Moduli spaces of local shtukas*

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Our goal is to find an incarnation of local Shimura varieties in terms of p-adic geometry, which we might hope to be applicable in more general settings than Rapoport-Zink spaces. These will be given by moduli spaces of local shtukas, analogous to function field shtukas, which are G-bundles over the curve in question together with certain additional arithmetic data. In the local setting, the analogue is given by G-torsors over the Fargues-Fontaine curve with additional structure, so understanding these torsors is our first goal. Next, we'll define local shtukas and sketch the argument that the moduli problem of such objects is representable by a reasonable geometric object, which in this setting is a locally spatial diamond. Finally, in the cases where we have Rapoport-Zink spaces the two theories should be compatible, and we will check that they are.

1. Torsors on the Fargues-Fontaine curve

In the function field setting, we work over a curve X over $\operatorname{Spec} \mathbb{F}_q$, and given a test object S, which is a suitable $\mathbb{F}_q$-scheme, we define our moduli problem by G-bundles with certain structure over $S \times_{\mathbb{F}_q} X$. We want to work over $\mathbb{Z}_p$ or $\mathbb{Q}_p$ (or more generally $\mathcal{O}_E$ or E for some finite $E/\mathbb{Q}_p$), and our test objects are perfectoid spaces of characteristic p; but then, in the adic setting, $\operatorname{Spa} \mathbb{Z}_p$ and $\operatorname{Spa} \mathbb{Q}_p$ are not over $\operatorname{Spa} \mathbb{F}_p$ and so the fiber product does not literally make sense.

This is fixed by passing to the category of v-sheaves, where we can take the *absolute* product $S \times \operatorname{Spd} \mathbb{Z}_p$ or $S \times \operatorname{Spd} \mathbb{Q}_p$. (The first gives more integral structure, which can be useful, but the second has the advantage of being a diamond). Both of these turn out to come from adic spaces: generally given suitable adic spaces A, B, the absolute product of their diamondization $A^\diamondsuit \times B^\diamondsuit$ lifts to an adic space $A \dot{\times} B$ whose diamondization is $A^\diamondsuit \times B^\diamondsuit$; but we'll work mostly in the setting of diamonds (or v-sheaves) and ignore this.

Thus the objects we're interested in classifying are torsors on $S \times \operatorname{Spd} \mathbb{Z}_p$ and $S \times \operatorname{Spd} \mathbb{Q}_p$. One can view the latter as the nonzero locus in the former: if ϖ is a uniformizer for S, for each point $x \in S \dot{\times} \operatorname{Spa} \mathbb{Z}_p$ we get an absolute value $|\varpi(x)|$ and thus a map $\kappa : S \dot{\times} \operatorname{Spa} \mathbb{Z}_p \to [0, \infty)$ sending $x \mapsto \log_{|p(x)|} |\varpi(x)|$, which is 0 on the unique characteristic 0 point. Write $\mathcal{Y}_I(S)$ for the preimage of any interval I under this map, so for example $\mathcal{Y}_{[0,\infty)}(S) = S \dot{\times} \operatorname{Spa} \mathbb{Z}_p$; in particular $\mathcal{Y}_{(0,\infty)}(S) = S \dot{\times} \operatorname{Spa} \mathbb{Q}_p$.

The Frobenius σ of S acts by $|\varpi(\sigma x)| = |\varpi(x)|^p$, and so takes $\mathcal{Y}_{[a,b]}(S)$ to $\mathcal{Y}_{pa,pb}(S)$. In particular, $\mathcal{Y}_{(0,\infty)}(S)$ has a free and totally discontinuous action of σ, by which we can quotient to get a (sousperfectoid) adic space $\mathcal{X}_S = \mathcal{Y}_{0,\infty}(S)/\sigma$, so that $\mathcal{X}_S^\diamondsuit \cong S/\sigma^{\mathbb{Z}} \times \operatorname{Spd} \mathbb{Q}_p$.

We call this the relative Fargues-Fontaine curve $\mathcal{X}_S$ over S. The simplest case is when $S = \operatorname{Spa} C$ for some algebraically closed perfectoid field C of characteristic p; in this case we call this the absolute Fargues-Fontaine curve $\mathcal{X}_C$. Our first goal is to understand G-torsors in this simplest case of $\mathcal{X}_C$.

We first consider the case $G = \mathrm{GL}_n$, so that we are just looking at vector bundles on $\mathcal{X}_C$.

*These notes are based on chapters 22-24 of [1].

Work of Fargues and Fontaine shows that there is a functor from the category of isocrystals over the residue field k of C to vector bundles on $\mathcal{X}_C$ which induces a bijection on the sets of isomorphism classes (and is further faithful and essentially surjective). The isomorphism classes of simple isocrystals correspond to rational numbers, and so the same holds for vector bundles, i.e. every vector bundle on $\mathcal{X}_C$ can be written as a sum of vector bundles $\mathcal{O}(\lambda)$ corresponding to the isocrystal corresponding to the rational number $-\lambda$. If $\lambda = \frac{d}{h}$ in lowest terms for $h > 0$, then $\mathcal{O}(\lambda)$ has rank h and degree d, so integers correspond to line bundles.

