Tate's thesis*

Avi Zeff

Our goal is to understand 'automorphic representations for GL_1,' i.e. representations of $K^\times\backslash\mathbb{A}_K^\times$ for number fields K (the theory also works for function fields, but we will focus on the number field case for simplicity) where $\mathbb{A}_K$ is the ring of adeles over K. Unlike the case for more complicated groups, this group is abelian, and so its irreducible representations are just the characters $\mathrm{Hom}(K^\times\backslash\mathbb{A}_K, \mathbb{G}_\mathrm{m})$; generally we take coefficients in $\mathbb{C}$, so $\mathbb{G}_\mathrm{m}$ can be thought of without harm as just $\mathbb{C}^\times$. This is a group, the Pontryagin dual of $K^\times\backslash\mathbb{A}_K^\times$, and so it is natural to attempt Fourier analysis on it. (Indeed, this is one of the main motivations for the adeles: Fourier analysis only works well on locally compact topological abelian groups, including the additive (and multiplicative) groups of local fields but not of global fields. We can nevertheless get a global object by taking the product of all local fields; to get something which is locally compact, we take the restricted product instead, and thus obtain the adeles.

Fourier transforms over $\mathbb{A}_K^\times$ decompose into factors given by the Fourier transform at each place (when the original function does). Adjusting these local integrals by local L-factors gives a natural functional equation, while Poisson summation gives a functional equation for the adelic Fourier transform; combining these gives a functional equation for the product of the local L-factors, i.e. the L-function corresponding to the chosen character.

Fix a character $\chi : K^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$. Consider the map $x \mapsto \frac{\chi(x)}{|x|^s}$, where $|x|$ is the absolute value on the adeles $|x| = \prod_v |x|_v$ (note any element of $K^\times$ in $\mathbb{A}_K^\times$ has absolute value 1, so this descends to $K^\times\backslash\mathbb{A}_K^\times$) and s is any complex number. This is also a character, since both χ and $|\cdot|^s$ are, and it has image in the unit circle in $\mathbb{C}^\times$, so we can write any character of $K^\times\backslash\mathbb{A}^\times$ as $x \mapsto \chi(x)\cdot|x|^s$ for some complex number s with $|\chi(x)| = 1$ for every x.

Thus if f is some well-behaved function on $K^\times\backslash\mathbb{A}_K^\times$, its Fourier transform at the character $\chi(\cdot)|\cdot|^s$ is

$$\hat{f}(\chi, s) = \int_{K^\times\backslash\mathbb{A}_K^\times} f(x)\chi(x)|x|^s\, d^\times x$$

where $d^\times x$ the Haar measure on $K^\times\backslash\mathbb{A}_K^\times$ with the right normalization to make Fourier inversion work. (We'll say more about this.) In fact, it is convenient to work over $\mathbb{A}_K^\times$ rather than its quotient by $K^\times$, where we can interpret χ as a K-invariant function; if we choose f to be nice enough on $\mathbb{A}_K^\times$ this will still converge, and we can recover the above integral by taking f to be $K^\times$-invariant.

Thus we're studying integrals of the form

$$\int_{\mathbb{A}_K^\times} f(x)\chi(x)|x|^s\, d^\times x$$

for a suitable Haar measure $d^\times x$ on $\mathbb{A}_K^\times$, which factors as $d^\times x = \prod_v d^\times x_v$ for some Haar measures $d^\times x_v$ on each $K_v^\times$; we fix the choice of $d^\times x_v$ such that for v nonarchimedean $\mathcal{O}_v^\times$ has volume 1, and for v archimedean $d^\times x_v = \frac{dx_v}{|x_v|_v}$; note that we normalize the absolute values (a priori defined only up to equivalence) such that multiplication by a multiplies the

*These notes are based on section 3.1 of [1].

measure by $|a|_v$, so in particular the real absolute value is the usual one but the complex absolute value is its square. We call this integral $\zeta(s, \chi, f)$.

To avoid issues of convergence and make Fourier inversion work reliably (keeping in mind we may want to do Fourier analysis work for either the additive or multiplicative groups), we want to find a good space of functions on $\mathbb{A}_K$ which is preserved under convolution and the Fourier transform. We work one place at a time. In the archimedean case, we can do this analytically: the space of Schwartz functions on $\mathbb{R}$ is the space of smooth functions $\mathbb{R} \to \mathbb{C}$ whose derivatives are all sub-polynomial, i.e. $x^a \frac{d^b f}{dx^b}$ is bounded for all nonnegative integers a, b. One can also do this for higher-dimensional vector spaces by requiring all derivatives bounded in this way, and in particular by viewing $\mathbb{C}$ as $\mathbb{R}^2$ we get a Schwartz space for $\mathbb{C}$. For nonarchimedean local fields F, we say that a function on F is smooth if it is locally constant, and define the Schwartz space for F to be the set of compactly supported smooth functions $F \to \mathbb{C}$. These all give $\mathbb{C}$-vector spaces, and we can take the restricted product over all places to get a space $\mathcal{S}(\mathbb{A}_K)$ of Schwartz functions on $\mathbb{A}_K$, where the restriction is that for all but finitely many places v the decomposable tensors $f = \prod_v f_v$ should have f_v equal to the characteristic function of the ring of integers $\mathcal{O}_v$ (possible only for v nonarchimedean). One can define a similar notion for vector spaces over $\mathbb{A}_K$.

