Local theory: background*

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Our goal in this section is to sketch the background in p-adic geometry we'll need to understand the theory of local Shimura varieties and moduli spaces of local shtukas. In particular, we'll study the general framework, that of adic spaces; specialize to the case of perfectoid spaces and study the tilting equivalence between characteristics 0 and p; and generalize to v-sheaves, a particularly strong notion of sheaves on the category of characteristic p perfectoid spaces, and in particular diamonds, a well-behaved type of v-sheaf analogous to algebraic spaces in the schematic setting.

1. Adic spaces

First, we recall two categories of geometric objects which we hope to generalize: formal schemes and rigid-analytic spaces.

When defining schemes, we look only at rings, with only algebraic structure. If we have a topology on our rings, we can sometimes say more. In particular, if A adic, i.e. a ring with the I-adic topology for some ideal I (called the ideal of definition), i.e. with basis of open neighborhoods of $x \in A$ given by $x + I^n$, then we can define the formal scheme $\operatorname{Spf} A$ to be the set of *open* prime ideals of A. As a set, this is in bijection with $\operatorname{Spec} A/I$ for any ideal of definition I (which is not part of the given data, only the topology is), but it is given a topology and structure sheaf as is done for $\operatorname{Spec} A$ and so is not in general isomorphic to $\operatorname{Spec} A/I$. Explicitly, for any $x \in A$ we write $D(x)$ for the nonvanishing locus of x, i.e. open prime ideals not containing x, and define the topology to have the $D(x)$ as a basis of open sets; we can then define a sheaf $\mathcal{O}_{\operatorname{Spf} A}$ by $\mathcal{O}_{\operatorname{Spf} A}(D(x)) = A[x^{-1}]^{\wedge}_I$, where the decoration with $^{\wedge}_I$ means I-completing. A formal scheme is a topologically ringed space which is locally of the form $\operatorname{Spf} A$ for some adic ring A. The functor sending an arbitrary ring A to A considered as an adic ring with the discrete topology and trivial ideal of definition gives a fully faithful functor $\operatorname{Spec} A \mapsto \operatorname{Spf} A$ from schemes to formal schemes.

For example, consider $X = \operatorname{Spf} \mathbb{Z}_p[[T]]$; this is a topological ring, and can be taken to be adic with ideal of definition (p, T). Its functor of points sends any $\mathbb{Z}_p$-algebra R to $\operatorname{Hom}(\mathbb{Z}_p[[T]], R)$, taken in the category of topological rings; such a homomorphism is determined by the image of T, which can be anything subject to the restriction that it must be topologically nilpotent, since $T^n \to 0$ for the topology on $\mathbb{Z}_p[[T]]$. Thus $X(R) = R^{\circ\circ}$, the ideal of topologically nilpotent elements. This for example gives a notion of an open disk: if $K/\mathbb{Q}_p$ is a p-adic field and $K^\circ \subset K$ is its ring of integers, then $X(K^\circ)$ is the set of topologically nilpotent elements of K° or equivalently of K°, i.e. points such that $|x| < 1$, i.e. the open unit disk over K.

Another way to generate such a disk is as a rigid-analytic space. Let K be a nonarchimedean field, i.e. complete with respect to a nontrivial nonarchimedean absolute value $|\cdot|$. For each $n \geq 0$ we have the Tate K-algebra $K\langle T_1, \ldots, T_n\rangle$, defined to be the completion of $K[T_1, \ldots, T_n]$ under the Gauss norm: if $f \in K[T_1, \ldots, T_n]$ is given by $\sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} T_1^{i_1} \cdots T_n^{i_n}$,

*These notes are based on chapters 2-10 of [2].

then the Gauss norm of f is $\max |a_{i_1,\ldots,a_n}|$. Thus the completion of $K[T_1,\ldots,T_n]$ can be described as the ring of power series in $T_1,\ldots,T_n$ whose coefficients tend to 0. An affinoid K-algebra is a topological K-algebra which is isomorphic to a quotient of $K\langle T_1,\ldots,T_n\rangle$ for some n.