For the case of general G, there is a similar description, now between G-torsors and G-isocrystals, which we have not yet defined. A G-isocrystal is an exact symmetric monoidal functor $\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathbf{Isoc}_k$. Keeping in mind that isocrystals over k are vector spaces over K_0 with a σ-linear automorphism, this is some decoration of the fiber functor $\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathbf{Vect}_{K_0}$, i.e. the forgetful functor $\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathbf{Vect}_{\mathbb{Q}_p}$ base changed to K_0. For G embedded in some GL_n, including any algebraic group, this is determined by the representation given by that embedding; in particular for GL_n it is determined by the natural action on $\mathbb{Q}_p^n$, which is carried to some isocrystal, and so a GL_n-isocrystal is just an isocrystal.

On the other hand, we have a functor as above from $\mathbf{Isoc}_k$ to the category $\mathrm{Bun}(\mathcal{X}_C)$ of vector bundles on $\mathcal{X}_C$. Composing with any G-isocrystal gives a functor $\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathrm{Bun}(\mathcal{X}_C)$, i.e. a G-torsor on $\mathcal{X}_C$ (since G is an algebraic group, one can look at global sections of its structure sheaf, which carry a G-action, and by modifying the functor for colimits one can evaluate on this representation to get a vector bundle whose relative spectrum is a G-torsor, similar to the natural representation construction above). Thus we get a functor from G-isocrystals to G-torsors. On the other hand, G-isocrystals are classified by $B(G)$: each G-isocrystal defines an automorphism of the fiber functor for $\mathrm{Rep}_{\mathbb{Q}_p}(G)$ over K_0, i.e. an element of $G(\mathbb{Q}_p^{\mathrm{unr}})$, and changing the trivialization changes this element up to σ-conjugacy. Thus we get a map from $B(G)$ to the set of isomorphism classes of G-torsors on $\mathcal{X}_C$, and a theorem of Fargues states that this is a bijection.

2. Local shtukas

Given a test object S (a perfectoid space of characteristic p, which we may as well assume is affinoid $S = \mathrm{Spa}(R, R^+)$ over an algebraically closed field k), let $\mathcal{P}$ be a $\mathcal{G}$-torsor on $S \dot{\times} \mathrm{Spa}\,\mathbb{Z}_p = \mathcal{Y}_{[0,\infty)}(S)$, where $\mathcal{G}$ is a smooth group scheme over $\mathbb{Z}_p$ with generic fiber G (and connected special fiber). By definition, this means that $\mathcal{P}$ is locally $\mathcal{G}$-equivariantly isomorphic to $\mathcal{G} \times \mathcal{Y}_{[0,\infty)}$, and in particular for any sufficiently large r we have an isomorphism $\iota_r : \mathcal{P}|_{\mathcal{Y}_{[r,\infty)}(S)} \xrightarrow{\sim} G \times \mathcal{Y}_{[r,\infty)}(S)$, where we can replace $\mathcal{G}$ by G since away from 0 we are over $\mathbb{Q}_p$ rather than $\mathbb{Z}_p$. Note that the Frobenius induces isomorphisms $\mathcal{Y}_{[r,\infty)}(S) \cong \mathcal{Y}_{[pr,\infty)}(S)$, so after taking the quotient by Frobenius the restriction of $\mathcal{P}$ to $\mathcal{Y}_{[r,\infty)}(S)$ induces a G-torsor $\mathcal{E}$ on $\mathcal{Y}_{(0,\infty)}(S)/\sigma^{\mathbb{Z}} = \mathcal{X}_S$, independent of r (and the chosen pseudo-uniformizer). Pulling back along any geometric point of S gives a G-torsor on $\mathcal{X}_C$ for a suitable field C, which is determined by an element $b \in B(G)$; since these should be compatible, we might hope that these will all give the same b.

Given some $b \in B(G)$, we can choose a representative in $G(\breve{\mathbb{Q}}_p)$, also denoted b. On $\mathcal{Y}_{[r,\infty)}(S)$, we don't have a canonical element, but we do have a canonical automorphism, given by the Frobenius on S as above; we denote this again by σ; and we can also view b as an automorphism by multiplication. Then we can pull back this automorphism (b, σ) along the isomorphism $\mathcal{P}|_{\mathcal{Y}_{[r,\infty)}(S)} \xrightarrow{\sim} G \times \mathcal{Y}_{[r,\infty)}(S)$ to get an automorphism of $\mathcal{P}_{\mathcal{Y}_{[r,\infty)}(S)}$.