Since every element of $\mathcal{S}(\mathbb{A}_K)$ can be written as a linear combination of decomposable tensors, it suffices to understand the theory for them. In that case we formally have

$$\zeta(s, \chi, f) = \int_{\mathbb{A}_K^\times} f(x)\chi(x)|x|^s \, d^\times x = \prod_v \int_{K_v^\times} f_v(x_v)\chi_v(x_v)|x_v|_v^s \, d^\times x_v$$

since every term in the integral factors (here χ_v is the restriction $K_v^\times \hookrightarrow \mathbb{A}_K^\times \xrightarrow{\chi} \mathbb{C}^\times$). We call these local integrals at each place v the local zeta integrals $\zeta_v(s, \chi_v, f_v)$. To see that this is a genuine identity, we need to know that these integrals converge, at least in some region.

Proposition 1. *For any Schwartz functions f on $\mathbb{A}_K$ and f_v on K_v for any place v, the local integral $\zeta_v(s, \chi_v, f_v)$ converges for $\operatorname{Re}(s) > 0$, and the global integral $\zeta(s, \chi, f)$ converges absolutely for $\operatorname{Re}(s) > 1$, in which case the decomposition*

$$\zeta(s, \chi, f) = \prod_v \zeta(s, \chi_v, f_v)$$

holds.

Proof. Since $|\chi_v(x)| = 1$, we have

$$|\zeta_v(s, \chi_v, f_v)| \leq \int_{K_v^\times} f_v(x_v)|x_v|_v^s \, d^\times x_v.$$

On the locus $|x_v|_v > 1$, since f_v is compactly supported (in the nonarchimedean case) or rapidly decaying (in the archimedean case) the integral is finite, so we restrict to the locus $|x_v|_v \leq 1$. This is a compact region, so since f_v is continuous it is bounded, so we can replace it by a constant; since we only need to show convergence, we assume this constant is 1. If v is nonarchimedean, this locus is precisely $\mathcal{O}_v = \bigsqcup_{k \geq 0} \pi^k \mathcal{O}_v^\times$ for a uniformizer π, with

$|x_v|_v = q^{-k}$ if x is in the k-component, where $q = \mathcal{O}_v/\mathfrak{m}_v$ is the cardinality of the residue field at v. Therefore

$$\int_{K_v^\times : |x_v|_v \leq 1} |x_v|_v^s \, d^\times x_v = \sum_{k=0}^{\infty} q^{-ks} \operatorname{vol}(\pi^k \mathcal{O}_v^\times)$$

where volume is computed with respect to $d^\times x_v$. The volume is independent of k, and so this series converges whenever $\operatorname{Re}(s) > 0$. Note that this proof also shows that the choice for $d^\times x_v$ giving volume 1 to $\mathcal{O}_v^\times$ is a natural one.

If v is real, the integral is bounded by the same argument by

$$\int_{-1}^{1} |x_v|_v^s \, d^\times x_v.$$

The chosen measure for $\mathbb{R}^\times$ is given by $\frac{dx_v}{x_v}$, so this is bounded up to a scalar by

$$\int_0^1 |x_v|_v^{s-1} \, dx_v = \frac{1}{s}$$

for $\operatorname{Re}(s) > 0$; the complex case is similar, with an added integral around the unit circle (using polar coordinates) which does not change the convergence.

For the global case, given any f there is a finite set S of places such that for $v \notin S$ we have f_v given by the characteristic function of $\mathcal{O}_v$. For any v not in S, it follows that

$$\zeta_v(s, \chi_v, f_v) = \int_{\mathcal{O}_v \setminus \{0\}} \chi_v(x_v) |x_v|_v^s \, d^\times x_v = \sum_{k=0}^{\infty} q^{-ks} \int_{\pi^k \mathcal{O}_v^\times} \chi_v(x_v) \, d^\times x_v.$$

For all but finitely many v, the character χ_v of $K_v^\times$ is unramified, i.e. trivial on $\mathcal{O}_v^\times$. To see this, consider the restriction of χ to $\mathbb{A}_f^\times$. Its kernel contains an open neighborhood of the identity, which necessarily contains some product of the $\mathcal{O}_v^\times$ over all but finitely many v. Thus choosing S appropriately we can assume that χ_v is trivial on $\mathcal{O}_v^\times$, so that $\chi_v(x_v) = \chi_v(\pi)^k$ for $x_v \in \pi^k \mathcal{O}_v^\times$. Therefore

$$\zeta_v(s, \chi_v, f_v) = \sum_{k=0}^{\infty} (\chi_v(\pi) q^{-s})^k = \frac{1}{1 - \chi_v(\pi) q^{-s}}.$$

This is precisely the local factor $L_v(s, \chi)$ of the L-function $L(s, \chi)$, and so

$$\prod_{v \notin S} \zeta_v(s, \chi_v, f_v) = L(s, \chi) \prod_{v \in S} (1 - \chi_v(\pi) q^{-s}).$$