Given a maximal ideal x of an affinoid K-algebra A, the residue field A/x is a finite extension of K and so carries a unique extension of its absolute value. For $f \in A$, write $f(x)$ for its image in this residue field, and $|f(x)|$ for the absolute value of $f(x)$ via this extension. If $(f_1,\ldots,f_n,g) = 1$, then we define the rational domain

$$U\left(\frac{f_1,\ldots,f_n}{g}\right) = \left\{x \in \operatorname{MaxSpec} A \middle| |f_i(x)| \leq g(x),\ i = 1,\ldots,n\right\}.$$

Taking the rational domains to be open defines a G-topology (i.e. a Grothendieck topology on a category with morphisms inclusions, finite products, and the empty set and whole space as objects) on $\operatorname{MaxSpec} A$, and one can define a sheaf of K-algebras on $\operatorname{MaxSpec} A$ by defining it on the rational domains. This is called a K-affinoid space; a rigid-analytic space over K is a G-topologized space with a sheaf of K-algebras which is locally isomorphic to a K-affinoid space.

For example, consider the closed rigid unit disk $X = \operatorname{MaxSpec} \mathbb{Q}_p\langle T\rangle$. Its functor of points on p-adic fields $K/\mathbb{Q}_p$ is given by $X(K) = \operatorname{Hom}(\mathbb{Q}_p\langle T\rangle, K)$ and so is determined by the image of T, which is arbitrary up to the restriction that $|T| = 1$ and so its image has absolute value at most 1; thus this is the closed unit disk K°, i.e. the ring of integers of K. The union of the rational domains $U(T^n/p)$ gives an open subset U defined by $T < 1$, which gives the open rigid unit disk, and $U(K)$ is the open unit disk in K.

There is a generic fiber functor from certain formal schemes over $\mathbb{Z}_p$ (those which are locally formal of finite type, i.e. a quotient of some $\mathbb{Z}_p[[X_1,\ldots,X_n]]\langle Y_1,\ldots,Y_m\rangle$) to rigid-analytic spaces over $\mathbb{Q}_p$. The image of the formal open disk $\operatorname{Spf}\mathbb{Z}_p[[T]]$ under this functor is the rigid open disk over $\mathbb{Q}_p$. (Note that this is not literally a generic fiber, since $\operatorname{Spf}\mathbb{Z}_p$ has only one point since it is in bijection with $\mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$).

We want to build a category subsuming these two, where the generic fiber can be formed as a fiber product (among other desirable properties). This is the category of adic spaces: it is built out of affinoid adic disks, associated to pairs (A, A^+) of certain rings: in particular A should be Huber and $A^+ \subset A^\circ$ should be an open and integrally closed subring of the ring A° of powerbounded elements of A.

We now define this condition: a topological ring A is Huber if it admits an open subring $A_0 \subset A$ which is adic with respect to a finitely generated ideal of definition, called a ring of definition for A.

For example, any discrete ring can be taken to be Huber, with $A_0 = A$; any adic ring with finitely generated ideal of definition is Huber with $A_0 = A$ again, and Tate algebras and therefore all K-affinoid algebras are Huber.

We say that a Huber ring is Tate if it contains a pseudo-uniformizer, i.e. a topologically nilpotent unit. A more general condition due to Kedlaya is being analytic: a Huber ring is analytic if the ideal generated by the topologically nilpotent elements is the unit ideal. Thus any Tate ring is analytic, but the converse is not necessarily true.

We say that a valuation $|\cdot| : A \to \Gamma \cup \{0\}$ on a topological ring A into a totally ordered abelian group Γ (satisfying $|ab| = |a|\cdot|b|$, $|a+b| \leq \max(|a|,|b|)$, $|1| = 1$, and $|0| = 0$) is continuous if for each γ in the image of $|\cdot|$ the set of $a \in A$ such that $|a| < \gamma$ is open in A. Two continuous valuations are equivalent if they induce the same order, i.e. $|a| \geq |b|$ if and only if $|a|' \geq |b|'$. Note that the kernel of any such valuation is a prime ideal of A.

Given a Huber pair (A, A^+), i.e. a pair where A is Huber and $A^+ \subset A^\circ$ is open and integrally closed, we define the adic spectrum $\mathrm{Spa}(A, A^+)$ to be the set of equivalence classes of continuous valuations $|\cdot|$ on A such that $|A^+| \leq 1$ (or more precisely $|a| \leq 1$ for each $a \in A^+$). For $x \in \mathrm{Spa}(A, A^+)$, we write the corresponding valuation as $g \mapsto |g(x)| = |g|_x$.

