Rapoport-Zink spaces*

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1. Lubin-Tate space

We start by trying to do explicit local class field theory. Given a p-adic field $F/\mathbb{Q}_p$, by local class field theory we can describe $\mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}}$ as $\mathcal{O}_F^\times \times \widehat{\mathbb{Z}}$. The fixed field of the $\mathcal{O}_F^\times$ factor is the maximal unramified extension of F, which is generated by roots of unity of order prime to p; what about the totally ramified extension $\overline{F}^{\widehat{\mathbb{Z}}}$?

In the case $F = \mathbb{Q}_p$, this is the field generated by the pth power roots of unity, i.e. the p-torsion in $\mathbb{G}_{\mathrm{m}}$. This suggests we should replace $\mathbb{G}_{\mathrm{m}}$ by some other object $\mathcal{G}$ for general p-adic fields F, carrying an action of $\mathcal{O}_F^\times$.

We can construct these objects as formal group laws. A (one-dimensional commutative) formal group law $\mathcal{G}$ over a ring A is a power series $\mathcal{G}(X,Y) \in A[[X,Y]]$ such that $\mathcal{G}(X,Y) = X + Y +$ higher order terms, $\mathcal{G}(X,Y) = \mathcal{G}(Y,X)$ (commutativity), and $\mathcal{G}(\mathcal{G}(X,Y),Z) = \mathcal{G}(X,\mathcal{G}(Y,Z))$ (associativity). From these axioms it is possible to derive the existence of an inverse $i(X) \in A[[X]]$ with $\mathcal{G}(X, i(X)) = 0$. This is analogous to addition on an abelian group, and we write it as $\mathcal{G}(X,Y) = X +_{\mathcal{G}} Y$; the simplest case is $\mathcal{G}(X,Y) = X + Y$, the additive group law $\mathbb{G}_{\mathrm{a}}$. The multiplicative group law is simply $X + Y + XY = (X+1)(Y+1) - 1$. Elliptic curves also have associated formal group laws. We define a homomorphism $f : \mathcal{G} \to \mathcal{G}'$ of formal groups to be a power series such that $f(X +_{\mathcal{G}} Y) = f(X) +_{\mathcal{G}} f(Y)$, or in other words $f(\mathcal{G}(X,Y)) = \mathcal{G}(f(X), f(Y))$.

For any natural number n, we write $[n]_{\mathcal{G}}(X)$ for the n-fold application of $\mathcal{G}$ to X, i.e. $X +_{\mathcal{G}} X +_{\mathcal{G}} \cdots +_{\mathcal{G}} X$; this satisfies $[n]_{\mathcal{G}}(X +_{\mathcal{G}} Y) = X +_{\mathcal{G}} Y +_{\mathcal{G}} X + \cdots +_{\mathcal{G}} X +_{\mathcal{G}} Y = [n]_{\mathcal{G}}(X) + [n]_{\mathcal{G}}(Y)$ by commutativity, so it is an endomorphism of $\mathcal{G}$. Via $i(X)$, we can do the same thing for negative numbers to get a map $[\cdot]_{\mathcal{G}} : \mathbb{Z} \to \operatorname{End}\mathcal{G}$, which is just the map coming from the fact that $\operatorname{End} G$ is a (possibly noncommutative) ring and $\mathbb{Z}$ is initial among rings.

More generally, for F a p-adic field and A an $\mathcal{O}_F$-algebra, we say that a formal group law $\mathcal{G}$ over A is a formal $\mathcal{O}_F$-module law when it has a homomorphism $[\cdot]_{\mathcal{G}} : \mathcal{O}_F \to \operatorname{End}\mathcal{G}$ extending the map from $\mathbb{Z}$ and such that $[a]_{\mathcal{G}} = aX + O(X^2)$ for $a \in \mathcal{O}_F$, where on the right we interpret a as its image in A.

For example, the additive group law becomes a formal $\mathcal{O}_F$-module over any $\mathcal{O}_F$-algebra A. In the case $F = \mathbb{Q}_p$, the multiplicative group law becomes a formal $\mathbb{Z}_p$-module because for $a \in \mathbb{Z}_p$ we can define $[a]_{\mathbb{G}_{\mathrm{m}}}(X) = (1+X)^a - 1$. In particular multiplication by p gives $[p]_{\mathbb{G}_{\mathrm{m}}}(X) = (1 + X^p) - 1 \equiv X^p \pmod{p}$.

Let π be a uniformizer for F, and q be the cardinality of the residue field k. It turns out that we can define a formal $\mathcal{O}_F$-module over $\mathcal{O}_F$ by choosing $[\pi]_{\mathcal{G}}$ and constructing the rest of $\mathcal{G}$ from it. Fix a power series $f(X) \in \mathcal{O}_F[[X]]$ such that $f(X) = \pi X + O(X^2)$ and $f(X) = X^q \pmod{\pi}$. (Note that in the case $F = \mathbb{Q}_p$ and $\pi = p$, $[p]_{\mathbb{G}_{\mathrm{m}}}$ has this property.) Then there exists a unique formal $\mathcal{O}_F$-module law $\mathcal{G}$ over $\mathcal{O}_F$ such that $[\pi]_{\mathcal{G}} = f(X)$.