In particular, the right-hand side converges absolutely for $\operatorname{Re}(s) > 1$; taking the finite product with the remaining $\zeta_v(s, \chi_v, f_v)$ gives the product over all v of the local zeta integrals, so this converges absolutely for $\operatorname{Re}(s) > 1$ and so the above manipulations are valid and this is equal to $\zeta(s, \chi, f)$. □

The above proof shows that the global zeta integral is closely related to the L-function $L(s,\chi)$: it agrees with it up to finitely many factors, which however may be complicated. It also shows that all but finitely many of the local zeta integrals have a very simple form, which in particular is a rational function and extends meromorphically to the whole complex plane, possibly with discrete poles along the imaginary axis. We might hope that similar properties hold for general f_v; this is true, but to even state the result we need the *additive* Fourier transform.

Fix a nontrivial additive character $\psi : \mathbb{A}_K \to \mathbb{C}^\times$ vanishing on the image of K. For each place, we get a restriction $K_v \to \mathbb{A}_K \to \mathbb{C}^\times$, with $\psi(x) = \prod_v \psi_v(x_v)$; any other character can be obtained from ψ by scaling by some $a \in \mathbb{A}_K$, i.e. $\psi_a(x) = \psi(ax)$, and so $a \mapsto \psi_a$ gives an isomorphism between the additive group of $\mathbb{A}_K$ and its Pontryagin dual. We fix Haar measures dx_v on the additive groups of the local fields K_v such that the volume of $\mathcal{O}_v$ is 1 for nonarchimedean v, and the standard measure on $\mathbb{R}$ and that induced from $\mathbb{R}^2$ for real and complex v respectively.

The Fourier transform of a Schwartz function f_v on K_v (identifying the character group with K_v as above) is

$$\hat{f}_v(x) = \int_{K_v} f_v(y)\psi_v(xy)\,dy,$$

for $dy = dy_v$ the chosen Haar measure on K_v. One can verify, using the self-duality and the fact that f_v is Schwartz, that $\hat{\hat{f}}_v(x) = f(-x)$, the Fourier inversion formula, and that $\hat{f}_v$ is Schwartz if f_v is.

Proposition 2. *For any Schwartz function f_v on K_v, the local zeta integral $\zeta_v(s,\chi_v,f_v)$ has a meromorphic extension to all of $\mathbb{C}$, with no poles on $\mathrm{Re}(s) > 0$. Further there is a functional equation*

$$\zeta_v(1-s,\chi_v^{-1},\hat{f}_v) = \gamma_v(s,\chi_v,\psi_v)\zeta_v(s,\chi_v,f_v)$$

for some meromorphic function $\gamma_v(s,\chi_v,\psi_v)$ independent of f_v, though depending on the choice of character ψ_v of K_v.

Proof. We know from Proposition 1 that both zeta integrals are holomorphic for $\mathrm{Re}(s) < 1$ on the left-hand side and $\mathrm{Re}(s) > 0$ on the right, and so the ratio

$$\frac{\zeta_v(1-s,\chi_v^{-1},\hat{f}_v)}{\zeta_v(s,\chi_v,f_v)}$$

is meromorphic on $0 < s < 1$. We want to show that it is in fact independent of f_v; we will then try to extend the resulting function to all of $\mathbb{C}$.

Suppose that f'_v is another Schwartz function on K_v, so that what we want to prove is that the above ratio gives the same meromorphic function for f_v and f'_v, i.e.

$$\zeta_v(1-s,\chi_v^{-1},\hat{f}_v)\zeta_v(s,\chi_v,f'_v) = \zeta_v(1-s,\chi_v^{-1},\hat{f}'_v)\zeta_v(s,\chi_v,f_v).$$

Indeed, the left-hand side is

$$\int_{K_v^\times} \chi_v(x)^{-1}|x|_v^{1-s} \int_{K_v} f_v(y)\psi_v(xy)\,dy\,d^\times x \int_{K_v^\times} f'_v(z)\chi_v(z)|z|_v^s\,d^\times z$$

where we omit some of the subscript v's for convenience. Rearranging, this is

$$\int_{K_v^\times}\int_{K_v}\int_{K_v^\times} f_v(y)f_v'(z)\chi_v(x^{-1}z)\psi_v(xy)|x|_v^{1-s}\cdot|z|_v^s\,d^\times x\,dy\,d^\times z.$$

Letting $w = xy$ for each fixed nonzero y (valid away from a set of measure 0, so we can ignore it), we have $d^\times w = d^\times x$ since they are Haar measures, so this is

$$\int_{K_v^\times}\int_{K_v^\times}\int_{K_v^\times} f_v(y)f_v'(z)\chi_v(w^{-1}yz)\psi(w)|w|_v^{1-s}|y|_v^{-1}|yz|_v^s\,d^\times x\,dy\,d^\times z.$$

