# The Siegel-Weil formula*

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## 1. Classical Siegel-Weil

Recall† that the Siegel-Weil formula gives an identity between a certain Eisenstein series and a weighted average of theta functions. Specifically, to any $\mathbb{Z}$-lattice in a quadratic space $V \simeq \mathbb{Q}^{2k}$ we can associate an Eisenstein series $E_\Lambda$, in explicit cases given as the Eisenstein series of weight $k$ for a certain quadratic Dirichlet character associated to $\Lambda$ by the induced pairing on $\Lambda$. On the other hand, we can associate to $\Lambda$ a theta function

$$\Theta_\Lambda(z) = \sum_{x \in \Lambda} e^{2\pi i n \langle x, x \rangle},$$

which for $\Lambda$ unimodular is a modular form of weight $k$ for $\Gamma_1(4)$. In certain simple cases, $E_\Lambda$ is a modular form of the same type and the space is one-dimensional, so we can check the normalization to verify that they agree: in the simplest case where $\Lambda = \mathbb{Z}^2 \subset \mathbb{Q}^2$, this gives, after comparing Fourier coefficients, that the number of ways an integer $n \geq 1$ can be written as the sum of two squares is

$$4 \sum_{d \mid n} \chi(d)$$

where $\chi$ is the unique odd Dirichlet character modulo 4. More generally, the two series are not equal, but there is a more complicated relationship:

$$E_\Lambda = \frac{\sum_{\Lambda'} \frac{\Theta_{\Lambda'}}{\operatorname{Aut} \Lambda'}}{\sum_{\Lambda'} \frac{1}{\operatorname{Aut} \Lambda'}},$$

where the sums are over lattices $\Lambda'$ in the same genus as $\Lambda$, i.e. such that $\Lambda$ and $\Lambda'$ are isomorphic after tensoring with $\mathbb{Q}_p$ and $\mathbb{R}$ (as quadratic spaces, so the induced pairings must give equivalent structure on the resulting vector spaces).

We can promote this relationship to an adelic one, where it generalizes to higher-rank groups. There is an action of the orthogonal group $\mathrm{O}(V)$ on $V$ compatible with the pairing which acts diagonally on any power $V^n$, which induces an action on the space of Schwartz functions on $V(\mathbb{A})^n$; there is also a more complicated action of $\mathrm{Sp}(2n)$, given by the Weil representation $\omega$, whose description we omit. This lets us derive the theta series

$$\theta(g, h, \varphi) = \sum_{x \in V(\mathbb{A})^n} \omega(g)\varphi(h^{-1}x)$$

for $\varphi$ a Schwartz function on $V(\mathbb{A})^n$ (so that the sum is convergent), $g \in \mathrm{Sp}(2n)(\mathbb{A})$ and $h \in \mathrm{O}(V)(\mathbb{A})$. This is automorphic under the action of $\mathrm{Sp}(2n)(\mathbb{Q}) \times \mathrm{SO}(V)(\mathbb{Q})$ and so gives an automorphic form for both $\mathrm{Sp}(2n)$ and $\mathrm{O}(V)$. Treating it as an integral kernel allows us

*These notes are based on the expository paper [1].

†See `http://www.math.columbia.edu/~avizeff/notes/SW-examples`.

to lift automorphic forms from one group to the other: for example, if $\phi$ is an automorphic form on $\mathrm{Sp}(2n)(\mathbb{A})$, then we can define its theta lifting

$$\theta_\varphi(\phi)(h) = \langle \theta(-, h, \varphi), \phi \rangle = \int_{\mathrm{Sp}(2n)(\mathbb{Q})\backslash \mathrm{Sp}(2n)(\mathbb{A})} \theta(g, h, \varphi)\overline{\phi(g)}\, dg.$$