It is too much to hope for in general for this automorphism to extend to an automorphism of $\mathcal{P}$ over all of $\mathcal{Y}_{[0,\infty)} = S \dot{\times} \operatorname{Spa}\mathbb{Z}_p$. However, it is possible away from finitely many points (or poles of this automorphism): choose S-points $x_1, \ldots, x_m : S \to S \dot{\times} \operatorname{Spa}\mathbb{Z}_p$, with graph $\Gamma_{x_i} \subset S \dot{\times} \operatorname{Spa}\mathbb{Z}_p$ (the graph can be similarly literally defined in the diamond setting, and then reinterpreted as an adic space). Then the preimage of (b, σ) gives an isomorphism

$$\varphi_{\mathcal{P}} : (\sigma^* \mathcal{P})|_{S \dot{\times} \operatorname{Spa}\mathbb{Z}_p \setminus \bigcup_i \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{P}|_{S \dot{\times} \operatorname{Spa}\mathbb{Z}_p \setminus \bigcup_i \Gamma_{x_i}}.$$

For fixed x_i, we can always choose r large enough that the isomorphism is defined on $\mathcal{Y}_{[r,\infty)}$, so the above construction works.

On the other hand, by definition a morphism $x : S \to S \dot{\times} \operatorname{Spa}\mathbb{Z}_p$ is the same thing as, on the level of diamonds, a morphism $S \to S \times \operatorname{Spd}\mathbb{Z}_p$ which is a section of the projection $S \times \operatorname{Spd}\mathbb{Z}_p \to S$, i.e. just a morphism $S \to \operatorname{Spd}\mathbb{Z}_p$, which by definition is the same thing as an untilt $S^\sharp$ of S. If x is defined over a field $\breve{E}/\breve{\mathbb{Q}}_p$ (necessarily of the form $\breve{E} = E \cdot \breve{\mathbb{Q}}_p$ for some finite $E/\mathbb{Q}_p$, since S is over k), then this is the same thing as an untilt $S^\sharp$ over $\breve{E}$. Such an untilt carries a map $S^\sharp \to S \dot{\times} \operatorname{Spa}\mathbb{Z}_p$, as can be seen on the level of diamonds: $\operatorname{Hom}(-, S^\sharp) \simeq \operatorname{Hom}(-, S) \hookrightarrow \operatorname{Hom}(-, S) \times \operatorname{Spd}\mathbb{Z}_p$, i.e $S^\sharp$ embeds into $S \dot{\times} \operatorname{Spa}\mathbb{Z}_p$ as adic spaces, and defines a Cartier divisor of $S \dot{\times} \operatorname{Spa}\mathbb{Z}_p$.

Thus in our case, we can take all the x_i to be in the nonzero locus and therefore over some fields $\breve{E}_i$, and so the data of the x_i is the same as the data of untilts $S_i^\sharp$ of S to $\breve{E}_i$, which each (and so collectively) define Cartier divisors in $S \dot{\times} \operatorname{Spa}\mathbb{Z}_p$. At each resulting Cartier divisor $S_i^\sharp$, choose trivializations of the $\mathcal{G}$-torsors $\phi_1 : \mathcal{P} \cong \mathcal{G} \times S^\sharp$ and $\phi_2 : \sigma^*\mathcal{P} \cong G \times S_i^\sharp$. Composing these gives a map of $\mathcal{G}$-torsors $\phi_2^{-1}\phi_1 : \mathcal{P} \to \sigma^*\mathcal{P}$, and restricting to $S^\sharp$ this is isomorphic to an automorphism of $\mathcal{G}$-torsors, i.e. an element of $\mathcal{G}$. This element depends on the chosen trivialization, but its image in the double quotient $\mathcal{G}(W)\backslash\mathcal{G}(\mathbb{Q}_p^{\mathrm{unr}})/\mathcal{G}(Q)$, which by the Cartan decomposition is in bijection with the set of positive weights, is not, and so we can define the relative position of σ^*P and $\mathcal{P}$ to be the equivalence class of this automorphism in the set of positive weights. In particular, for each i we pick a conjugacy class of cocharacters $\mu_i : \mathbb{G}_{\mathrm{m}} \to G_{\overline{\mathbb{Q}}_p}$, and we require that the relative position of σ^*P and $\mathcal{P}$ at $S_i^\sharp$ to be bounded by μ_i for the Bruhat ordering of weights. (The condition is more complicated when the x_i intersect, and should really be that the relative position is bounded by the sum of the μ_j for j such that $S_j^\sharp = S_i^\sharp$, but we'll mostly ignore this complication.) If μ_i is defined over $E_i/\mathbb{Q}_p$, then $S_i^\sharp$ will be defined over some extension, by the properties of reflex fields; in particular $S_i^\sharp$ is over $\breve{E}_i$.