To see this, recall that $[\pi]_{\mathcal{G}}$ is by definition an endomorphism of $\mathcal{G}$, i.e. a power series $f(X)$ such that $f(X +_{\mathcal{G}} Y) = f(X) +_{\mathcal{G}} f(Y)$. If we write $X +_{\mathcal{G}} Y = X + Y + a_{2,0}X^2 + a_{1,1}XY + a_{0,2}Y^2 + \cdots$ and $f(X) = \pi X + b_2 X^2 + b_3 X^3 + \cdots$, this is

*These notes are based on [2] and chapter 24 of [1].

$$\pi(X + Y + a_{2,0}X^2 + a_{1,1}XY + a_{0,2}Y^2 + \cdots) + b_2(X^2 + 2XY + Y^2 + \cdots) + \cdots$$
$$= \pi X + b_2X^2 + \pi Y + b_2Y^2 + a_{2,0}\pi^2X^2 + a_{1,1}\pi^2XY + a_{0,2}\pi^2Y^2 + \cdots$$

and so collecting terms we get that for coefficients of X^2 we have

$$a_{2,0}\pi + b_2 = b_2 + a_{2,0}\pi^2,$$

so $a_{2,0} = 0$, for XY we have

$$a_{1,1}\pi + 2b_2 = a_{1,1}\pi^2,$$

so $a_{1,1} = \frac{2b_2}{\pi^2-\pi}$ (note that we assume $f(X) \equiv X^q \pmod{\pi}$ so for $q \neq 2$ we always have b_2 divisible by π, so this is still in $\mathcal{O}_F$; if $q = 2$ then the numerator has a factor of 2, so it is again in $\mathcal{O}_F$), and for Y^2

$$a_{0,2}\pi + b_2 = b_2 + a_{0,2}\pi^2,$$

so $a_{0,2} = 0$. We can proceed similarly in higher degrees: the key point is that this formula gives a unique value of each coefficient of $\mathcal{G}$. Indeed, different choices of f give isomorphic $\mathcal{G}$.

Choose some f (for example, $f(X) = \pi X + X^q$ works) and the corresponding $\mathcal{G}$. Over a separable closure, let $\mathcal{G}[\pi^n]$ be the set of elements of the maximal ideal which are killed by $[\pi^n]_\mathcal{G}$; over k, this is just x^{q^n}, corresponding to q^nth roots of unity, and so lifts uniquely by Hensel's lemma to a set $\mathcal{G}[\pi^n]$ of order q^n. For each n, further action of $[\pi]_\mathcal{G}$ gives a surjection $\mathcal{G}[\pi^n] \to \mathcal{G}[\pi^n]$ with kernel $\mathcal{G}[\pi]$, of order q; each $\mathcal{G}[\pi^n]$ is a free $\mathcal{O}_F/\pi^n\mathcal{O}_F$-module of rank 1. We can define an extension F_π of F generated by the $\mathcal{G}[\pi^n]$ for all $n \geq 1$, and a Tate module

$$T_\pi(\mathcal{G}) = \varprojlim_n \mathcal{G}[\pi^n].$$

This is a free $\mathcal{O}_F$-module of rank 1 with a continuous Galois action by $\mathrm{Gal}(F_\pi/F)$; Lubin and Tate showed that this action induces an isomorphism $\mathrm{Gal}(F_\pi/F) \to \mathrm{Aut}(T_\pi(\mathcal{G})) = \mathcal{O}_F^\times$, as desired.

What we essentially did here was choose a formal group law over $k = \mathbb{F}_q$, and show that we could lift it uniquely to a formal $\mathcal{O}_F$-module over $\mathcal{O}_F$. In general, this is a bit more complicated, but is essentially what we want to do. The first thing is to study formal group laws over $\mathbb{F}_q$: it turns out that these are classified by positive integers, called their heights.

Fix a formal group law $\mathcal{G}$ over $\mathbb{F}_q$. Then we always have the action of integers $[n]_\mathcal{G}$; in particular we can look at $[p]_\mathcal{G}$, where $p|q$. We have $[p]_\mathcal{G}(X) = pX + O(X^2) = O(X^2)$ over $\mathbb{F}_q$. Differentiating $\mathcal{G}$ with respect to one variable, we have $(\partial_X\mathcal{G})(X, Y) = 1 + \cdots$, and in particular $(\partial_X\mathcal{G})(0, Y) = 1 + O(Y)$. Now $\frac{1}{(\partial_X\mathcal{G})(0,Y)}\,dY$ and $\frac{f'(Y)}{(\partial_X\mathcal{G})(0,f(Y))}$ are both $\mathcal{G}$-invariant differentials for any automorphism f of $\mathcal{G}$, and since this space is a free $\mathbb{F}_q$-module of dimension 1 they differ by a constant, which can be determined by the difference at 0, $\frac{f'(0)}{(\partial_X\mathcal{G})(0,f(0))}$. In the case $f = [p]_\mathcal{G}$, we have $f(0) = f'(0) = 0$, so $\frac{f'(Y)}{(\partial_X\mathcal{G})(0,f(Y))} = 0$ and therefore $f'(Y) = 0$, i.e. if $[p]_\mathcal{G}(X) = pX + b_2X^2 + \cdots$ then we must have every b_i equal to 0 in $\mathbb{F}_q$ unless i is divisible by p, i.e. $[p]_\mathcal{G}$ factors through $X \mapsto X^p$.

It is possible that $[p]_\mathcal{G}$ factors through $X \mapsto X^{p^h}$ for some $h > 1$, i.e. $[p]'_\mathcal{G}$ vanishes modulo p^h; in this case we say that $\mathcal{G}$ has height h for the maximal such h, and the above shows that every formal group law over $\mathcal{G}$ has a height. If this height is infinite then $\mathcal{G}(X,Y)$ is just $X+Y$, i.e. $\mathcal{G}=\mathbb{G}_{\mathrm{a}}$; in general by checking coefficients and using induction one can verify that two formal group laws with the same height are isomorphic. On the other hand the argument from before shows that we can always find a formal group law with $[p]_{\mathcal{G}}=X^{p^h}$ over $\mathbb{F}_q$ for any positive integer h, so the formal group laws over any finite field (or $\overline{\mathbb{F}_p}$) are classified by positive integers (plus infinity). (One can also prove this result via Dieudonné theory.) The case above is $h=1$ for $F=\mathbb{Q}_p$, and in general $h=[F:\mathbb{Q}_p]$.