Now, restricted to $K_v^\times$, for v nonarchimedean the difference between dy and $d^\times y$ is that the former gives volume 1 to $\mathcal{O}_v$, while the latter gives volume 1 to $\mathcal{O}_v^\times$. We can check that $\frac{dy}{|y|_v}$ is a Haar measure for the multiplicative group, since dy is for the additive group, and correspondingly $|y|_v\,d^\times y$ is an additive Haar measure, and so the question is only the scalar. We have

$$\int_{O_v}|y|_v\,d^\times y = \sum_{k=0}^{\infty} q^{-k}\int_{\pi^k\mathcal{O}_v^\times} d^\times y = \sum_{k=0}^{\infty} q^{-k} = \frac{1}{1-q^{-1}},$$

so $d^\times y = \frac{1}{1-q^{-1}}\cdot\frac{dy}{|y|_v}$. If v is archimedean, by definition this scalar is 1, and so if we define m_v to be $\frac{1}{1-q^{-1}}$ for v nonarchimedean and 1 for v archimedean the above is

$$m_v^{-1}\int_{K_v^\times}\int_{K_v^\times}\int_{K_v^\times} f_v(y)f_v'(z)\chi_v(w^{-1}yz)\psi(w)|w|_v^{1-s}|yz|_v^s\,d^\times x\,d^\times y\,d^\times z,$$

which is symmetric in f and f' upon switching y and z, so the desired equation holds.

Thus there is a function $\gamma_v(s,\chi_v,\psi_v)$ independent of f_v such that the functional equation above holds when $0 < \mathrm{Re}(s) < 1$. We next want to show that it extends to a meromorphic function on $\mathbb{C}$. Since it is independent of f_v, we can choose f_v freely. If we choose f_v such that $\hat{f}_v$ vanishes near 0 (actually vanishes on a neighborhood of 0 in the nonarchimedean case, and vanishes at 0 and is very small near 0 in the archimedean case) then $\zeta_v(1-s,\chi_v^{-1},\hat{f}_v)$ is actually convergent for $\mathrm{Re}(s)$ large: the danger is that for x near 0 when $\mathrm{Re}(s)$ is large $|x|_v^{1-s}$ will blow up, but in this case those terms are killed in any case. Therefore the ratio of zeta integrals is defined for all $\mathrm{Re}(s) > 0$. Similarly choosing f_v to vanish near 0 makes $\zeta_v(s,\chi_v,f_v)$ converge for $-\mathrm{Re}(s)$ large, and so the ratio, which is just $\gamma_v(s,\chi_v,\psi_v)$, is defined meromorphically for all s.

This defines the analytic continuation desired, for any f_v: for $\mathrm{Re}(s) > 0$ large, the right-hand side is defined and so gives a definition for $\zeta_v(1-s,\chi_v^{-1},\hat{f}_v)$, while for $\mathrm{Re}(s) < 1$ the left-hand side is defined and $\gamma_v(s,\chi_v,\psi_v)$ is defined as a meromorphic function, and so dividing by it gives a meromorphic extension of $\zeta_f(s,\chi_v,f_v)$ to all of $\mathbb{C}$. Proposition 1 shows that it has no poles on the region $\mathrm{Re}(s) > 0$. □

We can now turn to proving a similar functional equation and analytic continuation for the global zeta integral, which will in turn imply the functional equation for the L-functions. First, we need to discuss Poisson summation.

Similar to the local case, we can define the adelic Fourier transform by

$$\hat{f}(x) = \int_{\mathbb{A}_K} f(y)\psi(xy)\,dy.$$

Proposition 3 (Poisson summation formula)**.** *For any $x \in \mathbb{A}_K^\times$ and Schwartz function f on $\mathbb{A}_K$,*

$$\sum_{\alpha \in K} f(\alpha x) = \frac{1}{|x|} \sum_{\alpha \in K} \hat{f}(\alpha/x).$$

Since f is a Schwartz function (which we can assume decomposes as a product over places), away from finitely many places S it is the indicator function of $\mathcal{O}_v$ on each $v \notin S$, so we can restrict to α such that $|\alpha x|_v \leq 1$ for all $v \notin S$. This gives a subring of K finitely generated over $\mathcal{O}_K$, whose embedding in $\mathbb{A}_{K,f}$ is discrete; since f_v is compactly supported for every v, including those in S, the support for each nonarchimedean v is finite, and for the infinite places the exponential decay ensures that both sums converge.

Proof. We prove the result by generalizing: set

$$F(y) = \sum_{\alpha \in K} f((\alpha + y)x),$$

so that the left-hand side is $F(0)$. This is a continuous function on the compact abelian group $K \backslash \mathbb{A}_K$ (as additive groups), whose characters are in bijection with β by $y \mapsto \psi(-\beta y)$, and so it has Fourier expansion

$$F(y) = \sum_{\beta \in K} c_\beta \psi(-\beta y)$$

for some coefficients c_β. We can compute these by orthogonality:

$$\int_{K \backslash \mathbb{A}_K} F(y) \psi(\beta y)\, dy = \int_{K \backslash \mathbb{A}_K} \sum_{\beta' \in K} c_{\beta'} \psi((\beta - \beta')y)\, dy = c_\beta \cdot \operatorname{vol}(K \backslash \mathbb{A}_K)$$

since the integral of a nontrivial character is 0, while the integral of the trivial character is the volume. On the other hand the integral on the left is

$$\int_{K \backslash \mathbb{A}_K} \sum_{\alpha \in K} f((\alpha + y)x) \psi(\beta y)\, dy,$$

which since ψ is K-invariant is the same thing as

$$\int_{K \backslash \mathbb{A}_K} \sum_{\alpha \in K} f((\alpha + y)x) \psi(\beta(\alpha + y))\, dy.$$