Thus the data we want to classify is the following. We fix our group $\mathcal{G}$, an element $b \in B(G)$, and m conjugacy classes $\mu_1, \ldots, \mu_m : \mathbb{G}_{\mathrm{m}} \to G_{\overline{\mathbb{Q}}_p}$, each defined over some finite extension E_i of $\mathbb{Q}_p$. For each perfectoid space S in characteristic p we define a shtuka on S to be a tuple $(\mathcal{P}, \{S_i^\sharp\}, \varphi_{\mathcal{P}}, \iota_r)$, where $\mathcal{P}$ is a $\mathcal{G}$-torsor on $S \dot{\times} \operatorname{Spa}\mathbb{Z}_p$, $S_i^\sharp$ is an untilt of S to $\breve{E}_i$ for each i, $\varphi_{\mathcal{P}}$ is an isomorphism $\sigma^*\mathcal{P}|_{S \dot{\times} S \setminus \bigcup_i S_i^\sharp} \xrightarrow{\sim} \mathcal{P}|_{S \dot{\times} S \setminus \bigcup_i S_i^\sharp}$, and ι_r is an isomorphism $\mathcal{P}|_{\mathcal{Y}_{[r,\infty)}(S)} \xrightarrow{\sim} G \times \mathcal{Y}_{[r,\infty)}(S)$ for r sufficiently large carrying $\varphi_{\mathcal{P}}$ to (g, σ), where $\sigma^*\mathcal{P}$ and $\mathcal{P}$ have relative position at each $S_i^\sharp$ bounded by μ_i in the sense above.

The moduli problem is now clear: we send a perfectoid space S in characteristic p to the set of shtukas on S, and call this functor $\mathrm{Sht}_{\mathcal{G},b,\{\mu_i\}}$. This is a presheaf on $\mathbf{Perf}_k$, and is fibered over $\operatorname{Spd} \breve{E}_1 \times_{\operatorname{Spd} k} \cdots \times_{\operatorname{Spd} k} \operatorname{Spd} \breve{E}_m$ (which is a nontrivial product in diamonds!) by remembering only the untilts $S_i^\sharp$ (or equivalently the maps x_i), called the legs or paws. By a descent argument, it is possible to see that $\mathrm{Sht}_{\mathcal{G},b,\{\mu_i\}}$ is a v-sheaf; our goal is to prove that in fact it is a locally spatial diamond.

3. Moduli spaces of local shtukas

Consider first the case of shtukas with no legs, $m = 0$. In this case the data of $\{\mu_i\}$ is trivial, so we're looking at $\mathrm{Sht}_{\mathcal{G},b,\{\}}$, which classifies $\mathcal{G}$-torsors $\mathcal{P}$ on $S \dot\times \operatorname{Spa} \mathbb{Z}_p$ together with an isomorphism $\varphi_{\mathcal{P}} : \sigma^*\mathcal{P} \xrightarrow{\sim} \mathcal{P}$ and a trivialization on $\mathcal{Y}_{[r,\infty)}(S)$ for r sufficiently large sending $\varphi_{\mathcal{P}}$ to (b, σ). Since there are no legs, in this case this trivialization does extend to all of $\mathcal{Y}_{(0,\infty)}(S)$, with Frobenius acting by the isomorphism $\varphi_{\mathcal{P}}$ which is identified with (b, σ), so that we can quotient by the Frobenius action to get the G-torsor on the Fargues-Fontaine curve corresponding to b (i.e. the G-torsor whose restriction to any geometric point corresponds to b), so that $\mathcal{P}$ is the pullback of the G-torsor $\mathcal{E}$ corresponding to b. In particular, it gives an extension of $\mathcal{E}$ to a $\mathcal{G}$-torsor; but this is only possible when b is trivial, so the only nonempty moduli space is $\mathrm{Sht}_{\mathcal{G},1,\{\}}$. In this case, extensions of the trivial G-torsor are given by $\mathcal{G}(\mathbb{Z}_p)$-lattices in the trivial $G(\mathbb{Q}_p)$-torsor on S, i.e. S-points of $\mathcal{G}(\mathbb{Z}_p)/G(\mathbb{Q}_p)$, so $\mathrm{Sht}_{\mathcal{G},1,\{\}} = \mathcal{G}(\mathbb{Z}_p)/G(\mathbb{Q}_p)$, which is a perfectoid space and so certainly a spatial diamond.

In general, we need to pass through a (slightly) simpler object, namely a Beilinson-Drinfeld Grassmannian. For $E/\mathbb{Q}_p$ a finite extension, G an algebraic group over $\mathbb{Q}_p$, and $\mu : \mathbb{G}_{\mathrm{m}} \to G_{\overline{\mathbb{Q}}_p}$ a cocharacter, we define $\mathrm{Gr}_{G,\operatorname{Spd} E,\leq\mu}$ to be the presheaf sending S to the set of tuples $(\mathcal{P}, S^\sharp, \iota)$, where $\mathcal{P}$ is a G-torsor on $S \dot\times \operatorname{Spd} \mathbb{Q}_p$, $S^\sharp$ is an untilt of S over E (defining a Cartier divisor in $S \dot\times \mathbb{Q}_p$), and ι is a trivialization of $\mathcal{P}$ away from $S^\sharp$ bounded by μ at $S^\sharp$. This is reminiscent of the definition of $\mathrm{Sht}_{\mathcal{G},b,\{\mu_i\}}$ and one could imagine it would carry a natural map from it in the case of one leg, but there is one difference: now we're working with G-torsors instead of $\mathcal{G}$-torsors. To make the change, we need to reinterpret the moduli structure of $\mathrm{Sht}_{\mathcal{G},b,\{\mu\}}$.