For $V$ positive-definite, we can look at lifting the constant function in the other direction

$$\theta_\varphi(1)(h) = \int_{\mathrm{O}(V)(\mathbb{Q})\backslash \mathrm{O}(V)(\mathbb{A})} \theta(g, h, \varphi)\, dg.$$

This is an automorphic form on $\mathrm{Sp}(2n)$, closely related to the classical theta series: choosing $\varphi$ self-dual, given at finite places by the indicator function of the base change to $p$ of a fixed lattice and at infinite places by the corresponding Gaussian, by a series of expansions one can find that (for the Haar measure $dh$ suitably normalized, so that $\mathrm{vol}(\mathrm{O}(V)(\mathbb{Q})\backslash \mathrm{O}(V)(\mathbb{A})) = 1$) this integral, rescaled by some factors $\chi_\infty(\det a)^{-1}|\det a|^{-k}$ to agree with the adelic automorphic form associated to classical modular forms, this is equal to the weighted average of theta functions from above.

We can also define a Siegel-Eisenstein series

$$E(g, s, \varphi) = \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{Sp}(2n)(\mathbb{Q})} \Phi_\varphi(\gamma g, s)$$

for $g \in \mathrm{Sp}(2n)(\mathbb{A})$ and $s \in \mathbb{C}$, where $P$ is the subgroup of upper triangular matrices and $\Phi_\varphi(g, s) = \omega(g)\varphi(0) \cdot |\det a|^{s - s_0}$, with $a$ the upper left entry of $g$ which is in general an $n \times n$ invertible matrix. One can similarly define a classical version, upon which this reduces to classical Siegel-Weil; in the case $n = 1$, so $\mathrm{Sp}(2n) = \mathrm{SL}_2$, the Siegel-Weil section is

$$\Phi_\varphi(\gamma g, s) = \frac{\chi(d)\sqrt{y}}{cz + d}\, \mathrm{Im}(\gamma z)^{s/2}$$

where $z = x + iy = g \cdot i$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so this essentially recovers the classical Eisenstein series (up to a factor of $\sqrt{y}$).

The adelic Siegel-Weil formula relates these adelic versions, just as in the classical case:

**Theorem 1.** *With notation as above, suppose that $V$ is anisotropic or the maximal isotropic subspace has codimension greater than $n+1$, then $E(g, s, \varphi)$ is holomorphic at $s_0 = \frac{\dim V - n - 1}{2}$ and*

$$E(g, s_0, \varphi) = \kappa \int_{\mathrm{O}(V)(\mathbb{Q})\backslash \mathrm{O}(V)(\mathbb{A})} \theta(g, h, \varphi)\, dh$$

*where $\kappa$ is $1$ for $\dim V > n + 1$ and $2$ otherwise.*

When $V$ is positive-definite, it is anisotropic and so by taking the classical forms recovers the more concrete Siegel-Weil formula.

Our goal is to extend this idea of a correspondence between Eisenstein and theta series, in this more abstract sense, to give a "geometric" formula, where we replace the theta series by something coming more directly from geometry, and in particular from certain special cycles on Shimura varieties, an an "arithmetic" formula, where we try to do something similar over $\mathrm{Spec}\, \mathbb{Z}$; in this case we need to replace the value of the Eisenstein series at its central point with its central derivative.

## 2. Geometric Siegel-Weil

Our motivating example will be the observation that the Hurwitz class number relation can be viewed as an instantiation of a sort of Siegel-Weil formula. We first need to explain what this formula is.