To get formal group laws over rings such as $\mathcal{O}_F$, since the height is determined by the setting there is a unique such formal group law over the residue field k, and so the problem is just one of deforming formal group laws. This motivates the deformation problem for formal group laws: let $\mathcal{G}_0$ be the unique one-dimensional formal group law of height h over $k=\overline{\mathbb{F}_p}$, and let $W=W(k)$ and $K_0=W[1/p]=\widehat{\mathbb{Q}_p^{\mathrm{unr}}}$. If $\mathcal{C}$ is the category of complete local noetherian W-algebras with residue field k, we define M_0 to be the functor $\mathcal{C}\to\mathbf{Set}$ sending A to the set of isomorphism (really isogeny, see below) classes of one-dimensional formal group laws $\mathcal{G}$ over A together with an isomorphism $\mathcal{G}_0\to\mathcal{G}\otimes_A k$.

It turns out that this functor is representable by an adic ring non-canonically isomorphic to $W[[t_1,\ldots,t_{h-1}]]$, or equivalently the space $\operatorname{Spf} W[[t_1,\ldots,t_{h-1}]]$. In particular in the case $h=1$ we get that there is a unique lift of $\mathcal{G}_0$ to characteristic 0. We get a universal formal group $\tilde{\mathcal{G}}$ over $W[[t_1,\ldots,t_{h-1}]]$ whose special fiber is $\mathcal{G}_0$.

We can pass to the rigid generic fiber $\mathcal{M}_0$ of M_0; this is the rigid open unit ball. This is the Lubin-Tate deformation space at level 0.

Drinfeld showed how to add level structure to get a tower of moduli problems: in particular to a formal group law $\mathcal{G}$ over A with an isomorphism $\mathcal{G}_0\to\mathcal{G}\otimes_A k$, we add the data of an isomorphism $\alpha:(\mathbb{Z}/p^n\mathbb{Z})^{\oplus h}\to\mathcal{G}[p^n]$ of $\mathbb{Z}/p^n\mathbb{Z}$-modules. We call the corresponding moduli problem M_n, so that for $n=0$ the additional data is trivial and we obtain M_0 as expected; Drinfeld constructed rigid spaces $\mathcal{M}_n$ such that for finite extensions K/K_0 the K-points of $\mathcal{M}_n$ classify points of M_n over K. These give étale covers $\mathcal{M}_n\to\mathcal{M}_0$ with Galois group $G=\mathrm{GL}_h(\mathbb{Z}/p^n\mathbb{Z})$, which together form the Lubin-Tate tower. Using Drinfeld level structures, this can be extended to a formal model over W. At least on points (though not as rigid spaces) one can take the limit to get a space $\mathcal{M}$, which we can even think of just as the tower, which admits right actions of $\mathrm{GL}_h(\mathbb{Q}_p)$, the endomorphisms of $\mathcal{G}_0$ or more precisely $J=(\operatorname{End}\mathcal{G}_0\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)^\times$, and the Weil group $W_{\mathbb{Q}_p}$; these actions on cohomology realize the local Langlands correspondence.

When $h=1$, each $\mathcal{M}_n$ has dimension 0 and so $\mathcal{M}$ has dimension 0: every lift of $\mathcal{G}_0=\mathbb{G}_{\mathrm{m}}$ is $\mathbb{G}_{\mathrm{m}}$, and so $\mathcal{M}$ just classifies level structures $\alpha:\mathbb{Q}_p\xrightarrow{\sim}V(\mathbb{G}_{\mathrm{m}})=\mathbb{Q}_p(1)$, i.e. $\mathcal{M}$ gives the set $\mathbb{Q}_p(1)^\times$ of nonzero elements of $\mathbb{Q}_p(1)$. In this case $G=\mathrm{GL}_1(\mathbb{Q}_p)$ and $J=\operatorname{Aut}_{\mathbb{Q}_p}(\mathbb{G}_{\mathrm{m}})$ are both given by $\mathbb{Q}_p^\times$, though with inverse actions, and the action of the Weil group is via the reciprocity map $W_{\mathbb{Q}_p}\to W_{\mathbb{Q}_p}^{\mathrm{ab}}\xrightarrow{\sim}\mathbb{Q}_p^\times$.

2. Rapoport-Zink spaces

We can rephrase in terms of p-divisible groups. For the application to class field theory, we took our formal $\mathcal{O}_F$-module $\mathcal{G}$ and looked at the groups $\mathcal{G}[\pi^n]$ and their limit $T_\pi(\mathcal{G})$. A concrete definition is as a special kind of formal group law. First, we have to reinterpret in a more geometric way: a formal group over A can be thought of as a group object in the category of formal Lie varieties over A (i.e. a formal scheme locally isomorphic to $A[[X_1, \ldots, X_n]]$ for some n), after choosing a coordinate. In this case we write $A(\mathcal{G})$ for the coordinate ring of $\mathcal{G}$ as a formal scheme, so that each endomorphism of $\mathcal{G}$ induces an endomorphism of $A(\mathcal{G})$ as an A-module. In particular we can look at the endomorphism $[p]_{\mathcal{G}}$: we say that $\mathcal{G}$ is p-divisible if $[p]_{\mathcal{G}} : A(\mathcal{G}) \to A(\mathcal{G})$ makes $A(\mathcal{G})$ a finite module over itself, in which case it is locally free over itself of rank p^h for some positive integer h. We say that h is the height of the p-divisible group. More abstractly, a p-divisible group is an inductive system of finite commutative flat p-group schemes $\mathcal{G}[p^n]$ such that multiplication by p^{n-1} has kernel $\mathcal{G}[p^{n-1}]$. It follows that we have exact sequences

$$0 \to \mathcal{G}[p^m] \to \mathcal{G}[p^{m+n}] \to \mathcal{G}[p^n] \to 0,$$

and so if the order of $\mathcal{G}[p]$ is p^h (as it must be for some integer $h \geq 0$) then the order of $\mathcal{G}[p^n]$ is p^{nh} by induction, in which case we say that $\mathcal{G}$ has height h.

