Shimura data and varieties*

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1. Connected Shimura varieties

Let G be a reductive algebraic group over $\mathbb{Q}$. Fix an embedding $G \hookrightarrow \mathrm{GL}(n)$, and let $\Gamma(N) \subset G(\mathbb{Z}) \subset G(\mathbb{Q})$ be the subgroup of $g \in G(\mathbb{Z})$ whose image in $\mathrm{GL}_n(\mathbb{Z}) \subset \mathrm{GL}_n(\mathbb{Q})$ is in the kernel of the reduction map $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})$, i.e. $g \equiv I_n \pmod{N}$. We say that a subgroup $\Gamma \subset G(\mathbb{Q})$ is a congruence subgroup if it contains some $\Gamma(N)$ as a finite index subgroup. This turns out to be independent of the chosen embedding.

Proposition 1.1. *Let K be a compact open subgroup of $G(\mathbb{A}_f)$. Then $G(\mathbb{Q}) \cap K$ is a congruence subgroup of $G(\mathbb{Q})$, and every congruence subgroup is of this form.*

Proof. Fix an embedding $G \hookrightarrow \mathrm{GL}_n$. This corresponds to a surjection of group algebras $\mathbb{Q}[\mathrm{GL}_n] \twoheadrightarrow \mathbb{Q}[G]$, and we have the presentation

$$\mathbb{Q}[\mathrm{GL}_n] \simeq \mathbb{Q}[X_{11}, X_{12}, \ldots, X_{nn}, T]/(\det(X_{ij})T - 1)$$

and thus can write $\mathbb{Q}[G]$ as $\mathbb{Q}[x_{11}, x_{12}, \ldots, x_{nn}, t]$ with x_{ij} the image of X_{ij} and t the image of T (so there are some relations omitted from the notation). In particular, $G(\mathbb{Z}_\ell)$ is the subgroup of $G(\mathbb{Q}_\ell)$ consisting of elements which in this presentation are defined over $\mathbb{Z}_\ell$, i.e. $a_{11}x_{11} + a_{12}x_{12} + \cdots + a_{nn}x_{nn} + bt$ for $a_{ij}, b \in \mathbb{Z}_\ell$; in other words $G(\mathbb{Z}_\ell) = G(\mathbb{Q}_\ell) \cap \mathrm{GL}_n(\mathbb{Z}_\ell)$. For any positive integer N we can then define $K_\ell(N)$ to be $G(\mathbb{Z}_\ell)$ for $\ell \nmid N$ and the congruence subgroup of $G(\mathbb{Z}_\ell)$ consisting of g which are congruent to the identity modulo $\ell^{v_\ell(N)}$ for $\ell \mid N$, and $K(N) = \prod_\ell K_\ell(N)$. For any compact open subgroup K of $G(\mathbb{A}_f)$, by choosing N sufficiently large we can get $K \supseteq K(N)$, which is automatically of finite index, and so $G(\mathbb{Q}) \cap K$ contains $G(\mathbb{Q}) \cap K(N)$ as a finite index subgroup. But $G(\mathbb{Q}) \cap K(N)$ is just $\Gamma(N)$. □

Let $G^{\mathrm{ad}} = G/Z$ be the quotient of G by its center, e.g. if $G = \mathrm{GL}(2)$ then $G^{\mathrm{ad}} = \mathrm{PGL}(2)$. For any group H write H^+ for the connected component of the identity,[1] and if H is defined over $\mathbb{Q}$ let $H_\mathbb{R}$ be its base change to $\mathbb{R}$.

Definition 1.2. A connected Shimura datum is a pair (G, D) for a semisimple algebraic group G over $\mathbb{Q}$ and a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class D of homomorphisms $u : \mathrm{U}(1) \to G^{\mathrm{ad}}_\mathbb{R}$ satisfying the following conditions:

(1) for every $u \in D$, the only characters appearing in the representation $\mathrm{Ad} \circ u : \mathrm{U}(1) \to \mathrm{Lie}(G^{\mathrm{ad}})_\mathbb{C}$ induced by u and the adjoint representation $\mathrm{Ad} : G^{\mathrm{ad}} \to \mathrm{Lie}(G^{\mathrm{ad}})_\mathbb{C}$ are $z \mapsto z^i$ for $i \in \{-1, 0, 1\}$;

(2) for every $u \in D$, $\mathrm{ad}(u(-1))$ is a Cartan involution on $G^{\mathrm{ad}}_\mathbb{R}$;

*These notes are based on chapters 4-5 of [1].

[1]For non-connected rings like $\mathbb{Q}$, take $H(\mathbb{Q})^+ = H(\mathbb{Q}) \cap H(\mathbb{R})^+$.