Suppose we have a tuple $(\mathcal{P}, S^\sharp, \varphi_{\mathcal{P}}, \iota_r)$ giving an S-point of $\mathrm{Sht}_{\mathcal{G},b,\{\mu\}}$. Since $S^\sharp$ is away from 0, we can find some $\epsilon > 0$ such that $\mathcal{Y}_{(0,\epsilon)}(S)$ is disjoint from the Cartier divisor of $S^\sharp$ in $S \dot\times \mathbb{Q}_p = \mathcal{Y}_{(0,\infty)}(S)$. Then the restriction of $\mathcal{P}$ to $\mathcal{Y}_{(0,\epsilon)}(S)$ is φ-equivariant and so descends to $\mathcal{X}_S$, where it gives a G-torsor $\mathcal{E}$ corresponding to some $b \in B(G)$ on every geometric point away from $S^\sharp$. The pullback is an extension $\mathcal{P}$ to a $\mathcal{G}$-torsor, which as above corresponds to a $\mathcal{G}(\mathbb{Z}_p)$-lattice in the $\mathcal{G}(\mathbb{Q}_p)$-torsor $\mathcal{E}$. Thus the data of an S-point of $\mathrm{Sht}_{\mathcal{G},b,\{\mu\}}$ can be rephrased as a tuple $(\mathcal{E}, S^\sharp, \alpha, \mathbb{P})$ where $\mathcal{E}$ is a G-torsor on $\mathcal{X}_S$, $S^\sharp$ is an untilt of S to E, the field of definition of μ, α is an isomorphism away from $S^\sharp$ between $\mathcal{E}$ and the G-torsor $\mathcal{E}^b$ on $\mathcal{X}_S$ corresponding on each geometric point to b, which is bounded by μ, and $\mathbb{P}$ is a $\mathcal{G}(\mathbb{Z}_p)$-lattice in the $G(\mathbb{Q}_p)$-torsor $\mathcal{E}$.

Now this depends not on the model $\mathcal{G}$ but only on G and $\mathcal{G}(\mathbb{Z}_p)$. We can replace $\mathcal{G}(\mathbb{Z}_p)$ by other compact subgroups $K \subset G(\mathbb{Q}_p)$; in this case we replace the $\mathcal{G}(\mathbb{Z}_p)$-lattice $\mathbb{P}$ by a K-torsor, whose product to $\mathbb{Q}_p$ recovers $\mathcal{E}$. We can form a tower of moduli spaces $(\mathrm{Sht}_{G,b,\{\mu\},K})_K$ in this way, and taking the limit gives a moduli space $\mathrm{Sht}_{G,b,\{\mu\},\infty}$ classifying pairs $(S^\sharp, \alpha)$ where $S^\sharp$ is an untilt of S over E and α is an isomorphism away from $S^\sharp$ of the trivial torsor $\mathcal{E}^1$ with $\mathcal{E}^b$, bounded by μ. This tower, and its limit, have natural commuting actions of

$G(\mathbb{Q}_p)$ (via the action on $\mathcal{E}^1$) and the inner form $J_b(\mathbb{Q}_p)$, the σ-centralizer of b (via the action on $\mathcal{E}^b$).

Now we have a projection: send an S-point $(\mathcal{E}, S^\sharp, \alpha, \mathbb{P})$ of $\mathrm{Sht}_{G,b,\{\mu\},K}$ to $(\mathcal{E}, S^\sharp, \tilde{\alpha})$ where $\tilde{\alpha}$ is the trivialization away from $S^\sharp$ after pulling back $\mathcal{E}^b$ to the trivial G-torsor on $S \dot{\times} \operatorname{Spd} \mathbb{Q}_p$. This gives a period map $\pi_{GM} : \mathrm{Sht}_{G,b,\{\mu\},K} \to \mathrm{Gr}_{G,\operatorname{Spd} E,\leq\mu}$. As it turns out this map is étale, and $\mathrm{Gr}_{G,\operatorname{Spd} E,\leq\mu}$ is a spatial diamond, so it follows that $\mathrm{Sht}_{G,b,\{\mu\},K}$ and in particular $\mathrm{Sht}_{\mathcal{G},b,\{\mu\}}$ are locally spatial diamonds.