For any positive integer $D$, let $H(D)$ be the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of positive definite binary quadratic forms $ax^2 + bxy + cy^2$ with discriminant $b^2 - 4ac = -D$, with the count weighted by automorphisms: the forms $a(x^2 + y^2) = a(x + iy)(x - iy)$, with $D = 4a^2$, and $a(x^2 + xy + y^2) = a(x - \rho)(x - \bar{\rho})$, with $D = 3a^2$ (where $\rho = e^{2\pi i/3}$) have extra symmetry and so are given weight $\frac{1}{2}$ and $\frac{1}{3}$ respectively, related to the extra symmetry of $i$ and $\rho$ under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$. For example, there are two equivalence classes of discriminant $-12$, with representatives $2(x^2 + xy + y^2)$ and $x^2 + 3y^2$, so $H(12) = \frac{1}{3} + 1 = \frac{4}{3}$. If $D$ is not congruent to 0 or 3 modulo 4, then $H(D) = 0$ since $-D \equiv 2, 3 \pmod 4$ and $b^2 - 4ac \equiv b^2 \equiv 0, 1 \pmod 4$.

The Hurwitz class number relation is the following.

**Theorem 2.** *If $m$ is not a perfect square, then*

$$\sum_{\substack{t \in \mathbb{Z} \\ 4m - t^2 > 0}} H(4m - t^2) = \sum_{d \mid m} \max(d, m/d).$$

Hurwitz's proof proceeds by observing that the right-hand side is $\deg \Phi_m(x, x)$, where $\Phi_m(x, y)$ is the modular polynomial whose zero locus defines the modular curve $Y_0(m)$, and computing this degree in another way via Hurwitz class numbers. We can give a more geometric proof, again using modular curves.

Let $Y = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$ be the modular curve, whose complex points correspond to complex elliptic curves up to isomorphism, with $j$-invariant $j : Y \xrightarrow{\sim} \mathbb{A}^1$. Consider the surface $X = Y \times Y$, whose points parametrize isomorphism classes of pairs of elliptic curves. For each positive integer $m$ we get a correspondence $Z(m)$ over $X$ whose points parametrize triples $(E_1, E_2, \phi)$, where $\phi : E_1 \to E_2$ is an isogeny of degree $m$, and the map to $X$ is given by forgetting $\phi$. This gives a divisor of $X$, since the existence of $\phi$ determines a one-dimensional condition. For example, when $m = 1$ this is just the diagonal embedding $Y \hookrightarrow X = Y \times Y$. In particular, we expect that $Z(m) \cap Z(1)$ should be zero-dimensional, and so have a well-defined degree, which is the intersection number $\langle Z(m), Z(1) \rangle_X$.

Now, $j : Y \xrightarrow{\sim} \mathbb{A}^1$ has a natural compactification $j : \overline{Y} \xrightarrow{\sim} \mathbb{P}^1$ and so $j \times j$ gives a natural compactification $\overline{X} \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The diagonal on $\mathbb{P}^1$ is given by the motivic decomposition $Z(1) \sim \{*\} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{*\}$, and so

$$\left\langle \overline{Z(m)}, \overline{Z(1)} \right\rangle_{\overline{X}} = \left\langle \overline{Z(m)}, \{*\} \times \mathbb{P}^1 \right\rangle_{\overline{X}} + \left\langle \overline{Z(m)}, \mathbb{P}^1 \times \{*\} \right\rangle_{\overline{X}}.$$

Now, the intersection of $\overline{Z(m)}$ with $\{*\} \times \mathbb{P}^1$ consists of pairs $(E_1, E_2, \phi) \in \overline{Z(m)}$ consisting of generalized (i.e. possibly singular) elliptic curves with an isogeny of degree $m$ between them where $E_1$ is restricted (by the $j$-invariant) to the chosen point $*$, so this is the number of isogenies of degree $m$ from a fixed elliptic curve.

