal{M}^{\mathrm{int}}_{G,b,\mu,K}$ has generic fiber $\mathcal{M}_{G,b,\mu,K} = \mathrm{Sht}_{G,b,\{\mu\},K} = \mathrm{Sht}_{\mathcal{G},b,\{\mu\}}$ as desired.

The next thing to check is that in the simplest case, where $G = \mathrm{GL}_n$, $K = \mathrm{GL}_n(\mathbb{Z}_p)$, $\mu = (1, \ldots, 1, 0, \ldots, 0)$ with d occurrences of 1, and b corresponds to a p-divisible group X_b over k with dimension d and height n, this integral model for $\mathcal{M}_{G,b,\mu,K}$ is the canonical one defined by the Rapoport-Zink space $\mathcal{M}_{X_b}$.

*These notes are based on chapter 25 of [1].

Theorem 1. *With the above notation, there is a natural isomorphism of v-sheaves*

$$\mathcal{M}_{X_b}^{\diamondsuit} \cong \mathcal{M}^{\mathrm{int}}_{\mathrm{GL}_n,b,\mu,\mathrm{GL}_n(\mathbb{Z}_p)}$$

over $\operatorname{Spd} \breve{\mathbb{Z}}_p$.

Proof. Let $S = \operatorname{Spa}(R, R^+)$ for a perfectoid Tate pair (R, R^+) over $\breve{\mathbb{Z}}_p$ with a pseudo-uniformizer $\varpi \in R^+$ dividing p. An $S^\flat$-point of $\mathcal{M}_{X_b}^{\diamondsuit}$ is an S-point of $\mathcal{M}_{X_b}$, i.e. a p-divisible group X over R^+ with a quasi-isogeny $X \times_{\operatorname{Spf} R^+} \operatorname{Spec} R^+/\varpi \to X_b \times_{\operatorname{Spec} k} \operatorname{Spec} R^+/\varpi$, which is equivalent to a Breuil-Kisin-Fargues module M over $A_{\mathrm{inf}}(R^+) = W(R^{\flat+})$. If ξ generates the map $A_{\mathrm{inf}}(R^+) \to R^+$, then M has a Frobenius $\sigma : M[1/\xi] \cong M[1/\sigma(\xi)]$. This can be interpreted as a vector bundle over $\operatorname{Spa} A_{\mathrm{inf}}(R^+)$, which has $S \dot{\times} \operatorname{Spa} \mathbb{Z}_p$ as a subspace, so restricting to it gives a vector bundle on $S \dot{\times} \operatorname{Spa} \mathbb{Z}_p$; the Frobenius gives the desired isomorphism away from the untilt S of $S^\flat$, and the quasi-isogeny translated to M gives (over a suitable period ring) an isomorphism with the module coming from X_b, or in other words away from 0 a trivialization corresponding to b. Thus this gives an $S^\flat$-point of $\mathcal{M}^{\mathrm{int}}_{\mathrm{GL}_n,b,\mu,\mathrm{GL}_n(\mathbb{Z}_p)}$, and thus we've constructed a map

$$\mathcal{M}_{X_b}^{\diamondsuit} \to \mathcal{M}^{\mathrm{int}}_{\mathrm{GL}_n,b,\mu,\mathrm{GL}_n(\mathbb{Z}_p)}.$$

If we can show that this map induces a bijection on geometric points, it must be an isomorphism. Thus set $R^+ = \mathcal{O}_C$ for some algebraically closed nonarchimedean field C over $\mathbb{Z}_p$. If C is of characteristic 0, then shtukas with one leg at a C-point form a category equivalent to Breuil-Kisin-Fargues modules over $A_{\mathrm{inf}}(\mathcal{O}_{C^\flat})$ by work of Fargues; his argument works more generally, i.e. shtukas (in this setting) extend uniquely to modules over A_{inf} with a Frobenius, so if the module comes from a p-divisible group it corresponds to a unique shtuka, i.e. this map is injective on geometric points. We sketch the construction of the inverse functor.

Let $S = \operatorname{Spa}(C, \mathcal{O}_C)$ for such a C; then an $S^\flat$-point of $\mathcal{M}^{\mathrm{int}}_{\mathrm{GL}_n,b,\mu,\mathrm{GL}_n(\mathbb{Z}_p)}$ gives a vector bundle on $S \dot{\times} \operatorname{Spa} \mathbb{Z}_p$, which we can glue along the isomorphism ι_r with $\mathcal{E}^b$ over $\mathcal{Y}_{[r,\infty]}(S)$, i.e. adding the point at infinity in $\operatorname{Spa} A_{\mathrm{inf}}$ missing from $S \dot{\times} \operatorname{Spa} \mathbb{Z}_p$, to get a vector bundle $\mathcal{E}$ on $\operatorname{Spa} A_{\mathrm{inf}}$, or equivalently a projective module over A_{inf}. The data of $\varphi_{\mathcal{P}}$ away from the untilt of $S^\flat$ to S gives a Frobenius structure on this module after inverting ξ, i.e. a Breuil-Kisin-Fargues module as above; this is the equivalence referenced above.