In all the above, isomorphisms are isogenies, i.e. morphisms with finite kernel. We can also consider quasi-isogenies, which are isogenies up to factors of $\frac{1}{p}$.

This framework is much more amenable to discussing higher-dimensional objects than explicit formal group laws, but the concepts are essentially similar, and p-divisible groups over finite fields (or $\overline{\mathbb{F}_p}$) are classified by their dimension d and height h. One can also compute their deformation rings in a similar way: it is $W[[t_1, \ldots, t_{d(h-d)}]]$, recovering the deformation ring for $d = 1$ for one-dimensional formal groups. These can be identified with central simple algebras over $\mathbb{Q}_p$ by Dieudonné theory, which are classified in a similar way.

We can finally define Rapoport-Zink spaces. Let X_0 be a p-divisible group over $\overline{\mathbb{F}_p}$, and consider the moduli problem Def_{X_0}, i.e. the functor sending a formal scheme S over $\operatorname{Spf} W$ to the set of isomorphism classes of p-divisible groups X over S together with an isomorphism $X_0 \times_{\overline{\mathbb{F}_p}} \overline{S} \to X \times_S \overline{S}$, where $\overline{S} = S \times_{\operatorname{Spf} W} \operatorname{Spec} \overline{\mathbb{F}_p}$. Here isomorphisms are quasi-isogenies.

Rapoport and Zink proved that Def_{X_0} is representable by a formal scheme $\mathcal{M}_{X_0}$ over $\operatorname{Spf} W$, which is formally smooth and locally formally of finite type, and each irreducible component of the special fiber is proper over $\operatorname{Spec} \overline{\mathbb{F}_p}$.

We can again take the generic fiber as an adic space, written $\mathcal{M}_{X_0,K_0}$ (recall $K_0 = \operatorname{Frac} W$). This has a moduli interpretation: consider the functor sending complete Huber pairs (R, R^+) over (K_0, W) to the colimit over open bounded W-subalgebras $R_0 \subset R^+$ of $\operatorname{Def}_{X_0}(R_0)$. If we view this as a presheaf on the opposite category, its sheafification is represented by $\mathcal{M}_{X_0,K_0}$. In other words, a section of $\mathcal{M}_{X_0}$ over (R, R^+) is a covering of $\operatorname{Spa}(R, R^+)$ by rational subsets $\operatorname{Spa}(R_i, R_i^+)$ and a system of compatible (on overlaps) deformations to open bounded W-subalgebras $(R_i)_0 \subset R_i^+$ of X_0.

This Rapoport-Zink space parametrizing deformations of p-divisible groups, or equivalently p-divisible groups with fixed dimension and height, should be thought of as analogous to Shimura varieties parametrizing abelian varieties with certain invariants (such as dimension, with height corresponding to the choice of conjugacy class of cocharacters). To get an inverse system, we would need to add level structure; even so, we get only very simple Shimura varieties (roughly Hilbert modular varieties). To get more, e.g. PEL, Hodge, or abelian Shimura varieties, we need to classify more data, such as Hodge tensors.

Correspondingly, we want to generalize our moduli problem to get more general Rapoport-Zink spaces. To know how to do this, we need to say more precisely what the problem we're trying to solve is. We'll say more about this when we discuss moduli spaces of local shtukas, but broadly the goal is to construct local analogues of Shimura varieties, which we can then hope have similar applications to the local Langlands program as Shimura varieties do to the global setting. Recall from our discussion of the Langlands-Rapoport set that a (global) Shimura datum (G, X) at each prime p we obtain an element $b \in G(\overline{\mathbb{Q}}_p)$ or equivalently $G(\mathbb{Q}_p^{\mathrm{unr}})$, defined up to σ-conjugacy and a conjugacy class of cocharacters $\mu : \mathbb{G}_{\mathrm{m}} \to G_{\overline{\mathbb{Q}}_p}$. Each b defines a map $D \to \mathfrak{P}(p)^{\mathrm{unr}} \to E_{G/\mathbb{Q}_p^{\mathrm{unr}}} \to G(\mathbb{Q}_p^{\mathrm{unr}})$, and varying b gives a σ-equivariant map ν from the set $B(G)$ of σ-conjugacy classes in $G(\mathbb{Q}_p^{\mathrm{unr}})$ to the set $\mathbb{N}(G)$ of σ-invariant conjugacy classes of morphisms $D \to G_{\mathbb{Q}_p^{\mathrm{unr}}}$ (which can be identified with σ-invariant cocharacters). Any cocharacter projects to an element of the fundamental group, and from there to the σ-coinvariants; the Cartan decomposition $G(\mathbb{Q}_p^{\mathrm{unr}}) = \bigcup_\mu G(W)\mu(p)G(W)$ over dominant weights μ (or its generalization to other p-adic fields via other uniformizers) lets us send $b \in G(\mathbb{Q}_p^{\mathrm{unr}})$ to the μ labeling its stratum, and thence to $\pi_1(G)$. We call this map $K_G : B(G) \to \pi_1(G)$. For any dominant cocharacter μ of G, we define $B(G, \mu) \subset B(G)$ to be the subset of b such that $\nu(b) \le \bar{\mu}$ in the sense of weights and $K_G(b)$ is the class of μ in $\pi_1(G)_\sigma$, where $\bar{\mu}$ is given roughly by tracing down to the base field (say $\mathbb{Q}_p$ for simplicity): if G has splitting field of degree m over $\mathbb{Q}_p$, then

$$\bar{\mu} = \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(\mu).$$

In our case, the compatibility condition is that b must be in $B(G, \mu^{-1})$, i.e. $\nu(b) \le \mu^{-1}$ and $K_G(b) = -\mu$. This is because otherwise the corresponding local Shimura variety will turn out to be empty.