This carries a topology, generated by open subsets of the form

$$\left\{x \in \mathrm{Spa}(A, A^+) \middle| |f(x)| \leq |g(x)| \neq 0\right\}$$

for $f, g \in A$. In particular, both the set of x such that $|f(x)| \neq 0$ and the set of x such that $|f(x)| \leq 1$ are open, combining good properties of classical (formal) algebraic geometry and rigid geometry.

Huber showed that as a topological space $\mathrm{Spa}(A, A^+)$ is spectral, so it is at least as good as $\mathrm{Spec}\, A$ for arbitrary rings A.

We can restrict to complete Huber pairs: if $(\widehat{A}, \widehat{A}^+)$ is the completion of (A, A^+), then there is a natural homeomorphism $\mathrm{Spa}(\widehat{A}, \widehat{A}^+) \simeq \mathrm{Spa}(A, A^+)$. Given a complete Huber pair, we can give a description: $\mathrm{Spa}(A, A^+)$ is nonempty whenever $A \neq 0$, the ring A^+ is the subset of A defined by the condition $|f(x)| \leq 1$ for every x, and $f \in A$ is invertible if and only if $|f(x)| \neq 0$ for every x.

Like in rigid geometry, one can construct rational subsets: for $s \in A$ and $T \subset A$ finite with TA open in A, we define

$$U\left(\frac{T}{s}\right) = \left\{x \in \mathrm{Spa}(A, A^+) \middle| |t(x)| \leq |s(x)| \neq 0,\ \forall t \in T\right\}.$$

These are finite intersections of open subsets given by the rational subsets for singletons $T = \{t\}$, and so are open. In fact one can check that the intersection of rational subsets is again rational.

In fact, each rational subset U itself is an adic spectrum: by a theorem of Huber, we can find a complete Huber pair (A_U, A_U^+) over (A, A^+) such that the map $\mathrm{Spa}(A_U, A_U^+) \to \mathrm{Spa}(A, A^+)$ factors through U, is universal for such maps, and is a homeomorphism onto U. We can use this to define a structure presheaf of Huber pairs on $X = \mathrm{Spa}(A, A^+)$: for U a rational subset, we define $\mathcal{O}_X(U) = A_U$ and $\mathcal{O}_X^+(U) = A_U^+$. For an arbitrary open subset $W \subset X$, we define

$$\mathcal{O}_X(W) = \varprojlim_{U \subset W} \mathcal{O}_X(U)$$

where the limit is over rational subsets of X contained in W, and similarly for $\mathcal{O}_X^+$.

For any open $U \subset X$, we can describe $\mathcal{O}_X^+(U)$ as the set of $f \in \mathcal{O}_X(U)$ such that $|f(x)| \leq 1$ for every $x \in U$. Thus $\mathcal{O}_X^+$ is a sheaf if $\mathcal{O}_X$ is.

It is not always true that $\mathcal{O}_X$ is a sheaf. However, it is always true in the examples we've seen, i.e. when (A, A^+) is a complete Huber pair coming from a scheme, a formal scheme (when finitely generated over a noetherian ring of definition), or a rigid space (when Tate and strongly noetherian, i.e. $A\langle T_1, \dots, T_n\rangle$ is noetherian for every n).

For example, we might be concerned that $(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$ does not work, since $\mathcal{O}_{\mathbb{C}_p}$ is not noetherian; but $\mathbb{C}_p \langle T_1, \dots, T_n \rangle$ is noetherian for every n, so it does work.

We say that a Huber pair (A, A^+) is sheafy if $\mathcal{O}_X$ (and therefore also $\mathcal{O}_X^+$) is a sheaf. This will generally hold in practice.