(3) G has no $\mathbb{Q}$-factor H such that $H(\mathbb{R})$ is compact.

Given a connected Shimura datum (G, D), D has an action of $G^{\mathrm{ad}}(\mathbb{R})^+$ by conjugation with compact kernel. In particular, every arithmetic subgroup Γ of $G^{\mathrm{ad}}(\mathbb{R})^+$ acts on D to give a quotient $\Gamma\backslash D$ with the structure of an algebraic variety $\mathrm{Sh}^\circ_\Gamma(G, D)$, called the connected Shimura variety of level Γ attached to (G, D). Inclusions $\Gamma \subset \Gamma'$ induce regular maps $\Gamma'\backslash D \to \Gamma\backslash D$, giving an inverse system denoted $\mathrm{Sh}^\circ(G, D)$, the connected Shimura variety attached to (G, D).

The strong approximation theorem says that $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_f)$ provided that G is semisimple, simply connected, and of noncompact type. We can use this to find an adelic description of $\mathrm{Sh}_\Gamma(G, D)$.

Proposition 1.3. *Let (G, D) be a connected Shimura datum with G simply connected, K be a compact open subgroup of $G(\mathbb{A}_f)$, and $\Gamma = G(\mathbb{Q}) \cap K$. The map $x \mapsto [x, 1]$ gives a homeomorphism*

$$\Gamma\backslash D \simeq G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K,$$

where $G(\mathbb{Q})$ acts on $D \times G(\mathbb{A}_f)$ by $h \cdot (x, g) = (hx, hg)$ for the action of $G(\mathbb{Q}) \subset G(\mathbb{R})$ on D by $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugation and K acts by $(x, g) \cdot k = (x, gk)$.

Proof. By strong approximation, we have $G(\mathbb{A}_f) = G(\mathbb{Q}) \cdot K$, so every $(x, g) \in D \times G(\mathbb{A}_f)$ can be written as (x, hk) for $h \in G(\mathbb{Q})$ and $k \in K$, which is thus in $[x, 1]$, so this map is surjective (since $\Gamma \subset G(\mathbb{Q})$, restricting to the D part gives a surjection $\Gamma\backslash D \to G(\mathbb{Q})\backslash D$). To see that it is injective, suppose that $[x, 1] = [x', 1]$. By definition this means that there is some $h \in G(\mathbb{Q})$ and $k \in K$ such that $(hx, hk) = (x', 1)$. The second coordinate implies that $K \ni k = h^{-1} \in G(\mathbb{Q})$, so h and k are both in $G(\mathbb{Q}) \cap K = \Gamma$, so x and x' differ by an element of Γ as desired.

Since K is open, $G(\mathbb{A}_f)/K$ is discrete, so since the map $D \to D \times G(\mathbb{A}_f)/K$ sending $x \mapsto (x, [1])$ is continuous and injective it must be a homeomorphism from D onto its image, which is an open subset. Quotienting by a discrete group does not change this, so $x \mapsto [x, 1]$ is a homeomorphism onto its image, which is the whole right-hand side since this map is surjective. □

There are several reasons we might want to rewrite the apparently simpler quotient on the left as the more complicated double quotient on the right. One is that it makes clear that both sides carry a natural action of $G(\mathbb{A}_f)$, which is not obvious from the left-hand side. Another is to understand what happens in the limit as we let Γ, or equivalently K, range over all congruence/compact open subgroups.

Proposition 1.4. *Taking the limit over compact open subgroups K of $G(\mathbb{A}_f)$ gives*

$$\varprojlim_K G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f).$$

Correspondingly both sides are homeomorphic to the limit $\varprojlim_\Gamma \Gamma\backslash D$ over congruence subgroups Γ of $G(\mathbb{Q})$.

The proposition follows from the following topological lemma.

Lemma 1.5. *Let G be a topological group acting continuously on a topological space X, with a directed system $\{G_i\}$ of subgroups of G. Then the canonical map $h : X/\bigcap G_i \to \varprojlim X/G_i$ is continuous. Further, if the stabilizer of every $x \in X$ in every G_i is compact then h is injective, and if the orbit xG_i of every $x \in X$ under every G_i is compact then h is surjective.*

If we can verify these conditions in our case, then we can apply the lemma to see that the left-hand side is canonically isomorphic to $G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/\bigcap K = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)$ since the intersection of all compact subgroups of $G(\mathbb{A}_f)$ is trivial.