We also require μ to be minuscule, i.e. all its weights have multiplicity 1. This translates to our condition that the action of $\mathbb{G}_{\mathrm{m}}$ on $\mathrm{Lie}(G_{\mathbb{C}})$ by $\mathrm{ad} \circ \mu$ has only characters of weight -1, 0, or 1.

The Rapoport-Zink spaces are supposed to be incarnations of this notion of local Shimura varieties. The case we considered above, where $\mathcal{M}_{X_0}$ classifies deformations of a p-divisible group X_0 over $\overline{\mathbb{F}_p}$ which is determined by its dimension d and height h, corresponds to $G = \mathrm{GL}_h$ and b corresponding to the map $D \to G$ giving the isocrystal of X_0 determined by Dieudonné theory. In this case μ has weight 1 on dimensions spanned by the isocrystal in question, i.e. up to dimension d, and 0 otherwise, i.e. μ corresponds to the weight $(1, \ldots, 1, 0, \ldots, 0)$ with d occurrences of 1.

More generally, let's consider the case of Shimura data of PEL type. In this case we have a semisimple $\mathbb{Q}$-algebra B with involution $*$, and a finite-dimensional B-module V with an alternating pairing ψ compatible with the involution. To get local data, we simply complete at p to get (by an abuse of notation) a semisimple $\mathbb{Q}_p$-algebra B with involution $*$ and a finite-dimensional B-module V with an alternating pairing ψ compatible with the involution; let $F/\mathbb{Q}_p$ be the center of B, which we may as well assume is a field (after taking factors), and G is the B-similitudes of (V, ψ) over $\mathbb{Q}_p$ which we assume connected, with a character $c : G \to \mathbb{G}_{\mathrm{m}}$ defined by the similitude. This has a degree 2 subfield F_0, which is the subfield fixed by $*$.

In the local setting, we also need to fix an integral structure. First we fix a maximal order $\mathcal{O}_B \subset B$; then let $\mathcal{L}$ be a chain of $\mathcal{O}_B$-lattices in V, i.e. a set of $\mathcal{O}_B$-lattices in V

such that for any two $\Lambda, \Lambda' \in \mathcal{L}$ either $\Lambda \subset \Lambda'$ or $\Lambda' \subset \Lambda$, and if $x \in B^\times$ normalizes $\mathcal{O}_B$ (i.e. $x\mathcal{O}_B x^{-1} = \mathcal{O}_B$) then it preserves $\mathcal{L}$, i.e. $x\Lambda \in \mathcal{L}$ if $\Lambda \in \mathcal{L}$. We further require $\mathcal{L}$ to be self-dual, i.e. $\mathcal{L}$ is closed under duals: if $\Lambda \in \mathcal{L}$ then $\Lambda^* \in \mathcal{L}$, with respect to the pairing ψ. We define the corresponding group scheme $\mathcal{G}$ to be the group of compatible isomorphisms of all $\Lambda \in \mathcal{L}$. To make this precise, we introduce a new notation.

For any $\mathbb{Z}_p$-algebra R, we say that a chain of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-lattices of type $(\mathcal{L})$ is a functor $\Lambda \mapsto M_\Lambda$ from $\mathcal{L}$ to $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-**Mod** equipped with isomorphisms $\theta_x : M_\Lambda^x \xrightarrow{\sim} M_{x\Lambda}$ for any $x \in B^\times$ normalizing $\mathcal{O}_B$, where M_Λ^x is M_Λ with the $\mathcal{O}_B$-action conjugated by x, such that M_Λ is locally (over $\operatorname{Spec} R$) isomorphic to $\Lambda \otimes_{\mathbb{Z}_p} R$ as $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-modules; M is compatible with quotients, i.e. $M_\Lambda / M_{\Lambda'} \cong \Lambda/\Lambda' \otimes_{\mathbb{Z}_p} R$ for $\Lambda' \subset \Lambda$ adjacent (whatever that means—finite quotient?); the isomorphisms θ_x are functorial, i.e. commute with the transition maps $M_\Lambda \to M_{\Lambda'}$; and for $x \in B^\times \cap \mathcal{O}_B$ normalizing $\mathcal{O}_B$, θ_x is just multiplication by x. A polarized chain of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-lattices of type $(\mathcal{L})$ is a chain M_Λ of type $(\mathcal{L})$ together with an isomorphism $M_\Lambda \simeq M_{\Lambda^*}^* \otimes_R L$ of chains for some invertible R-module L.

Note that the identity functor makes $\mathcal{L}$ a chain of lattices of type $(\mathcal{L})$ for $R = \mathbb{Z}_p$, with polarization coming from ψ. We can then define $\mathcal{G}$ to be the automorphisms of $\mathcal{L}$ as a polarized chain of lattices of type $(\mathcal{L})$. This is a smooth group scheme over $\mathbb{Z}_p$ extending G, and for any $\mathbb{Z}_p$-algebra R we get a natural equivalence between $\mathcal{G}$-torsors over R and the groupoid of polarized chains of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-lattices of type $(\mathcal{L})$.