To define adic spaces, we need to define a "topologically ringed space with valuations." Consider the category (V) whose objects are triples $(X, \mathcal{O}_X, (|\cdot|_x)_{x \in X})$ for X a topological space, $\mathcal{O}_X$ a sheaf of topological rings, and $|\cdot|_x$ an equivalence class of continuous valuations on $\mathcal{O}_{X,x}$ for each $x \in X$. (By the description above, this data determines $\mathcal{O}_X^+$.) The morphisms are maps of topologically ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, so each map $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ is continuous for opens $U \subset V$, such that the diagram

$$\begin{array}{ccc} \mathcal{O}_{Y,f(x)} & \xrightarrow{f_x^*} & \mathcal{O}_{X,x} \\ \Big\downarrow {\scriptstyle |\cdot|_{f(x)}} & & \Big\downarrow {\scriptstyle |\cdot|_x} \\ \Gamma_{f(x)} \cup \{0\} & \longrightarrow & \Gamma_x \cup \{0\} \end{array}$$

commutes up to equivalence for every $x \in X$. Note that each sheafy Huber pair gives an object $\mathrm{Spa}(A, A^+)$ of (V) via the valuation interpretation (in fact this is a fully faithful functor). An adic space is an object of (V) which is locally isomorphic to $\mathrm{Spa}(A, A^+)$ for some sheafy Huber pair (A, A^+), i.e. admits a cover by objects U_i of V isomorphic to $\mathrm{Spa}(A_i, A_i^+)$ for sheafy Huber pairs (A_i, A_+^i).

We say that $\mathrm{Spa}(A, A^+)$, as an object of (V), is an affinoid adic space, which we continue to write in the same way.

Given an adic space $(X, \mathcal{O}_X, (|\cdot|_x)_{x \in X})$, often we will just write it as X; in this case $|X|$ denotes the underlying topological space.

One can also consider "pre-adic" spaces, with the same definition but where $\mathcal{O}_X$ may fail to be a sheaf and so is replaced by the sheafification of $\mathcal{O}_X$ in the category of ind-topological rings.

Note that in general, fiber products fail to exist; with finiteness restrictions (A finitely generated over a ring of definition), they exist in pre-adic spaces.

Let $D = \mathrm{Spa}\, \mathbb{Z}[[T]] = \mathrm{Spa}(\mathbb{Z}[[T]], \mathbb{Z}[[T]])$. Then

$$\begin{aligned} D_K &= D \times \mathrm{Spec}_K \\ &= (D \times \mathrm{Spa}\, \mathcal{O}_K) \times_{\mathrm{Spa}\, \mathcal{O}_K} \mathrm{Spa}\, K \\ &= \mathrm{Spa}\, \mathcal{O}_K[[T]] \times_{\mathrm{Spa}\, \mathcal{O}_K} \mathrm{Spa}\, K \\ &= \bigcup_{n \geq 1} \mathrm{Spa}\, K \left\langle T, \frac{T^n}{\varpi} \right\rangle \end{aligned}$$

where we work over pre-adic spaces (since we have finiteness conditions) so as not to have to worry about whether the products exist as adic spaces. This gives the open unit disk over K. If we repeat the construction with $D^* = \mathrm{Spa}\, \mathbb{Z}((T))$, we get $D_K^* = D_K \setminus \{T = 0\}$, the punctured open unit disk.

We say that a Huber ring A is uniform if A° is bounded; for complete analytic Huber pairs (A, A^+), we can define a stronger condition: we say that (A, A^+) is stably uniform if $\mathcal{O}_X(U)$

is uniform for all rational $U \subset \mathrm{Spa}(A, A^+)$. It turns out that a complete analytic Huber pair is sheafy if it is stably uniform. One can also compute that for a complete analytic Huber pair (A, A^+), if it is sheafy then $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.

We say that an adic space X is uniform if for each open affinoid $\mathrm{Spa}(A, A^+) \subset X$, the Huber ring A is uniform.

If X is a uniform analytic adic space, then we define a Cartier divisor on X to be an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ that is locally free of rank 1, with support the support of $\mathcal{O}_X/\mathcal{I}$. If (A, A^+) is a stably uniform analytic Huber pair, $X = \mathrm{Spa}(A, A^+)$, and $\mathcal{I} \subset \mathcal{O}_X$ is a Cartier divisor, then the support is a nowhere dense closed subset of X, and an invertible ideal I of A whose vanishing locus in X is nowhere dense defines a Cartier divisor $I\mathcal{O}_X$; this map is a bijection.

Let $\mathcal{I}$ be a Cartier divisor on a uniform analytic adic space X with support Z, and let $j : U = X \setminus Z \hookrightarrow X$ be the complement. There are injections

$$\mathcal{O}_X \hookrightarrow \varinjlim_n \mathcal{I}^{\otimes n} \hookrightarrow j_* \mathcal{O}_U.$$

(This presupposes some notion of pushforward, defined as usual for topological spaces I guess, i.e. $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$.)