The general case is similar; we again have towers $\mathrm{Sht}_{G,b,\{\mu_i\},K}$, and again build an étale map to a Grassmannian, which is again a spatial diamond and so the result follows. In general the Grassmannian $\mathrm{Gr}_{G,\operatorname{Spd} E_1 \times_{\operatorname{Spd} k} \cdots \times_{\operatorname{Spd} k} \operatorname{Spd} E_m, \leq(\mu_1,\ldots,\mu_m)}$ is more complicated: generically, it is just the product of the $\mathrm{Gr}_{G,\operatorname{Spd} E_i,\leq\mu_i}$, but in general (where the points may intersect) it is more complicated and should really be a 'twisted' Grassmannian, which one can define as a moduli problem in a similar fashion to the moduli space of shtukas. The infinite level space of shtukas parametrizes tuples of untilts over the E_i with a trivialization of $\mathcal{E}^b$ away from the Cartier divisor of the (union of the) untilts, bounded in a certain complicated way by the μ_i. Each $\mathrm{Sht}_{G,b,\{\mu_i\},K}$ lives over $\operatorname{Spd} E_1 \times_{\operatorname{Spd} k} \cdots \times_{\operatorname{Spd} k} \operatorname{Spd} E_m$, via the map to the paws; this factors through π_{GM}.

4. Comparison to Rapoport-Zink spaces

In some cases, such as PEL Shimura data, we already have an interpretation of local Shimura varieties, via Rapoport-Zink spaces. If moduli spaces of local shtukas are to give a notion of local Shimura varieties, they should certainly be compatible with this preexisting notion.

Recall that a local Shimura datum is given by a triple (G, b, μ) as above (in particular, we will be concerned only with shtukas with one leg), subject to certain compatibility conditions: $b \in B(G, \mu^{-1})$. (Absent this condition, we can still define $\mathrm{Sht}_{G,b,\{\mu\},K}$, but it will be empty.) Fix additionally a model $\mathcal{G}$ of G over $\mathbb{Z}_p$, or equivalently a compact open subgroup $K \subset G(\mathbb{Q}_p)$; to get a good theory of extensions of torsors, the technical condition that we need on $\mathcal{G}$ is that it be parahoric, which we do not explain here. If E is the field of definition of μ, then $\mathrm{Gr}_{G,\operatorname{Spd} \breve{E},\leq\mu}$ is just the diamondization of the flag variety for $(G, \mu, \breve{E})$ when μ is minuscule. In particular since $\pi_{GM} : \mathrm{Sht}_{G,b,\{\mu\},K} \to \mathrm{Gr}_{G,\operatorname{Spd} \breve{E},\leq\mu}$ is étale and diamondization induces an equivalence of étale sites $X_{\text{ét}} \cong X^\diamondsuit_{\text{ét}}$, it follows that π_{GM} is the diamondization of some analytic space over the flag variety, i.e. $\mathrm{Sht}_{G,b,\{\mu\},K} = \mathcal{M}^\diamondsuit_{G,b,\mu,K}$ for a unique adic (in fact, rigid) space $\mathcal{M}_{G,b,\mu,K}$ with a map to the flag variety, and the transition maps for varying K are finite étale. We define the local Shimura variety over $\breve{E}$ to be the tower $(\mathcal{M}_{G,b,\mu,K})_K$.

Recall that in the simplest case the Rapoport-Zink space associated to a local Shimura datum (GL_n, b, μ) is just the moduli space $\mathcal{M}_{X_b}$ of deformations of a p-divisible group X_b over k corresponding to $b \in B(G, \mu^{-1})$. We can take its generic fiber $\mathcal{M}_{X_b,\breve{\mathbb{Q}}_p}$ to get a rigid (or adic) space over $\breve{\mathbb{Q}}_p$.

Theorem 4.1. *There is a natural isomorphism of diamonds*

$$\mathcal{M}^\diamondsuit_{X_b,\breve{\mathbb{Q}}_p} \simeq \mathcal{M}^\diamondsuit_{\mathrm{GL}_n,b,\mu,\mathrm{GL}_n(\mathbb{Z}_p)} = \mathrm{Sht}_{\mathrm{GL}_n,b,\{\mu\},\mathrm{GL}_n(\mathbb{Z}_p)}$$

over $\operatorname{Spd} \breve{\mathbb{Q}}_p$.

Proof. We sketch a proof based on the period maps; next time we'll give another proof using the integral structure.

The first thing to do is to construct a map $\mathcal{M}_{X_b,\breve{\mathbb{Q}}_p} \to \mathrm{Sht}_{\mathrm{GL}_n,b,\{\mu\},\mathrm{GL}_n(\mathbb{Z}_p)}$, which we do by taking a p-divisible group X lifting X_b and building from it a GL_n-shtuka.