If α varies over K and y over $K \backslash \mathbb{A}_K$, this is the same thing as letting a single variable, say z, range over all of $\mathbb{A}_K$, i.e.

$$\int_{\mathbb{A}_K} f(zx) \psi(\beta z)\, dz.$$

Letting $w = zx$, this is

$$\int_{\mathbb{A}_K} f(w) \psi(w\beta/x) \frac{dw}{|x|} = \frac{1}{|x|} \hat{f}(\beta/x),$$

so in all

$$c_\beta = \frac{1}{|x| \operatorname{vol}(K\backslash\mathbb{A}_K)}\hat{f}(\beta/x).$$

In particular, setting $y = 0$ in the Fourier expansion of $F(y)$ gives

$$\sum_{\alpha\in K} f(\alpha x) = F(0) = \frac{1}{|x| \operatorname{vol}(K\backslash\mathbb{A}_K)} \sum_{\alpha\in K} \hat{f}(\alpha/x).$$

Thus it suffices to show that $\operatorname{vol}(K\backslash\mathbb{A}_K) = 1$.

To see this, we apply this relation twice, say with $x = 1$. Then this gives

$$\sum_{\alpha\in K} f(\alpha) = \frac{1}{\operatorname{vol}(K\backslash\mathbb{A}_K)} \sum_{\alpha\in K} \hat{f}(\alpha) = \frac{1}{\operatorname{vol}(K\backslash\mathbb{A}_K)^2} \sum_{\alpha\in K} \hat{\hat{f}}(\alpha).$$

By Fourier inversion, $\hat{\hat{f}}(\alpha) = f(-\alpha)$, and since we sum over all of K we have

$$\sum_{\alpha\in K} f(-\alpha) = \sum_{\alpha\in K} f(\alpha)$$

by reordering the sum, so since we can choose f such that the sum is nonzero we get $\operatorname{vol}(K\backslash\mathbb{A}_K)^2 = 1$, which implies $\operatorname{vol}(K\backslash\mathbb{A}_K) = 1$ since it is nonnegative. □

We are now ready to prove the global functional equation.

Theorem 4. *Let f be any Schwartz function on $\mathbb{A}_K$. Then $\zeta(s, \chi, f)$ has a meromorphic continuation to all $s \in \mathbb{C}$, and it satisfies the functional equation*

$$\zeta(s, \chi, f) = \zeta(1 - s, \chi^{-1}, \hat{f}).$$

Proof. Integrating over $\mathbb{A}_K^\times$ is the same as integrating over $K^\times\backslash\mathbb{A}_K^\times$ and summing over $K^\times$, i.e.

$$\zeta(s, \chi, f) = \int_{\mathbb{A}^\times} f(x)\chi(x)|x|^s \, d^\times x = \sum_{\alpha\in K^\times} \int_{K^\times\backslash\mathbb{A}_K^\times} f(\alpha x)\chi(\alpha x)|\alpha x|^s \, d^\times x.$$

Since χ and $|\cdot|$ are trivial on the image of $K^\times$ and the integral converges absolutely when $\operatorname{Re}(s) > 1$, under that assumption this is the same as

$$\int_{K^\times\backslash\mathbb{A}_K^\times} \left(\sum_{\alpha\in K^\times} f(\alpha x)\right) \chi(x)|x|^s \, d^\times x.$$

By the Poisson summation formula,

$$\sum_{\alpha\in K^\times} f(\alpha x) = -f(0) + \sum_{\alpha\in K} f(\alpha x) = -f(0) + \frac{1}{|x|} \sum_{\alpha\in K} \hat{f}(\alpha/x).$$

Therefore

$$\zeta(s, \chi, f) = \int_{K^\times\backslash\mathbb{A}_K^\times} \frac{1}{|x|} \left(\sum_{\alpha\in K} \hat{f}(\alpha/x)\right) \chi(x)|x|^s \, d^\times x - f(0) \int_{K^\times\backslash\mathbb{A}_K^\times} \chi(x)|x|^s \, d^\times x.$$

We focus on the first term for now. If we replace x by x^{-1}, this becomes

$$\int_{K^\times\backslash\mathbb{A}_K^\times}\left(\sum_{\alpha\in K}\hat{f}(\alpha x)\right)\chi(x)^{-1}|x|^{1-s}\,d^\times x,$$

which we can expand as

$$\int_{K^\times\backslash\mathbb{A}_K^\times}\left(\sum_{\alpha\in K}\hat{f}(\alpha x)\right)\chi(x)^{-1}|x|^{1-s}\,d^\times x+\hat{f}(0)\int_{K^\times\backslash\mathbb{A}_K^\times}\chi(x)^{-1}|x|^{1-s}\,d^\times x.$$

Ignoring the two simpler integrals, this is the same formula we got earlier but with the arguments replaced: the same argument shows that the main term is just $\zeta(1-s,\chi^{-1},\hat{f})$ as desired.