The support of a Cartier divisor may or may not be an adic space; if it is, with structure sheaf $\mathcal{O}_X/\mathcal{I}$, we say that the Cartier divisor is closed. This is equivalent to $\mathcal{I}(U) \hookrightarrow \mathcal{O}_X(U)$ having closed image for every open affinoid $U \subset X$.

2. Perfectoid spaces

Let R be a Huber ring. We want a pseudo-uniformizer ϖ, so we assume that it is Tate, and complete for good measure; we would further like to have ϖ such that quotienting by it sends us to characteristic p, which we formalize as $\varpi^p | p$ in R°. We then get a pth power map, which we write as the Frobenius

$$\Phi : R^\circ/\varpi \to R^\circ/\varpi.$$

We say that a complete Tate ring R with pseudo-uniformizer ϖ satisfying $\varpi^p | p$ in R° is perfectoid if Φ is an isomorphism.

There is a more general way of defining perfectoid rings for analytic, rather than Tate, rings; the category of perfectoid spaces we end up with is equivalent. Dropping further is not possible, however: we need R to be at least analytic.

In general, for any ring R and ideals $I, J \subset R$ containing p with $I^p \subset J$, we write $\Phi : R/I \to R/J$ for the pth power map.

Note that this condition on Φ being an isomorphism is independent of ϖ: injectivity is automatic from the other assumptions, and surjectivity reduces to the case of R°/p.

For example, $\mathbb{Q}_p$ is not perfectoid, even though the Frobenius on its residue field is an isomorphism, because there is no pseudo-uniformizer ϖ such that $\varpi^p | p$. More generally any discretely valued nonarchimedean field cannot be perfectoid: K°/ϖ and K°/ϖ^p are Artin local rings of different lengths.

Examples which are perfectoid Tate rings include $\mathbb{Q}_p^{\mathrm{cycl}}$, the completion of $\mathbb{Q}_p(\mu_{p^\infty})$; the completion of $\mathbb{F}_p((t))(t^{1/p^\infty})$, which we write as $\mathbb{F}_p((t^{1/p^\infty}))$; $\mathbb{Q}_p^{\mathrm{cycl}} \left\langle T^{1/p^\infty} \right\rangle = \mathbb{Z}_p^{\mathrm{cycl}}[T^{1/p^\infty}]_p^\wedge[1/p]$; an example not over a field, $\mathbb{Z}_p^{\mathrm{cycl}}[[T^{1/p^\infty}]] \left\langle (p/T)^{1/p^\infty} \right\rangle [1/T]$.

If R is already in characteristic p, then if it is complete Tate being perfectoid is the same thing as being perfect.

We define a perfectoid field to be a perfectoid Tate ring which is a field. A nonarchimedean field K is perfectoid if and only if K is not discretely valued, $|p| < 1$, and $\Phi : \mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective.

Suppose that (R, R^+) is a Huber pair with R perfectoid. Then $\mathcal{O}_X(U)$ is also perfectoid for every rational $U \subset X = \operatorname{Spa}(R, R^+)$, so in particular (R, R^+) is stably uniform and therefore sheafy. The proof of this uses tilting.

The main reason to be interested in perfectoid rings is the phenomenon of tilting. For a perfectoid Tate ring, we define the tilt of R to be

$$R^\flat = \varprojlim_{x \mapsto x^p} R,$$

with the inverse limit topology. Since $x \mapsto x^p$ is in general only a multiplicative homomorphism, $R^\flat$ is a priori only a topological multiplicative monoid. We can equip it with a ring structure via

$$(x_0, x_1, \dots) + (y_0, y_1, \dots) = (z_0, z_1, \dots)$$

where

$$z_i = \lim_{n \to \infty} (x_{i+n} + y_{i+n})^{p^n} \in R,$$

which exist essentially by completeness. This defines a ring structure making $R^\flat$ a topological $\mathbb{F}_p$-algebra which is a perfect complete Tate ring. The power-bounded elements $R^{\flat\circ}$ are given by the limit

$$R^{\flat\circ} = \varprojlim_{x \mapsto x^p} R^\circ \simeq \varprojlim_{\Phi} R^\circ/p \simeq \varprojlim_{\Phi} R^\circ/\varpi$$

for a pseudo-uniformizer $\varpi \in R$ dividing p in R°, and we can choose ϖ such that $\varpi^p | p$ in R° so that there is a corresponding element $\varpi^\flat = (\varpi, \varpi^{1/p}, \varpi^{1/p^2}, \dots) \in R^{\flat\circ}$, which is a pseudo-uniformizer of $R^\flat$ such that $R^\flat = R^{\flat\circ}[1/\varpi^\flat]$.