**Lemma 3.** *There are $\sum_{d \mid m} d$ isogenies from a fixed elliptic curve over $\mathbb{C}$.*

*Proof, due to Kyle Hofmann.* Write $g(m)$ for the number of such isogenies. Our strategy is to find

$$f(m) = \sum_{d|m} \mu(d) g(m/d),$$

and compute $g$ by Möbius inversion. Counting isogenies is the same as counting subgroups of order $m$, whose elements are $m$-torsion and so are contained in $E[m] \simeq (\mathbb{Z}/m\mathbb{Z})^2$, so $g(m)$ is the number of order $m$ (or equivalently index $m$) subgroups of $(\mathbb{Z}/m\mathbb{Z})^2$. This is multiplicative, by the classification of finite abelian groups, so it suffices to reduce to the case where $m = p^k$ is a prime power, where

$$f(m) = g(p^k) - g(p^{k-1}).$$

The map $(\mathbb{Z}/m\mathbb{Z})^2 \to (\mathbb{Z}/m\mathbb{Z})^2$ sending $(a, b) \mapsto (pa, b)$ sends a subgroup of index $p^{k-1}$ to a (unique) subgroup of index $p^k$, and so the set of subgroups of $(\mathbb{Z}/p^k\mathbb{Z})^2$ of index $p^{k-1}$ injects into the set of subgroups of index $p^k$, contributing a term of $g(p^{k-1})$. The remaining subgroups are those containing a (unique) element of the form $(1, y)$ for some $y \in \mathbb{Z}/p^k\mathbb{Z}$, of which there are $p^k$, so $g(p^k) = g(p^{k-1}) + p^k$ and therefore $f(p^k) = p^k$, i.e. $f$ is the identity. Therefore

$$g(m) = \sum_{d|m} f(d) = \sum_{d|m} d.$$

□

Essentially the same argument works for isogenies with fixed target, so

$$\left\langle \overline{Z(m)}, \overline{Z(1)} \right\rangle_{\overline{X}} = 2 \sum_{d|m} d = \sum_{d|m} \left( d + \frac{m}{d} \right).$$

It remains to account for the points at infinity, which for reasons I don't as yet fully understand contribute $\min(d, m/d)$ at each $d$ and therefore leave a total of

$$\sum_{d|m} \max(d, m/d).$$

On the other hand, the intersection of $Z(m)$ and $Z(1)$ classifies isogenies $E \to E$ of degree $m$. If $m$ is a square, such an isogeny is given by multiplication by its square root, but if not, as we assume, then such isogenies only exist when $E$ has complex multiplication by some imaginary quadratic order $\mathbb{Z}[\alpha]$ with $\alpha\overline{\alpha} = m$. Thus we want to count elliptic curves with complex multiplication by some such order $\mathbb{Z}[\alpha]$. If $\alpha$ has defining equation $\alpha^2 + t\alpha + m = 0$ then $t^2 - 4m < 0$, and the number of isomorphism classes of orders of each such discriminant is $H(t^2 - 4m)$ and so the total number of orders, and thus the total number of isogenies $E \to E$ of order $m$, is

$$\sum_{t: 4m - t^2 > 0} H(4m - t^2).$$

Therefore these two sums are equal and so we get the Hurwitz class number relation.

We can reinterpret again: the numbers $H(4m - t^2)$ appear as Fourier coefficients of a certain Eisenstein series. In particular, for $G = \mathrm{Sp}(4)$ we have the Siegel-Eisenstein series of weight 2

$$E(\tau, s) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in P \backslash \mathrm{Sp}(4)} \det(C\tau + D)^{-2} \frac{\det(\operatorname{Im} \tau)^{s - \frac{1}{2}}}{|\det(C\tau + D)|^{2s-1}}$$

for $\tau$ in the Siegel upper half-plane, which has a Fourier expansion of the form

$$E(\tau, s) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}} E_T(\tau, s) q^T,$$

where the sum is over symmetric positive-definite matrices over $\mathbb{Z}$. When $T = \begin{pmatrix} m & t/2 \\ t/2 & 1 \end{pmatrix}$, the Fourier coefficient at $s = s_0 = \frac{1}{2}$ is $H(4m - t^2)$. In other words, the Hurwitz class number relation comes from a relationship between a special value of an Eisenstein series (rather than the whole series as before) and a *geometric* theta series, i.e. generated by intersection numbers of special cycles on Shimura varieties. In this case the algebraic group on the theta series side is $\mathrm{O}(2, 2)$, which has an accidental isomorphism to $\mathrm{SL}_2 \times \mathrm{SL}_2 / \{\pm 1\}$ and therefore has Shimura variety given by a product of modular curves.