There are two issues. The first is the remaining integrals: these are not obviously symmetric. The other is more subtle: in order to do the manipulations above, we needed to assume $\mathrm{Re}(s)>1$; but to do the same thing in reverse on the other side, we would need to assume $\mathrm{Re}(1-s)>1$, i.e. $\mathrm{Re}(s)<0$!

To fix this problem, we split the integral:

$$\zeta(s,\chi,f)=\int_{\mathbb{A}_K^\times}f(x)\chi(x)|x|^s\,d^\times x=\int_{\mathbb{A}^\times:|x|<1}f(x)\chi(x)|x|^s\,d^\times x+\int_{\mathbb{A}^\times:|x|>1}f(x)\chi(x)|x|^s\,d^\times x.$$

(The locus with $|x|=1$ is measure 0 and so we can omit it, and doing so makes some of the symmetry arguments easier later.) Call the first integral ζ_0 and the second ζ_1. In particular ζ_1 is convergent for all s: we know that it is convergent for $\mathrm{Re}(s)>1$, and when $|x|>1$ decreasing $\mathrm{Re}(s)$ only makes the integral converge faster. Therefore to prove the extension to $\mathbb{C}$ we can restrict attention to ζ_0; we'll come back to ζ_1 for the functional equation.

Since the image of $K^\times$ has absolute value 1, we can still pass to the quotient and do all manipulations as above with the restriction to x with $|x|<1$; the change of variables to x^{-1} switches the restriction to $|x|>1$, where the same argument proves the convergence of the main term. Thus we have

$$\zeta_0(s,\chi,f)=\zeta_1(1-s,\chi^{-1},\hat{f})-f(0)\int\limits_{\substack{K^\times\backslash\mathbb{A}_K^\times\\|x|<1}}\chi(x)|x|^s\,d^\times x+\hat{f}(0)\int\limits_{\substack{K^\times\backslash\mathbb{A}_K^\times\\|x|>1}}\chi(x)^{-1}|x|^{1-s}\,d^\times x.$$

By multiplying by a real number, we have a bijection $\mathbb{A}_{K,t}^\times\xrightarrow{\sim}\mathbb{A}_{K,1}^\times$ for every $t>0$, where $\mathbb{A}_{K,t}^\times$ denotes the ideles with absolute value t, which descends to the quotient by $K^\times$ since the absolute value is trivial on its image. In particular we can reinterpret these two integrals as

$$f(0)\int_0^1 t^s\int_{K^\times\backslash\mathbb{A}_{K,1}^\times}\chi(tx)\,d^\times x\,d^\times t$$

and

$$\hat{f}(0)\int_1^\infty t^{1-s}\int_{K^\times\backslash\mathbb{A}_{K,1}^\times}\chi(tx)^{-1}\,d^\times x\,d^\times t.$$

If χ is nontrivial on $\mathbb{A}_{K,1}^\times$ (or equivalently its quotient, since it is trivial on $K^\times$) then the integrals both vanish, and we have $\zeta_0(s,\chi,f) = \zeta_1(1-s,\chi^{-1},\hat{f})$, for all s. By applying this to $(1-s,\chi^{-1},\hat{f})$, we get $\zeta_1(s,\chi,f) = \zeta_0(1-s,\chi^{-1},\hat{f})$, and so combining these gives

$$\zeta(s,\chi,f) = \zeta_0(s,\chi,f)+\zeta_1(s,\chi,f) = \zeta_1(1-s,\chi^{-1},\hat{f})+\zeta_0(1-s,\chi^{-1},\hat{f}) = \zeta(1-s,\chi^{-1},\hat{f}),$$

and further shows that in this case this gives $\zeta(s,\chi,f)$ as an entire function on $\mathbb{C}$.

If χ is trivial restricted to $\mathbb{A}_{K,1}^\times$, it descends to the quotient (which is isomorphic to $\mathbb{R}_{>0}^\times$), i.e. it factors through the absolute value map $x \mapsto |x|$ of which $\mathbb{A}_{K,1}^\times$ is the kernel. Thus $\chi(x) = |x|^\lambda$ for some complex number λ. Since we've assumed $|\chi(x)| = 1$, we have $\big||x|^\lambda\big| = |e^{\lambda \log|x|}| = |e^{\mathrm{Re}(\lambda)\log|x|}| = 1$ for all x, i.e. $\mathrm{Re}(\lambda) = 0$, so λ is an imaginary number. In this case, if $|x| = 1$ then $\chi(tx) = |tx|^\lambda = t^\lambda$ and so our integrals above are

$$f(0)\int_0^1 t^{s+\lambda}\,\mathrm{vol}(K^\times\backslash\mathbb{A}_{K,1}^\times)\,\frac{dt}{t} = f(0)\frac{\mathrm{vol}(K^\times\backslash\mathbb{A}_{K,1}^\times)}{s+\lambda}$$

and

$$\hat{f}(0)\int_1^\infty t^{1-s-\lambda}\,\mathrm{vol}(K^\times\backslash\mathbb{A}_{K,1}^\times)\,\frac{dt}{t} = \hat{f}(0)\frac{\mathrm{vol}(K^\times\backslash\mathbb{A}_{K,1}^\times)}{s+\lambda-1}.$$