In the global setting, we'd want our Shimura variety to parametrize abelian varieties with a B-action, polarization, and level structure. In the local case, we replace abelian varieties by p-divisible groups and add the integral structure: we define a chain of $\mathcal{O}_B$-p-divisible groups of type $(\mathcal{L})$ over a $\mathbb{Z}_p$-algebra R on which p is nilpotent to be a functor $\Lambda \mapsto X_\Lambda$ from $\mathcal{L}$ to the category of p-divisible groups over R with $\mathcal{O}_B$-action (corresponding to formal $\mathcal{O}_B$-module laws), together with an isomorphism $\theta_x : X_\Lambda^x \xrightarrow{\sim} X_{x\Lambda}$ for all $x \in B^\times$ normalizing $\mathcal{O}_B$, with X_Λ^x similarly to above X_Λ with the x-conjugate action of $\mathcal{O}_B$, such that the θ_x are functorial, for $x \in B^\times \cap \mathcal{O}_B$ normalizing $\mathcal{O}_B$ the θ_x are just multiplication by x, and if EX_Λ is the universal vector extension of X_Λ (i.e. the universal extension by $\mathbb{G}_\mathrm{a}$), then $\Lambda \mapsto \mathrm{Lie}(EX_\Lambda)$ defines a chain of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-modules of type $(\mathcal{L})$. A polarized chain of $\mathcal{O}_B$-p-divisible groups of type $(\mathcal{L})$ is a chain as above together with a $\mathbb{Z}_p$-local system $\mathbb{L}$ on $\operatorname{Spec} R$ and an isomorphism

$$X_\Lambda \cong X_{\Lambda^*}^* \otimes_{\mathbb{Z}_p} \mathbb{L}.$$

These have a natural notion of isogeny between them, and similarly quasi-isogeny; we consider quasi-isogenies to be isomorphisms.

Finally we require an admissibility condition: a chain X_Λ of $\mathcal{O}_B$-p-divisible groups of type $(\mathcal{L})$ over R is admissible if it satisfies a determinant condition: fix a conjugacy class of minuscule cocharacters $\mu : \mathbb{G}_\mathrm{m} \to G_{\overline{\mathbb{Q}}_p}$. We have the requirements that $c \circ \mu : \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$ is the identity and the weights of this character on $V \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ are only 0 and 1, so we get a decomposition $V = V_0 \oplus V_1$; thus any $x \in \mathcal{O}_B$ acts on both $\operatorname{Lie} X_\Lambda$ and V_1, and the determinant condition is that

$$\det_R(x | \operatorname{Lie} X_\Lambda) = \det_{\overline{\mathbb{Q}}_p}(x | V_1)$$

for all $x \in \mathcal{O}_B$. Note that although the right-hand side a priori only has values in $\overline{\mathbb{Q}}_p$, by the weight condition the action is actually defined over the reflex field E, and by integrality the values are in $\mathcal{O}_E$ which maps to R.

In the case we looked at above, b determined the p-divisible group X_0 and μ was chosen to be compatible with b (or equivalently X_0). In the more general PEL setting the idea is the same: $b \in B(G)$ corresponds to a map $D \to G$ giving an isocrystal with B-structure. Fixing a chain $\mathcal{L}$ and a maximal order $\mathcal{O}_B$, for each $\Lambda \in \mathcal{L}$ we get an integral structure and thus a p-divisible group over $\overline{\mathbb{F}_p}$ with an $\mathcal{O}_B$-action, which extends to a functor, i.e. a chain of $\mathcal{O}_B$-p-divisible groups of type $(\mathcal{L})$ up to quasi-isogeny, with polarization coming from the dualization with respect to ψ. We write this chain as $X_{b,\Lambda}$. Fix μ such that $b \in B(G, \mu^{-1})$.

We can finally define the moduli problem in the PEL setting. Fix a datum $\mathcal{D} = (B, V, \mathcal{O}_B, \mathcal{L}, \psi, *, b, \mu)$ as defined above. Let $\breve{\mathcal{O}}$ be the ring of integers of $W \otimes_{\mathbb{Z}_p} F$; everything will be over this ring. The moduli problem $\mathcal{M}_{\mathcal{D}}$ over $\operatorname{Spf} \breve{\mathcal{O}}$ sends a p-torsion $\breve{\mathcal{O}}$-algebra R to the set of polarized chains of $\mathcal{O}_B$-p-divisible groups X_Λ of type $(\mathcal{L})$ over R which are admissible with respect to μ with quasi-isogenies

$$X_{b,\Lambda} \times_{\operatorname{Spec} \overline{\mathbb{F}_p}} \operatorname{Spec} R/p \to X_\Lambda \times_{\operatorname{Spec} R} \operatorname{Spec} R/p,$$

i.e. deformations of the chain $X_{b,\Lambda}$.

For each $\Lambda \in \mathcal{L}$, we get a map $\mathcal{M}_{\mathcal{D}} \to \mathcal{M}_{X_{b,\Lambda}}$ in the sense above, deformations of the single $\mathcal{O}_B$-p-divisible group $X_{b,\Lambda}$ by forgetting the rest of the structure. Taking the product gives a map

$$\mathcal{M}_{\mathcal{D}} \to \prod_{\Lambda \in \mathcal{L}} \mathcal{M}_{X_{b,\Lambda}}.$$

It turns out (by results of Rapoport and Zink) that this is a closed immersion, and the same is true taking a sufficiently large finite subset of $\mathcal{L}$. In particular, each factor on the right-hand side is representable by a formal scheme locally formally of finite type over $\operatorname{Spf} \breve{\mathcal{O}}$, and so so is $\mathcal{M}_{\mathcal{D}}$. One can again take the generic fiber and try to understand its moduli structure.