More precisely, let $S = \mathrm{Spa}(R, R^+)$ be a perfectoid space over $\breve{\mathbb{Q}}_p$, and let X be a p-divisible group over R^+ with a quasi-isogeny $X \times_{\mathrm{Spf}\, R^+} \mathrm{Spec}\, R^+/p \to X_b \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, R^+/p$. If EX is the universal vector extension of X, then this quasi-isogeny induces an isomorphism on tangent spaces $\mathrm{Lie}\, EX[1/p] \cong M(X) \otimes_{\mathbb{Q}_p} R$, where $M(X)$ is the Dieudonné module of X. Recall that X_b/k is determined by its height, here n since we take $G = \mathrm{GL}_n$, and dimension d, so that $\mu = (1, \dots, 1, 0, \dots, 0)$ has d occurrences of 1; then $M(X) \cong \breve{\mathbb{Q}}_p^n$ with σ-linear action $b\sigma$. On the other hand, we have a surjection $\mathrm{Lie}\, EX \to \mathrm{Lie}\, X$ induced by $EX \to X$, or equivalently a point of $\mathrm{Gr}(d, n)_{\breve{\mathbb{Q}}_p}$ classifying d-dimensional quotients of a vector space (here $\mathrm{Lie}\, EX$) over $\breve{\mathbb{Q}}_p$ dimension n. Combining these gives an R-point of $\mathrm{Gr}(d, n)_{\breve{\mathbb{Q}}_p}$.

The data of $S = \mathrm{Spa}(R, R^+)$ associates to its tilt $S^\flat$ a specified untilt S over $\mathbb{Q}_p$, i.e. a map $S^\flat \to \mathrm{Spd}\, \mathbb{Q}_p$ or equivalently a map $S^\flat \to S^\flat \dot{\times} \mathrm{Spa}\, \mathbb{Q}_p$ which on diamonds is a section of the projection $S^\flat \times \mathrm{Spd}\, \mathbb{Q}_p \to S^\flat$. The data of an R-point of $\mathrm{Gr}(d, n)_{\breve{\mathbb{Q}}_p}$ further associates to each point of S a d-dimensional quotient of an n-dimensional space, and in particular along this morphism gives rise to a vector bundle $\mathcal{E}$ on $S^\flat \times \mathrm{Spd}\, \mathbb{Q}_p$; since this comes via the Cartier divisor given by S, away from this divisor this has a natural isomorphism with the pullback by Frobenius. For r sufficiently large, restricting to $\mathcal{Y}_{[r,\infty)}(S^\flat)$ gives a vector bundle isomorphic to $\mathcal{E}^b$ for our fixed b, in this case a vector bundle of rank n and degree d. This is true away from S; more generally, if $i : S \to S^\flat \times \mathrm{Spd}\, \mathbb{Q}_p$ is the embedding of the Cartier divisor, $\mathrm{Lie}\, X$ lives over R and thus over S so we can push forward to i_*, which is a quotient of $\mathcal{E}^b$; call the kernel $\mathcal{F}$. It is σ-equivariant for the Frobenius σ on $S^\flat$, and so descends to $\mathcal{X}_{S^\flat}$. This gives us our desired $\mathrm{GL}_n(\mathbb{Q}_p)$-torsor.

To get the integral lattice, we look at the Tate module $\mathcal{T}$ of X. This is a $\mathbb{Z}_p$-local system with a map $\mathcal{T} \times_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{X}_{S^\flat}} \to \mathcal{E}^b$. It turns out that this factors through the subbundle $\mathbb{F}$, onto which it is an isomorphism; this gives the desired $G(\mathbb{Z}_p)$-torsor in the $G(\mathbb{Q}_p)$-torsor. Thus we've built a map $\mathcal{M}_{X_b,\breve{\mathbb{Q}}_p} \to \mathrm{Sht}_{\mathrm{GL}_n,b,\{\mu\},\mathrm{GL}_n(\mathbb{Z})}$.

Since this map is built out of the map to $\mathrm{Gr}(d, n)_{\breve{\mathbb{Q}}_p}$, which in this case is equal to the Grassmannian $\mathrm{Gr}_{\mathrm{GL}_n,\mathrm{Spd}\, \breve{\mathbb{Q}}_p,\le\mu}$ with μ as above corresponding to d, it commutes with the period morphisms. The fibers of $\mathcal{M}_{X_b,\breve{\mathbb{Q}}_p}$ over points $x \in \mathrm{Gr}(d, n)_{\breve{\mathbb{Q}}_p}$ are the lattices in the universal n-vector space, and so in bijection with $\mathrm{GL}(\mathbb{Q}_p)/\mathrm{GL}(\mathbb{Z}_p)$, if x is admissible, and empty otherwise. But this is the same description as the fibers of π_{GM}: if the admissibility condition is not satisfied they are empty, and if it is then the data of the vector bundle is fixed and what remains is the lattices, again in this case classified by $\mathrm{GL}(\mathbb{Q}_p)/\mathrm{GL}(\mathbb{Z}_p)$. Since the diagram commutes and the images and fibers in the Grassmannian agree, the map constructed above $\mathcal{M}_{X_b,\breve{\mathbb{Q}}_p} \to \mathrm{Sht}_{\mathrm{GL}_n,b,\{\mu\},\mathrm{GL}_n(\mathbb{Z}_p)}$ must be an isomorphism. □

In fact, the same proof shows that when we put level structures on Rapoport-Zink spaces and take the diamondization of the generic fiber, we again get $\mathrm{Sht}_{\mathrm{GL}_n,b,\{\mu\},K}$.