We can generalize this picture. Let $F$ be a totally real number field and $V$ a vector space of dimension $m$ over $F$ equipped with a quadratic form; fix a real place $w$ of $F$, and suppose that $V_v$ is positive-definite at $v | \infty$ for every $v \neq w$, and has signature $(m-2, 2)$ at $w$. Rather than the orthogonal Shimura variety, it is better for technical reasons to use the Shimura variety associated to $G = \mathrm{GSpin}(V)$, which is an extension of $\mathrm{O}(V)$ by $\mathbb{G}_\mathrm{m}$ (this gives a Shimura variety of Hodge type, rather than merely of abelian type). The associated Shimura variety $\mathrm{Sh}_G$ has a model over $F$ (embedded into $\mathbb{C}$ by $w$) of dimension $m - 2$, with complex uniformization

$$\mathrm{Sh}_{G,K}(\mathbb{C}) = G(F) \backslash \mathcal{D} \times G(\mathbb{A}_{F,f})/K$$

for $\mathcal{D}$ a certain Hermitian symmetric domain (given by a certain Grassmannian in $V_w$) and $K$ is any compact open subgroup $K$ of $G(\mathbb{A}_{F,f})$. We want to build special divisors $Z(m)$ on $\mathrm{Sh}_{G,K}$ in analogy to those above. In the case where $\mathrm{Sh}_{G,K}$ is one-dimensional, this will recover the theory of Heegner points; in the case above we get modular correspondences.

For any $v \in V$ with $\langle v, v \rangle > 0$, its orthogonal complement $v^\perp$ in $V$ has codimension 1 and so we get an inclusion $G_v = \mathrm{GSpin}(v^\perp) \hookrightarrow \mathrm{GSpin}(V) = G$, giving a Shimura subvariety $\mathrm{Sh}_{G_v}$ for $v$ of codimension 1. For any $x \in V(\mathbb{A}_{F,f})$ with $\langle x, x \rangle$ a totally positive element of $F$, we can find $g \in G(\mathbb{A}_f)$ such that $gx = v$ and so can associate to $x$ another subvariety $\mathrm{Sh}_{G_v} g$ in $\mathrm{Sh}_G$. We call this divisor $Z(x)$, the special divisor for $x$. For $\underline{x} = (x_1, \dots, x_n) \in V(\mathbb{A}_{F,f})^n$ whose associated matrix $\langle \underline{x}, \underline{x} \rangle$, with entries $\langle x_i, x_j \rangle$, is symmetric and positive-definite over $F$, we associate the cycle $Z(\underline{x}) = Z(x_1) \cap \cdots \cap Z(x_n)$, which is codimension $n$ in $\mathrm{Sh}_G$. Here $\cap$ denotes the fiber product over $\mathrm{Sh}_G$.

More generally, for any $K$-invariant Schwartz function $\varphi$ on $V(\mathbb{A}_{F,f})^n$ we could look at the weighted sum $\sum_{\underline{x} \in K \backslash V(\mathbb{A}_{F,f})^n} \varphi(\underline{x}) Z(\underline{x})$; we can partition this sum into terms such that $\langle \underline{x}, \underline{x} \rangle$ is a fixed positive-definite symmetric matrix $T$, and we define this to be the weighted special cycle

$$Z_\varphi(T) = \sum_{\substack{\underline{x} \in K \backslash V(\mathbb{A}_{F,f})^n \\ \langle \underline{x}, \underline{x} \rangle = T}} \varphi(\underline{x}) Z(\underline{x}).$$