Now, to show that our map is surjective, we need to check that this module in fact comes from a p-divisible group. This boils down to checking a certain condition on the module, which holds due to good properties required of $\varphi_{\mathcal{P}}$. Thus our given $S^\flat$-shtuka lifts to a p-divisible group over S, and the trivialization ι_r with the correspondence to b translates to a quasi-isogeny of the p-divisible group after base change with the base change of X_b. □

As a corollary, we can also prove the comparison in the case of PEL Rapoport-Zink spaces. Fix an integral PEL datum $\mathcal{D} = (B, V, \mathcal{O}_B, \mathcal{L}, \psi, *, b, \mu)$, which defines a local Shimura datum (G, b, μ) and a model $\mathcal{G}$ of G, which we assume parahoric.

Theorem 2. *With the above notation, there is a natural isomorphism of v-sheaves*

$$\mathcal{M}_{\mathcal{D}}^{\diamondsuit} \cong \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$$

over $\operatorname{Spd} \breve{\mathcal{O}}_E$.

Proof. The same argument as in the proof of Theorem 1 (now on the level of $\mathcal{L}$-chains and with $\mathcal{O}_B$-structure, both of which transfer) gives a map $\mathcal{M}_{\mathcal{D}}^{\diamondsuit} \to \mathcal{M}_{\mathcal{G},b,\mu}^{\text{int}}$, so we just need to see that this is a bijection on geometric points.

As last time, $\mathcal{M}_{\mathcal{D}}^{\diamondsuit}$ has a closed embedding into $\prod_{\Lambda\in\mathcal{L}} \mathcal{M}_{X_{b,\Lambda}}^{\diamondsuit}$, which by Theorem 1 is isomorphic to $\prod_{\Lambda\in\mathcal{L}} \mathcal{M}_{\mathrm{GL}_\Lambda,b,\mu,\mathrm{GL}_\Lambda(\mathbb{Z}_p)}^{\text{int}}$. The same argument as last time shows that $\mathcal{M}_{\mathcal{G},b,\mu}$ has a closed embedding into this latter product compatible with the map from $\mathcal{M}_{\mathcal{D}}^{\diamondsuit}$, i.e. the diagram

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{D}}^{\diamondsuit} & \longrightarrow & \mathcal{M}_{\mathcal{G},b,\mu}^{\text{int}} \\ \big\downarrow & & \big\downarrow \\ \prod_{\Lambda\in\mathcal{L}} \mathcal{M}_{X_{b,\Lambda}}^{\diamondsuit} & \xrightarrow{\sim} & \prod_{\Lambda\in\mathcal{L}} \mathcal{M}_{\mathrm{GL}_\Lambda,b,\mu,\mathrm{GL}_\Lambda(\mathbb{Z}_p)}^{\text{int}} \end{array}$$

and so in particular the top map must be injective.

Given a $\mathcal{G}$-torsor $\mathcal{P}$ on $S \dot{\times} \operatorname{Spa} \mathbb{Z}_p$ with a trivialization corresponding to $\mathcal{E}^b$ over $\mathcal{Y}_{[r,\infty)}(S)$, we can extend as above to a $\mathcal{G}$-torsor $\tilde{\mathcal{P}}$ over $\mathcal{Y}_{[0,\infty]}(S) = \operatorname{Spa} A_{\text{inf}}(S)$ with a trivialization over $\mathcal{Y}_{[r,\infty]}(S)$ to $\mathcal{E}^b$, and so a projective $A_{\text{inf}}(S)$-module; the remaining structure transfers as above, and so we get in a similar way an $\mathcal{L}$-chain of Breuil-Kisin-Fargues modules with $\mathcal{O}_B$-structure, which just recovers the chain of $\mathcal{O}_B$-p-divisible groups corresponding to a preimage of the shtuka corresponding to $\mathcal{P}$, so the map is surjective on geometric points. □

References

[1] Peter Scholze and Jared Weinstein. Berkeley lectures on p-adic geometry. In *Berkeley Lectures on p-adic Geometry*. Princeton University Press, 2020.