For example, consider the case $\mathbb{Q}_p^{\mathrm{cycl}}$, and let $\zeta_p, \zeta_{p^2}, \dots$ be a compatible system of pth power roots of unity. Then $t = (1, \zeta_p, \zeta_{p^2}, \dots) - 1$ is a pseudo-uniformizer of $(\mathbb{Q}_p^{\mathrm{cycl}})^\flat$, and $(\mathbb{Q}_p^{\mathrm{cycl}})^\flat = \mathbb{F}_p((t^{1/p^\infty}))$. The Galois action of $\mathbb{Z}_p^\times$ on $\mathbb{Q}_p^{\mathrm{cycl}}$ acts on $\mathbb{F}_p((t^{1/p^\infty}))$ by $\gamma \cdot t = (1+t)^\gamma - 1$.

There is a continuous multiplicative lifting map $R^\flat \to R$ given by projecting onto the first coordinate, which we write as $x \mapsto x^\sharp$; it is not in general a ring homomorphism (since $R^\flat$ is characteristic p and R may be characteristic 0). However it defines an isomorphism $R^{\flat\circ}/\varpi^\flat \simeq R^\circ/\varpi$, which lifts to a bijection of rings of integral elements of R and its tilt.

The weak version of tilting equivalence is that when applied to adic spaces we get a homeomorphism: if (R, R^+) is a perfectoid Huber pair with tilt $(R^\flat, R^{\flat+})$, then there is a homeomorphism of the underlying topological spaces $|\operatorname{Spa}(R, R^+)| \cong |\operatorname{Spa}(R^\flat, R^{\flat+})|$, which preserves rational subsets. For any rational $U \subset X$ with image $U^\flat \subset X^\flat$, the complete Tate ring $\mathcal{O}_X(U)$ is perfectoid with tilt $\mathcal{O}_{X^\flat}(U^\flat)$.

The stronger version of tilting equivalence is the following theorem.

Theorem 2.1 (Tilting equivalence)**.** *Let R be a perfectoid ring with tilt $R^\flat$. Then there is an equivalence of categories $S \mapsto S^\flat$ between perfectoid R-algebras S and perfectoid $R^\flat$-algebras.*

This recovers Fontaine-Winterberger as a special case.

We say that a complete Tate $\mathbb{Z}_p$-algebra is sousperfectoid if there is an injection $R \hookrightarrow \tilde{R}$ into a perfectoid Tate ring $\tilde{R}$ such that the injection splits in the category of topological R-modules. Any perfectoid ring is clearly sousperfectoid; for a non-perfectoid example, take $R = \mathbb{Q}_p \langle T \rangle$, and $\tilde{R} = \mathbb{Q}_p^{\mathrm{cycl}} \langle T^{1/p^\infty} \rangle$. It turns out that this assumption suffices for sheafiness.

We now turn to perfectoid spaces. A perfectoid space is an adic space covered by affinoid adic spaces $\mathrm{Spa}(R, R^+)$ with R perfectoid. Unlike adic spaces, perfectoid spaces do have all fiber products.

Note that in general not every open affinoid in a perfectoid space is perfectoid; by an affinoid perfectoid space, we always mean a space of the form $\mathrm{Spa}(R, R^+)$ with R perfectoid.

We can glue together tilts to get a tilting functor $X \mapsto X^\flat$ from perfectoid spaces to perfectoid spaces in characteristic p. We can then restate the tilting equivalence in this global setting:

Theorem 2.2 (Tilting equivalence, global case)**.** *For any perfectoid space X with tilt $X^\flat$, the functor $Y \mapsto Y^\flat$ defines an equivalence between the category of perfectoid spaces over X and the category of perfectoid spaces over $X^\flat$.*

We can generalize a bit further, to the étale site: $X_{\text{ét}} \cong X^\flat_{\text{ét}}$.