This is valued in the Chow group tensored with $\mathbb{C}$ in codimension $n$, i.e. $\mathrm{CH}^n(\mathrm{Sh}_{G,K}) \otimes \mathbb{C}$. As before, we can assemble these into a Fourier series

$$Z_\varphi(\tau) = \sum_{T \in \mathrm{Sym}_n(\mathbb{Z})_{\geq 0}} Z_\varphi(T) q^T,$$

for $\tau$ in the Siegel upper half-plane and $q^T$ denoting $\prod_{v|\infty} e^{2\pi i \operatorname{tr}(T\tau_v)}$, valued in $\mathrm{CH}^n(\mathrm{Sh}_{G,K}) \otimes \mathbb{C}$. If this is a good analogy to classical theta series, we should expect it to be modular:

**Conjecture 4.** *The generating series $Z_\varphi(\tau)$ converges absolutely to a modular form on the Siegel upper half-plane $\mathcal{H}_n$ of weight $m/2$ valued in $\mathrm{Ch}^n(\mathrm{Sh}_G) \otimes \mathbb{C}$.*

This is known in some special cases, but not in general. One can also formulate a similar story in the unitary rather than orthogonal setting.

In the special case $n = \dim \mathrm{Sh}_G$, the series $Z_\varphi$ is valued in 0-cycles, and so composing with the degree map gives a numerical generating series $\deg Z_\varphi(\tau)$, whose terms encode intersection numbers between special cycles, generalizing the case $\mathrm{Sh}_G = Y \times Y$. In this case $n = \dim \mathrm{Sh}_G = m - 2$ and so the central point $s_0 = \frac{m-n-1}{2} = \frac{1}{2}$, where the geometric Siegel-Weil formula, due to Kudla, is the following.

**Theorem 5.** *Suppose that $V$ is anisotropic, or equivalently that $\mathrm{Sh}_G$ is projective, and let $n = \dim \mathrm{Sh}_G$. Then for any Schwartz function $\varphi$ on $V(\mathbb{A}_{F,f})^n$ we have (for a suitable choice of measures)*

$$Z_\varphi(\tau) = E(\tau, 1/2, \varphi \cdot \varphi_\infty)$$

*for $\varphi_\infty$ a certain explicit Schwartz function on $V(\prod_{v|\infty} F_v)^n$.*

In particular we get a concrete relationship between this geometric theta series for $\mathrm{GSpin}(V)$ and the Siegel-Eisenstein series for $\mathrm{Sp}(2n)$.

In fact, Kudla proved a version of this statement for arbitrary codimension, but we focus on the case of 0-cycles for concreteness.

## 3. Arithmetic Siegel-Weil

We can try and do the same story over $\operatorname{Spec} \mathbb{Z}$ rather than $\mathbb{Q}$ (or $F$). Let's try to do the motivating example of $Y \times Y$ from the previous section in this setting. We can give an integral model of the modular curve $Y$ over $\operatorname{Spec} \mathbb{Z}$, and the product $X = Y \times_{\operatorname{Spec} \mathbb{Z}} Y$ is an arithmetic threefold. One can define the special divisors $Z(m)$ in the same way; now to get down to something of expected dimension 0 we need to take the intersection of three divisors rather than two. At each prime $p$ the fibers of $Y$ are one-dimensional, so the fibers of the product are two-dimensional and $Z(m)_p$ is a one-dimensional variety over $\mathbb{F}_p$, so the threefold intersection is nonempty for all but finitely many primes and we can define the arithmetic intersection number

$$\langle Z(m_1), Z(m_2), Z(m_3)\rangle_X = \sum_p \langle Z(m_1)_p, Z(m_2)_p, Z(m_3)_p\rangle_{X_p} \log p$$

so long as the divisors intersect properly (or more generally so long as their intersection is supported in finitely many fibers).