For example, consider the simplest situation $B = \mathbb{Q}_p$ with trivial involution and $V = \mathbb{Q}_p^2$ with the standard alternating pairing ψ. In this case there is a unique maximal order $\mathcal{O}_B$, namely the integers $\mathbb{Z}_p$ whose action is automatic, and $G = \operatorname{GSp}_2(V, \psi) \simeq \operatorname{GL}_2(\mathbb{Q}_p)$. The cocharacters we consider must have weights 0 and 1, so the only possibilities are for weights $(0,0)$, $(1,0)$, or $(1,1)$, i.e.

$$\mu_0(t) = 1, \qquad \mu_1(t) = \begin{pmatrix} t & \\ & 1 \end{pmatrix}, \qquad \mu_2(t) = t$$

in some basis. The weight decomposition for μ_i gives $V_1 \subset V$ of dimension i, with multiplication by $x \in \mathcal{O}_B = \mathbb{Z}_p$ just the scalar action, so $\det(x|V_1) = x^i$, so the action on $\operatorname{Lie} X_\Lambda$ must be the same for the determinant condition, i.e. $\dim X_\Lambda = i$. In the dimension 0 case, there is a unique lift to characteristic 0 and so X_Λ is a constant functor and in particular does not depend on $\mathcal{L}$, and so this recovers the trivial case of the above case for GL_2 parametrizing deformations of the zero-dimensional p-divisible group over $\overline{\mathbb{F}_p}$ of height 2.

The remaining cases are more nontrivial. However, we can note that the unique p-divisible group X_0 over $\overline{\mathbb{F}_p}$ of height 2 and dimension i has a Dieudonné module defining a $\mathbb{Z}_p$-lattice in V, which generates a canonical $\mathcal{L}$, with polarization corresponding to ψ; thus we recover the case of the deformations of X_0 as a special case.

Note that the extra data of $\mathcal{O}_B$ and $\mathcal{L}$ we use to define the integral structure gives rise to the level structure: $\mathcal{G}(\mathbb{Z}_p)$ gives a compact open subgroup, and varying the $\mathcal{O}_B$ and $\mathcal{L}$ gives different extensions $\mathcal{G}$. We could take the limit over different choices to get a tower of Rapoport-Zink spaces.

It should be possible to do similar constructions at least for Shimura varieties of Hodge even abelian type, but this is complicated enough for now.

3. p-adic uniformization

Remarkably, it is possible to use these local analogues of Shimura varieties to say something in the global setting. In particular, by slightly changing the moduli problem we can get models of Shimura varieties over W, and it is possible to describe the points of these models in a way analogous to how we describe the $\mathbb{C}$-points of Shimura varieties by a double quotient.

First, let's define our moduli problem. We won't go more general than the PEL setting, though again it's possible to generalize to Hodge and even abelian type. Let (G, X) be a Shimura datum of PEL type, coming from data $(B, *, V, \psi)$, and fix a compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$. Let E be the reflex field and ν a place over p, with completion E_ν, and fix integral data $\mathcal{O}_B, \mathcal{L}$ as above (but now globally, over $\mathbb{Z}$!). We define $\mathcal{S}_{K^p}$ to be the moduli problem sending an $\mathcal{O}_{E_\nu}$-scheme S to the set of isomorphism classes of tuples $(A, \bar{\lambda}, \eta K^p)$, where A is a chain of $\mathcal{O}_B$-abelian varieties of type $(\mathcal{L})$ (i.e. a functor from $\mathcal{L}$ to abelian varieties with $\mathcal{O}_B$-action with isomorphisms θ_x satisfying conditions as for chains of $\mathcal{O}_B$-p-divisible groups), $\bar{\lambda}$ is a $\mathbb{Q}$-homogeneous principle polarization of A (i.e. a compatible system of principle polarizations), and ηK^p is a K^p-orbit of isomorphisms $\eta : H_1(A, \mathbb{A}_f^p) \xrightarrow{\sim} V(\mathbb{A}_f^p)$ compatible with λ up to a constant in $(\mathbb{A}_f^p)^\times$, such that $\det_{\mathcal{O}_S}(x|\operatorname{Lie} A_\Lambda) = \det_{\overline{\mathbb{Q}}_p}(x|V_1)$. This is representable by a quasiprojective scheme over $\mathcal{O}_{E_\nu}$, which form a projective system with a right action of $G(\mathbb{A}_f^p)$ and gives a model, up to the level structure at p, for the corresponding PEL Shimura variety $\operatorname{Sh}(G, X)$ over $\mathcal{O}_{E_\nu}$.

Suppose we want to understand $\mathcal{S}_{K^p}(\overline{\mathbb{F}_p})$, as we have been interested in in the past. In particular, as before, we fix some point $x_0 = (A_0, \bar{\lambda}_0, \eta_0 K^p) \in \mathcal{S}_{K^p}(\overline{\mathbb{F}_p})$ and want to understand the set of points in $\mathcal{S}_{K^p}(\overline{\mathbb{F}_p})$ isogenous to this initial point. To A_0 we can associate an isocrystal, on which $\bar{\lambda}$ induces a polarization; this corresponds to some $b \in G(\mathbb{Q}_p^{\mathrm{unr}})$ compatible with the conjugacy class of cocharacters μ corresponding to X (already part of the Shimura datum).