More generally, fix PEL data with integral structure $\mathcal{D} = (B, V, \mathcal{O}_B, \mathcal{L}, \psi, *, b, \mu)$ and corresponding group scheme $\mathcal{G}$, with generic fiber G. We can take the generic fiber to get a rigid space $\mathcal{M}_{\mathcal{D},\breve{E}}$, which is independent of the integral structure other than the compact open $\mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$.

Theorem 4.2. *There is a natural isomorphism of smooth rigid spaces*

$$\mathcal{M}_{\mathcal{D},\breve{E}} \simeq \mathcal{M}_{G,b,\mu,\mathcal{G}(\mathbb{Z}_p)}$$

over $\breve{E}$.

Proof sketch. Since $\mathcal{M}_{\mathcal{D}}$ has a closed immersion into the product of $\mathcal{M}_{X_{b,\Lambda}}$ over $\Lambda \in \mathcal{L}$, the same is true of the generic fibers. On the other side, for each Λ write G_Λ for the group $R \mapsto \mathrm{Aut}(\Lambda \otimes_{\mathbb{Z}_p} R)$ for $\mathbb{Z}_p$-algebras R, so that in particular $G_\Lambda(\mathbb{Q}_p) = \mathrm{GL}(V)$. There is a natural map $\rho_\Lambda : G \to \mathrm{GL}_\Lambda$ for each Λ given by the action of G on $\Lambda \subset V$. Then $\mathcal{M}_{G,b,\mu,\mathcal{G}(\mathbb{Z}_p)}$ has a natural map to $\mathcal{M}_{\mathrm{GL}_\Lambda,\rho_\Lambda(b),\rho_\Lambda\circ\mu,\mathrm{GL}_\Lambda(\mathbb{Z}_p)}$, and taking the product of the Λ again gives a closed immersion. For each fixed Λ, by Theorem 4.1 we get an isomorphism between these two sides and so

$$\prod_\Lambda \mathcal{M}_{X_{b,\Lambda}} \simeq \prod_\Lambda \mathcal{M}_{\mathrm{GL}_\Lambda,\rho_\Lambda(b),\rho_\Lambda\circ\mu,\mathrm{GL}_\Lambda(\mathbb{Z}_p)},$$

so it is enough to see that the geometric points of each side (on the level of diamonds) agree under these embeddings.

To that end, fix an algebraically closed perfectoid field C over $\breve{E}$, with ring of integers $\mathcal{O}_C$, and let A_{inf} be the Witt vectors $W(\mathcal{O}_{C^\flat})$. An $\mathcal{O}_C$-point of $\mathcal{M}_{\mathcal{D},\breve{E}}$ is a chain of $\mathcal{O}_B$-p-divisible groups of type $(\mathcal{L})$ over $\mathcal{O}_C$, whose associated Breuil-Kisin-Fargues module (roughly an upgrading of the Dieudonné module $\mathrm{Hom}(X, W)$ for a p-divisible group X over k) inherits an $\mathcal{O}_B$-action over A_{inf}, i.e. it is a chain of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}$-modules, which one can verify from the conditions on the chain of $\mathcal{O}_B$-p-divisible groups of type $(\mathcal{L})$ is itself a chain of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}$-modules of type $(\mathcal{L})$. The automorphisms of this chain are then given at each Λ by the algebraic subgroup of $\mathrm{GL}(V)$ commuting with the $\mathcal{O}_B$-structure, which is precisely G, and so this is the data of a G-torsor over A_{inf}, with the Breuil-Kisin-Fargues structure giving an isomorphism away from the kernel of the map $A_{\mathrm{inf}} \to \mathcal{O}_C$ with the pullback by Frobenius. The condition $b \in B(G, \mu^{-1})$ translates to the relative position of this torsor and its Frobenius pullback being bounded by μ; but descending to $\mathrm{Spa}\, C^\flat \dot{\times} \mathrm{Spa}\, \mathbb{Z}_p$ loses no data and so this is equivalent to the data parametrized by geometric points in $\mathcal{M}_{G,b,\mu,\mathcal{G}(\mathbb{Z}_p)}$. ☐

References

[1] Peter Scholze and Jared Weinstein. Berkeley lectures on p-adic geometry. In *Berkeley Lectures on p-adic Geometry*. Princeton University Press, 2020.