Finally, we define the étale site of perfectoid spaces. A morphism $f : X \to Y$ of perfectoid spaces is finite étale if for all $\mathrm{Spa}(B, B+) \subset Y$ open, the pullback $X \times_Y \mathrm{Spa}(B, B^+)$ is $\mathrm{Spa}(A, A^+)$ for A a finite étale B-algebra, with A^+ the integral closure of B^+ in A. A morphism is étale if it factors as an open inclusion together with a finite étale morphism, and an étale cover is a jointly surjective family of étale maps. This has all the expected properties.

We can also define the pro-étale site: pro-étale maps are cofiltered limits of étale maps (at least on affinoids), and the rest is as expected.

3. Diamonds

Consider the category **Perf** of perfectoid spaces in characteristic p as a subcategory of the category **Perfd** of all perfectoid spaces. We can topologize this via the pro-étale topology, and look at pro-étale sheaves on **Perf**. The pro-étale topology is subcanonical, so a perfectoid space X of characteristic p defines a sheaf $\mathrm{Hom}_{\mathbf{Perf}}(-, X)$ on **Perf**, which we also write as X. We can extend this to perfectoid spaces of any characteristic: if Y is any perfectoid space, its tilt $Y^\flat$ is in **Perf** and so defines a sheaf on **Perf**, and any Y tilting to the same $Y^\flat$ defines the same sheaf. Fixing $X = Y^\flat$, for any untilt Y, which we write as $X^\sharp$, we get an equivalence $\mathrm{Hom}_{\mathbf{Perf}}(T, X) \simeq \mathrm{Hom}_{\mathbf{Perfd}}(T^\sharp, X^\sharp)$ for a compatible untilt $T^\sharp$ of T. Thus we can think of $Y = X^\sharp$ as defining a sheaf sending T to the set of untilts $T^\sharp$ of T over Y, i.e. maps $T^\sharp \to Y$ untilting to $T \to X$.

Thus we can associate a pro-étale sheaf not just to characteristic p perfectoid spaces, but arbitrary perfectoid spaces. We'd like to be able to do this for arbitrary adic spaces Y, and the above suggests that we define the sheaf associated to Y on **Perf**, written $Y^\diamondsuit$, as sending a characteristic p perfectoid space T to the set of untilts of T over Y.

We'd like to have some geometric understanding of what these sheaves $Y^\diamondsuit$ are, and in fact when Y is a reasonable space they are very good objects: these are diamonds.

A diamond is an analogue of algebraic spaces for perfectoid spaces. Namely, for X a perfectoid space of characteristic p we define a pro-étale equivalence relation $R \subset X \times X$ is an equivalence relation which forms a perfectoid space such that the projections onto each factor are pro-étale. We define a diamond to be any pro-étale sheaf on **Perf** which can be written as the quotient of a perfectoid space by a pro-étale equivalence relation.

If $Y = X/R$, then $R \to X \times_Y X$ is a diamond, and the map $X \to X/R = Y$ is quasi-pro-étale (i.e. its pullback to strictly totally disconnected covers is pro-étale).

Diamonds have all fiber products. In fact they also have absolute products, induced by the (nonobvious) fact that **Perf** has absolute products (though **Perfd** may not).