On the other hand, associated to the data $(B, *, V, \psi, \mathcal{O}_B, \mathcal{L})$ and each choice of b, μ (fixed by the choice of x_0), we get a Rapoport-Zink space $\breve{\mathcal{M}}$ over $\operatorname{Spf} \breve{\mathcal{O}}_{E_\nu}$, which we can descend to a formal proscheme $\mathcal{M}$ over $\operatorname{Spf} \mathcal{O}_{E_\nu}$. Our goal is to construct a morphism between these, $\mathcal{M} \to \mathcal{S}_{K^p}$, whose image on $\overline{\mathbb{F}_p}$-points gives the points of $\mathcal{S}_{K^p}(\overline{\mathbb{F}_p})$ isogenous to x_0.

To have any hope of being natural such a map should respect the $G(\mathbb{A}_f^p)$ action, but none exists on the left; we fix this by adding a factor of $G(\mathbb{A}_f^p)$ with the action by right multiplication, so we now hope to build a suitable map $\mathcal{M} \times G(\mathbb{A}_f^p) \to \mathcal{S}_{K^p}$. This should of course be K^p-invariant, so we hope it descends to $\mathcal{M} \times G(\mathbb{A}_f^p)/K^p \to \mathcal{S}_{K^p}$.

The functor $\mathcal{M}$ arises via deformations of a polarized p-divisible group X. Since $\mathcal{S}_{K^p}$ is built around the same structures as $\mathcal{M}$, they have the same invariants and in particular we can find an element A of the isogeny class of A_0 whose p-divisible group is X. The quasi-isogeny $A \to A_0$ induces a polarization and level structure. By Serre-Tate, deformations of X correspond to deformations of A, and this map $\tilde{X} \mapsto \tilde{A}$ with the corresponding additional structure gives a map $\mathcal{M} \to \mathcal{S}_{K^p}$; to make it $G(\mathbb{A}_f)$-equivariant, we send $(\tilde{X}, g)$ to the twist $(\tilde{A}, \tilde{\lambda}, \tilde{\eta} g K^p)$ over $\operatorname{Spec} \breve{\mathcal{O}}_{E_\nu}$, which is manifestly K^p-invariant.

Let $I(\mathbb{Q})$ be the automorphism group of A, i.e. the group of quasi-isogenies respecting the polarization. This has a homomorphism to $G(\mathbb{A}_f^p)$ by $\xi \mapsto \eta \circ V^p(\xi) \circ \eta^{-1}$, induced from the action on the Tate module of A composed with the level structure. It extends to an algebraic group I over $\mathbb{Q}$ which acts on the left but not on the right, and makes the map descend to

$$I(\mathbb{Q}) \backslash \breve{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \to \mathcal{S}_{K^p} \times_{\operatorname{Spec} \mathcal{O}_{E_\nu}} \operatorname{Spec} \breve{\mathcal{O}}_{E^\nu}.$$

This is the uniformization map; one can check that it is compatible with the descent data to $\operatorname{Spec} \mathcal{O}_{E_\nu}$ on both sides. After viewing $\mathcal{S}_{K^p}$ as a formal scheme over $\mathcal{O}_{E_\nu}$, in a somewhat nontrivial way (completing along the images of irreducible components of $I(\mathbb{Q})$-orbits), we get a $G(\mathbb{A}_f^p)$-equivariant isomorphism of formal schemes over $\mathcal{O}_{E_\nu}$

$$I(\mathbb{Q}) \backslash \mathcal{M} \times G(\mathbb{A}_f^p)/K^p \to \mathcal{S}_{K^p}.$$

By construction, the image on $\overline{\mathbb{F}_p}$-points consists of elements in the isogeny class of x_0; the fact that it gives all of them, like the proof that this is an isomorphism, we omit.

One can also take rigid generic fibers to get an isomorphism of rigid spaces, classifying a similar problem over $\mathbb{A}_f$ instead of $\mathbb{A}_f^p$ where we add level structure at p corresponding to the choice of integral data $\mathcal{O}_B$ and $\mathcal{L}$.

One example is counting supersingular elliptic curves. Let B be the quaternion algebra over $\mathbb{Q}$ split away from p and ∞, so that $B \cong \operatorname{End}^0(E)$ for E a supersingular elliptic curve over $\overline{\mathbb{F}_p}$, acting on $H_1(E, \mathbb{Q})$. This gives rise to a Shimura variety for $G = B^\times$ classifying polarized elliptic curves with an action of B and some level structure; we take $K^p = \mathcal{O}_B(\mathbb{A}_f^p) = \operatorname{End}(E_0) \otimes_{\mathbb{Z}} \mathbb{A}_f^p$. Fixing E_0 supersingular over $\overline{\mathbb{F}_p}$, its p-divisible group has dimension 1 and height 2 and so the Rapoport-Zink space $\mathcal{M}$ parametrizes deformations; in particular the $\overline{\mathbb{F}_p}$-points are just the single point E_0. Finally $I(\mathbb{Q})$ is the set of self-isogenies of E_0 respecting the polarization corresponding to the involution and so is just $\operatorname{End}^0(E_0) = B^\times(\mathbb{Q})$. Therefore the isogeny class of E_0, and thus the set of all supersingular curves over $\overline{\mathbb{F}_p}$, is in bijection with

$$B^\times(\mathbb{Q}) \backslash B^\times(\mathbb{A}_f)/\mathcal{O}_B(\mathbb{A}_f)$$

(the factor at p has trivial contribution).

References

[1] Peter Scholze and Jared Weinstein. Berkeley lectures on p-adic geometry. In *Berkeley Lectures on p-adic Geometry*. Princeton University Press, 2020.

[2] Jared Weinstein. The geometry of Lubin-Tate spaces.