Like in the rational case, we want to put the indices of the special cycles on the diagonal of a symmetric matrix, analogous to $T = \begin{pmatrix} m & t/2 \\ t/2 & 1 \end{pmatrix}$, and so we're looking to sum over $\mathrm{Sym}_3(\mathbb{Z})_{\geq 0}$. This corresponds to the Eisenstein series for $\mathrm{Sp}(6)$ rather than $\mathrm{Sp}(4)$. The central point is at $s = 0$, which we might hope to be a rational number given by the intersection number as before. However, there are two problems: first, the arithmetic intersection number involves factors of $\log p$, and so is not rational in the first place. Second, the functional equation for this Siegel-Eisenstein series has odd sign and so $E(\tau, 0) = 0$, so we can't hope to learn anything from studying this value.

Given that this value vanishes, though, together with the appearance of logarithmic terms, it's natural to look instead at the first derivative of the Eisenstein series. After making this replacement and restricting to $m_1, m_2, m_3$ not all represented by a single positive-definite binary quadratic form (analogous to requiring $m$ not be a perfect square, ensuring that the divisors intersect properly), we get the following formula analogous to the (reinterpreted) Hurwitz relation.

**Theorem 6.** *With notations and assumption as above, we have (up to an explicit constant)*

$$\langle Z(m_1), Z(m_2), Z(m_3)\rangle_X = \sum_{T = \begin{pmatrix} m_1 & * & * \\ * & m_2 & * \\ * & * & m_3 \end{pmatrix} \in \mathrm{Sym}_3(\mathbb{Z})_{>0}} E'_T(\tau, 0).$$

This is an instance of an identity between an arithmetic intersection number and a value of the derivative of a Siegel-Eisenstein series. We can similarly generalize it through an *arithmetic* theta series, whose coefficients encode arithmetic intersection numbers: we replace $X$ by an integral model of a Shimura variety over $\operatorname{Spec}\mathbb{Z}$ for an orthogonal (or unitary) group, and define special divisors $Z(m)$ on $X$ as above. Again, we can take the intersection of $n = \dim X$ special divisors (note that this is one greater than in the rational case), which we can similarly decompose into special cycles $Z_T$ from each $T$ as above, which itself decomposes into parts from each prime. Then the arithmetic Siegel-Weil formula on each Fourier coefficient is a (conjectural) identity, up to a constant depending on choices of measure,

$$\langle Z(m_1), \ldots, Z(m_n)\rangle_T = E'_T(\tau, 0).$$

In general, the intersection is not proper and we need to use derived intersections to get something in the expected dimension.

This has been proven by Chao Li and Wei Zhang in the unitary case, with slightly weaker results in the orthogonal case.

One should be able to find an archimedean term as well giving a full arithmetic Siegel-Weil formula

$$\deg Z_\varphi(\tau) = \sum_T \deg Z_\varphi(T) q^T = E'(\tau, 0, \varphi \otimes \varphi_\infty)$$

for some suitable $\varphi_\infty$, where $Z_\varphi(T)$ is the weighted special cycle from the contribution at $T$. This has been shown in some special cases but is still open in general.

To prove this formula, one reduces to the local terms: the intersection number decomposes into local terms $\langle Z(m_1), \ldots, Z(m_n)\rangle_{Z(T)_p}$, which can be reinterpreted as intersection numbers on Rapoport-Zink spaces. On the other side, the Eisenstein series has an Euler product expansion, and so its derivative can be expressed as a sum over primes; therefore the theorem reduces to the claim that the contributions at each prime are the same. The local contribution on the Eisenstein side can be further reduced to a local density, which can be compared to the intersection number via local harmonic analysis, and in particular the uncertainty principle: one can show that the difference between the two sides has small support in a certain sense, and at least in principle has computable Fourier transform which can also be shown to have small support. But the uncertainty principle states that a function with small support whose Fourier transform also has small support must be zero. There are significant complications coming from singularities, which are resolved via deformation theory and some explicit calculations.

## References

[1] Chao Li. From sum of two squares to arithmetic Siegel-Weil formulas. *arXiv preprint arXiv:2110.07457*, 2021.