For example, take $X = (\operatorname{Spa} \mathbb{Q}_p^{\mathrm{cycl}})^\flat = \operatorname{Spa} \mathbb{F}_p((t^{1/p^\infty}))$, and let R be the relation coming from the Galois action of $\mathbb{Z}_p^\times$ induced from the action on $\mathbb{Q}_p^{\mathrm{cycl}}$. If we quotient by this action, we get something that intuitively should be something like the tilt of $\operatorname{Spa} \mathbb{Q}_p$, except that this is not perfectoid and has no tilt. Instead, we suggestively write it as $\operatorname{Spd} \mathbb{Q}_p = X/R$; it will turn out that this is the sheaf $(\operatorname{Spa} \mathbb{Q}_p)^\diamondsuit$. Roughly, given an element $Y^\sharp \to \operatorname{Spa} \mathbb{Q}_p$ of $(\operatorname{Spa} \mathbb{Q}_p)^\diamondsuit(Y)$, we can take its fiber product $\tilde{Y}^\sharp \approx Y^\sharp \times_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{cycl}}$; then $\tilde{Y}^\sharp \to Y^\sharp$ is a pro-étale $\mathbb{Z}_p^\times$-torsor, whose tilt $\tilde{Y} \to Y$ is a pro-étale $\mathbb{Z}_p^\times$-torsor with a $\mathbb{Z}_p^\times$-equivariant map $\tilde{Y} \to \operatorname{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat$, i.e. an element of $((\operatorname{Spa} \mathbb{Q}_p^{\mathrm{cycl}})^\flat / \mathbb{Z}_p^\times)(Y) = (\operatorname{Spd} \mathbb{Q}_p)(Y)$. In the reverse direction, from such a torsor $\tilde{Y} \to Y$ with an equivariant map $\tilde{Y} \to \operatorname{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})^\flat$ we can untilt to a unique $\tilde{Y}^\sharp \to \operatorname{Spa} \mathbb{Q}_p^{\mathrm{cycl}}$, also $\mathbb{Z}_p^\times$-equivariant, and therefore with a descent datum down to some $Y^\sharp$ over $\mathbb{Q}_p$ providing an element of $(\operatorname{Spa} \mathbb{Q}_p)^\diamondsuit(Y)$.

We can apply our construction $Y^\diamondsuit$ to any pre-adic space Y over $\operatorname{Spa} \mathbb{Z}_p$; if $Y = \operatorname{Spa}(R, R^+)$, we write $Y^\diamondsuit = \operatorname{Spd}(R, R^+)$. If Y is perfectoid, this agrees with the sheaf associated to Y by Yoneda by the discussion opening this section. The "absolute" case is $\operatorname{Spd} \mathbb{Z}_p$, which is not a diamond but is a pro-étale sheaf, which classifies untilts.

We can say a little more by defining a more refined topology. This is the v-topology: it is generated by open covers and all (jointly) surjective maps of affinoids. Remarkably this is still subcanonical, and more is true: any diamond is a v-sheaf. Although not a diamond, $\operatorname{Spd} \mathbb{Z}_p$ is a v-sheaf; for any pre-adic space Y, the associated presheaf $Y^\diamondsuit$ is a v-sheaf. If (A, A^+) is a Huber pare with A a Tate $\mathbb{Z}_p$-algebra, then $\operatorname{Spd}(A, A^+)$ is a diamond (in fact a locally spatial diamond, which we'll come to presently). For any analytic adic space Y over $\mathbb{Z}_p$, $Y^\diamondsuit$ is a locally spatial diamond.

What is this locally spatial condition? We first need to discuss the underlying topological space of a diamond. If $Y = X/R$ is a diamond, then its underlying topological space is $|Y| = |X|/|R|$ (it is not obvious this is well-defined, but it turns out to be). We say that Y is spatial if it is quasicompact and quasiseparated, and $|Y|$ admits a basis of open subsets given by $|U|$ for quasicompact open immersions $U \subset Y$. We say that Y is locally spatial if it admits an open cover by spatial diamonds. Finite étale covers of (locally) spatial diamonds by diamonds are (locally) spatial.

Unlike the category of diamonds, the category of v-sheaves does have a terminal object: this is the trivial sheaf $*$ sending any object to a point, and is represented as $\operatorname{Spd} \mathbb{F}_p$. Note that there is a map $\operatorname{Spd} \mathbb{Z}_p \to \operatorname{Spd} \mathbb{F}_p$ (on objects X the unique map $\operatorname{Spd} \mathbb{Z}_p(X) = \{X^\sharp\} \to \{*\}$), even though there is no map $\mathbb{F}_p \to \mathbb{Z}_p$; this is some incarnation of a 'deeper base,' and is what makes constructions (in diamonds, where this incarnates as an absolute product even though the base is not a diamond) like $\operatorname{Spd} \mathbb{Q}_p \times \operatorname{Spd} \mathbb{Q}_p$ meaningful and interesting.

It is possible to build a six-functors formalism resembling that for schemes, though different in some important aspects, for diamonds and v-sheaves (and more generally v-stacks); see [1].

References

[1] Peter Scholze. Étale cohomology of diamonds. *arXiv preprint arXiv:1709.07343*, 2017.

[2] Peter Scholze and Jared Weinstein. Berkeley lectures on p-adic geometry. In *Berkeley Lectures on p-adic Geometry*. Princeton University Press, 2020.