Since every K is compact, the orbits are necessarily compact. If further the orbits are Hausdorff, then the stabilizer of any x is a closed subgroup of K and therefore also compact, so we just need to show that $[x, g] \cdot K$ is Hausdorff at least for K sufficiently small, i.e. that any two points $[x, g]$ and $[x, g']$ in this orbit (so $g' = gk$ for some $k \in K$) are separated. We would like to find neighborhoods in $G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)$ of $[x, g]$ and $[x, g']$ respectively which are disjoint; to do so, we'll find a neighborhood V of x such that $V \times Kg$ and $V \times Kg' = V \times Kgk$ have images modulo $G(\mathbb{Q})$ giving the desired neighborhoods. Since only $G(\mathbb{Q})$ acts nontrivially on D, we want to restrict to $G(\mathbb{Q})$ intersected with some compact open K'; call this Γ. By shrinking K' to a finite index open subgroup, we can ensure that Γ is torsion-free and acts nicely on D, i.e. in particular for every x there is a neighborhood V such that $h \cdot V \cap V = \{\}$ for every $1 \neq h \in \Gamma$; this will be our V. For $h \in G(\mathbb{Q})$, suppose that $h \cdot (V \times gK) \cap (V \times g'K)$ is nonempty, i.e. contains some element $(z, j') \in V \times g'K$ which is equal to (hz, hj) for some $(z, j) \in V \times gK$. We have $hgK \ni hj = j' \in g'K = gkK = gK$, so $h \in gKg^{-1}$; if we set $K' = gKg^{-1}$, it follows that $h \in \Gamma$ and therefore $hz = z$ is impossible unless $h = 1$, so $h \cdot (V \times gK) \cap (V \times g'K) = \{\}$ unless $h = 1$. In particular this means that the images of $V \times Kg$ and $V \times Kg'$ separate $[x, g]$ and $[x, g']$.

It remains only to prove the lemma.

Proof of Lemma 1.5. The first assertion is straightforward: the inverse limit of continuous maps is continuous. Suppose that the stabilizer of every x for every G_i is compact. For every $x, x' \in X$, let $G_i(x, x')$ be the set of $g \in G_i$ such that $xg = x'$. For some fixed $g_0 \in G_i(x, x')$, note that every other $g \in G_i(x, x')$ gives rise to a unique element gg_0^{-1} such that $xgg_0^{-1} = x'g_0^{-1} = x$, i.e. a unique element of the stabilizer of x, so if $G_i(x, x')$ is nonempty it is homeomorphic to the stabilizer of x in G_i and so compact by assumption. If x and x' have the same image in $\lim X/G_i$, then (for all sufficiently small G_i) the $G_i(x, x')$ are nonempty, so their intersection is nonempty (since in this direction it is an inverse limit, and the inverse limit of nonempty compact sets is nonempty). Thus there exists some g in this intersection such that $xg = x'$, i.e. the map h is injective.

Finally, suppose that each orbit xG_i is compact. For any $(x_iG_i)_i \in \varprojlim X/G_i$, it follows as above that the inverse limit of the orbits is nonempty, i.e. there is some $x \in \varprojlim x_iG_i$. Then the image of x modulo the intersection of the G_i is the element $(x_iG_i) \in \varprojlim X/G_i$, i.e. it is in the image of h. □

We can also define connected Shimura data in a slightly different way which will generalize better. Let $\mathbb{S} = \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ be the Deligne torus, i.e. the real algebraic group with $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $S(\mathbb{C}) = \mathbb{C} \times \mathbb{C}$. This is equipped with a map $w : \mathbb{G}_m \to \mathbb{S}$ given by inversion. The short exact sequence

$$1 \to \mathbb{R}^\times \xrightarrow{x \mapsto x^{-1}} \mathbb{C}^\times \xrightarrow{z \mapsto z/\bar{z}} \mathrm{U}(1) \to 1$$

then comes from the short exact sequence of tori

$$0 \to \mathbb{G}_{\mathrm{m}} \xrightarrow{w} \mathbb{S} \to \mathrm{U}(1) \to 0.$$

For any semisimple real algebraic group H with trivial center (for example, $G^{\mathrm{ad}}_{\mathbb{R}}$), a homomorphism $u : \mathrm{U}(1) \to H$ gives a homomorphism $h : \mathbb{S} \to H$ by composing, i.e. $h(z) = u(z/\bar{z})$; then the condition that $\mathrm{U}(1)$ act on $\mathrm{Lie}(H)_{\mathbb{C}}$ via characters of weight -1, 0, or 1 is the same as requiring that $\mathbb{S}$ act on $\mathrm{Lie}(H)_{\mathbb{C}}$ through the characters $z/\bar{z}$, 1, or $\bar{z}/z$. Conversely, if $h : \mathbb{S} \to H$ is a homomorphism such that $\mathbb{S}$ acts on $\mathrm{Lie}(H)_{\mathbb{C}}$ through these characters, then $w(\mathbb{G}_{\mathrm{m}})$ acts trivially and so (since the adjoint representation $H \to \mathrm{Lie}(H)$ is faithful) it follows that h is trivial on $w(\mathbb{G}_{\mathrm{m}})$ and therefore descends to a homomorphism from $\mathbb{S}/w(\mathbb{G}_{\mathrm{m}}) = \mathrm{U}(1)$, i.e. a homomorphism $\mathrm{U}(1) \to H$ with the above properties.