Thus

$$\zeta_0(s,\chi,f) = \zeta_1(1-s,\chi^{-1},\hat{f}) - f(0)\frac{\mathrm{vol}(K^\times\backslash\mathbb{A}_{K,1}^\times)}{s+\lambda} - \hat{f}(0)\frac{\mathrm{vol}(K^\times\backslash\mathbb{A}_{K,1}^\times)}{1-s-\lambda}.$$

Replacing (s,χ,f) with $(1-s,\chi^{-1},\hat{f})$ and rearranging, keeping in mind that if $\chi(x) = |x|^\lambda$ then $\chi(x)^{-1} = |x|^{-\lambda}$, gives

$$\zeta_1(s,\chi,f) = \zeta_0(1-s,\chi^{-1},\hat{f}) + \hat{f}(0)\frac{\mathrm{vol}(K^\times\backslash\mathbb{A}_{K,1}^\times)}{1-s-\lambda} + f(0)\frac{\mathrm{vol}(K^\times\backslash\mathbb{A}_{K,1}^\times)}{s+\lambda},$$

so

$$\zeta(s,\chi,f) = \zeta_0(s,\chi,f)+\zeta_1(s,\chi,f) = \zeta_1(1-s,\chi^{-1},\hat{f})+\zeta_0(1-s,\chi^{-1},\hat{f}) = \zeta(1-s,\chi^{-1},\hat{f}).$$

(Note we do not need to compute the volume, though one can and Tate does; for us it is enough to see that it is finite, which is clear by compactness.)

In this case, our formulas above show that $\zeta(s,\chi,f)$ is *not* entire, but has poles at $s=\lambda$ and $s = 1-\lambda$. □

Corollary 5. *Let χ be a character of $K^\times\backslash\mathbb{A}_K^\times$, and S be a finite set of places including the archimedean ones and any at which χ is ramified. Write $L_S(s,\chi) = \prod_{v\notin S} L_v(s,\chi)$ for the partial L-function away from s. Then $L_S(s,\chi)$ extends meromorphically to the complex plane, and satisfies the functional equation*

$$L_S(s,\chi) = L_S(1-s,\chi^{-1})\prod_{v\in S}\gamma_v(s,\chi_v,\psi_v)$$

where the γ_v are the meromorphic factors from Proposition 2. Further $L_S(s,\chi)$ is entire unless $\chi(x) = |x|^\lambda$ for some complex number λ, in which case it has poles at λ and $1-\lambda$.

Proof. Choose a Schwartz function $f = \prod_v f_v$ on $\mathbb{A}_K$. The proof of Proposition 1 shows that

$$L_S(s,\chi)\prod_{v\in S}\zeta_v(s,\chi_v,f_v) = \prod_v \zeta_v(s,\chi_v,f_v) = \zeta(s,\chi,f).$$

By Theorem 4, the right-hand side extends meromorphically to $\mathbb{C}$, and by Proposition 2 so do each of the local factors for $v \in S$ on the left; therefore so does $L_S(s,\chi)$, and the above identity holds for all s except possibly at discrete poles. The functional equation for ζ gives

$$L_S(s,\chi)\prod_{v\in S}\zeta_v(s,\chi_v,f_v) = \zeta(s,\chi,f) = \zeta(1-s,\chi^{-1},\hat{f}) = L_S(1-s,\chi^{-1})\prod_{v\in S}\zeta_v(1-s,\chi_v^{-1},\hat{f}_v).$$

On the other hand the local functional equation from Proposition 2 shows that the right-hand side is equal to

$$L_S(1-s,\chi^{-1})\prod_{v\in S}\gamma_v(s,\chi_v,\psi_v)\zeta_v(s,\chi_v,f_v),$$

so canceling the local factors we obtain the claimed functional equation.

If $\zeta(s,\chi,f)$ is entire, then in order for $L_S(s,\chi)$ to have a pole at s_0 we would have to have $\zeta_v(s_0,\chi_v,f_v) = 0$ for some v. But by choosing f_v judiciously for each of the finitely many places in v we can ensure that this does not happen: if f_v has compact support and has positive real part on a small neighborhood of $x = 1$, then the integral defining $\zeta_v(s_0,\chi_v,f_v)$ must have positive real part. Therefore in this case $L_S(s,\chi)$ is also entire. If $\zeta(s,\chi,f)$ has a pole, by the proof of Theorem 4 this can only happen when $\chi(x) = |x|^\lambda$, in which case the pole is at $-\lambda$ or $1-\lambda$; thus these are the only possible poles of $L_S(s,\chi)$. □

We would like to extend this result to all of $L(s,\chi)$, rather than just the factors away from S. To do so, we need to define the local L-factors at v ramified or archimedean. At the ramified places v, we make the simplest choice and set $L_v(s,\chi_v) = 1$; for s real, we have $\chi_v(x) = (x/|x|_v)^\epsilon$ for some complex ϵ; since $|\chi_v(x)| = 1$ by assumption, ϵ must be real, and all $\epsilon \in \mathbb{R}^\times$ give the same character so there are essentially only two possibilities, which we write as $\epsilon = 0$ or $\epsilon = 1$. In this case we define $L_v(s,\chi_v) = \pi^{-(s+\epsilon)/2}\Gamma((s+\epsilon)/2)$. Finally if v is complex, χ_v is of the form $|x|_v^\mu \cdot (x/\sqrt{|x|_v})^k$ for some complex number μ and integer k (keeping in mind that $|x|_v$ is the square of the usual complex absolute value, so to recover the sign we need to take the square root). Since this has absolute value 1, we must have μ imaginary. In this case, we set $L_v(s,\chi_v) = 2(2\pi)^{s+\mu+|k|/2}\Gamma(s+\mu+|k|/2)$.

Recall that $\zeta_v(s,\chi_v,f_v)$ may not be entire, and for all but finitely many v for a given $f = \prod_v f_v$ will be equal to $L_v(s,\chi_v)$. Thus it is sometimes more natural to divide by the L-factor: $\frac{\zeta_v(s,\chi_v,f_v)}{L_v(s,\chi_v)}$ is entire. The proof is somewhat tedious and so omitted: in the nonarchimedean case, one can decompose the integral by the absolute value, and then for large values it vanishes by compact support and for small values it vanishes in the ramified case and gives an explicit geometric series whose quotient by the L-factor is entire, and in the archimedean case one studies the residues at the poles and verifies that they agree with those of the L-factors.

If we take the functional equation from Corollary 5 and add the remaining L-factors, we get

$$L(s,\chi) = L(1-s,\chi^{-1})\prod_{v\in S}\frac{\gamma_v(s,\chi_v,\psi_v)L_v(s,\chi_v)}{L_v(1-s,\chi_v^{-1})},$$

where $L(s,\chi) = \prod_v L_v(s,\chi_v)$ over all v is the completed L-function. Define these factors on the right to be

$$\epsilon_v(s,\chi_v,\psi_v) = \frac{\gamma_v(s,\chi_v,\psi_v)L_v(s,\chi_v)}{L_v(1-s,\chi_v^{-1})}.$$

In the unramified setting, since $L_v(s,\chi_v) = \frac{1}{1-\chi_v(\pi)q^{-s}}$, changing χ_v by multiplying by a factor of $|\cdot|_v^\lambda$ gives $\frac{1}{1-\chi_v(\pi)q^{-s-\lambda}} = L_v(s+\lambda,\chi_v)$ since $|\pi|_v = q^{-1}$. Therefore changing χ by a factor of $|\cdot|^\lambda$ does not meaningfully change the result, and in particular we can assume without any real loss that if $\chi(x) = |x|^\lambda$, as in Corollary 5, then $\lambda = 0$, i.e. χ is the trivial character.

Theorem 6. *The completed L-function $L(s,\chi)$ has analytic continuation to all of $\mathbb{C}$ if χ is nontrivial, and meromorphic continuation to $\mathbb{C}$ with simple poles at 0 and 1 for χ trivial. There is a function $\epsilon(s,\chi) = A\cdot B^s$ for $A \in \mathbb{C}^\times$ and $B \in \mathbb{R}$ such that*

$$L(s,\chi) = \epsilon(s,\chi)L(1-s,\chi^{-1}).$$

Proof. We fix f and S as above; setting $\epsilon(s,\chi) = \prod_{v\in S}\epsilon_v(s,\chi_v,\psi_v)$ gives the desired functional equation. Since $L(s,\chi)$ and $L(1-s,\chi^{-1})$ are independent of ψ, so is $\epsilon(s,\chi)$. It remains to show that it is of the claimed form.

We do this by proving that each ϵ_v is of the claimed form; since ϵ is a finite product, the result follows. By the definition of γ_v, we have

$$\epsilon_v(s,\chi_v,\psi_v) = \frac{\zeta_v(1-s,\chi_v^{-1},\hat{f}_v)}{L_v(1-s,\chi_v^{-1})}\cdot\frac{L_v(s,\chi_v)}{\zeta_v(s,\chi_v,f_v)}$$

for any Schwartz function f_v on K_v. In the nonarchimedean case, taking f_v to be the indicator function of $\mathcal{O}_v$, which is the indicator function (up to some scalar) of the inverse different ideal of $\mathcal{O}_v$; then one can compute $\zeta_v(s,\chi_v,f_v) = A_vL_v(s,\chi)$ for some constant $A_v \in \mathbb{C}^\times$, and $\zeta_v(1-s,\chi_v^{-1},\hat{f}) = A_v'B^{-s}L_v(1-s,\chi_v)$, from which the claim follows. In the archimedean case, choosing $f(x) = e^{-\pi x^2}$ in the real case or $e^{-2\pi|x|^2}$ in the complex case one can compute explicitly that ϵ_v is constant (indeed, for good choices it will be 1).

Finally, the location of the poles follows from Corollary 5 and our assumptions on χ. ☐

References

[1] Daniel Bump. *Automorphic forms and representations*. Number 55. Cambridge University Press, 1998.