Therefore if G is a semisimple algebraic group over $\mathbb{Q}$, a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class D of homomorphisms $u : \mathrm{U}(1) \to G^{\mathrm{ad}}_{\mathbb{R}}$ satisfying the conditions (1), (2), and (3) of Definition 1.2 is the same thing as a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class X^+ of homomorphisms $h : \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ satisfying the following conditions:

(1) for every $h \in X^+$, only the characters $z/\bar{z}$, 1, and $\bar{z}/z$ occur in the representation of $\mathbb{S}$ on $\mathrm{Lie}(H)_{\mathbb{C}}$ defined by $\mathrm{Ad} \circ h$;

(2) for every $h \in X^+$, $\mathrm{ad}(h(i))$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$;

(3) G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Therefore we can give the following equivalent definition to Definition 1.2.

Definition 1.6. A connected Shimura datum is a pair (G, X^+) for a semisimple algebraic group G over $\mathbb{Q}$ and a $G^{\mathrm{ad}}(R)^+$-conjugacy class X^+ of homomorphisms $h : \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ satisfying conditions (1), (2), and (3) above.

2. Shimura varieties

We now want to remove the adjective "connected." There are a few reasons to do this: for one thing, it lets us work with reductive groups instead of only semisimple ones; for another, it lets us get models over $\mathbb{Q}$ instead of some number field depending on the level. For example, the (models of the) modular curves $X(N) = \Gamma(N)\backslash\mathscr{H}$ are defined over $\mathbb{Q}[\zeta_N]$, and so the limit is not defined over any number field. This is typical of the situation for the variety structure of any $\Gamma\backslash D$. To fix this situation, we need multiple connected components (over $\mathbb{C}$) so that the total variety is defined over $\mathbb{Q}$.

The first result we need is the following.

Proposition 2.1. *For a reductive group G over $\mathbb{R}$, $G(\mathbb{R})$ has finitely many connected components.*

Proof. We first need a lemma (Lemma 2.2 below): a surjection of algebraic groups $G \twoheadrightarrow H$ induces a surjection $G(\mathbb{R})^+ \twoheadrightarrow H(\mathbb{R})^+$, i.e. $G \mapsto G(\mathbb{R})^+$ is right-exact and so $G \mapsto G(\mathbb{R})/G(\mathbb{R})^+ = \pi_0(G)$ is . Therefore for any short exact sequence of algebraic groups

$$1 \to N \to G' \to G \to 1$$

taking $H \mapsto H(\mathbb{R})^+$ gives an exact sequence

$$N(\mathbb{R})^+ \to G'(\mathbb{R})^+ \to G(\mathbb{R})^+ \to 1.$$

Suppose that $N \subset Z(G')$, so in particular it is abelian (so group cohomology is well-behaved). Taking group cohomology for $G_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$, we get an exact sequence

$$\pi_0(G'(\mathbb{R})) \to \pi_0(G(\mathbb{R})) \to H^1(G_{\mathbb{R}}, N).$$

If T is the largest commutative quotient of G, we define G^{der} to be the kernel of $G \twoheadrightarrow T$. This is isogenous to G^{ad} with finite kernel $Z \cap G^{\mathrm{der}}$, the center of G^{der}, giving rise to a short exact sequence

$$1 \to Z \cap G^{\mathrm{der}} \to Z \times G^{\mathrm{der}} \to G \to 1.$$

Applying the above machinery gives an exact sequence

$$\pi_0(Z(\mathbb{R}) \times G^{\mathrm{der}}(\mathbb{R})) \to \pi_0(G) \to H^1(G_{\mathbb{R}}, Z \cap G^{\mathrm{der}}),$$

and since both $G_{\mathbb{R}}$ and $Z \cap G^{\mathrm{der}}$ are finite so is the cohomology group. The same holds true if we replace G^{der} by any finite cover, so in particular if we replace it by its universal cover $\tilde{G}$ we get an exact sequence

$$\pi_0(Z(\mathbb{R}) \times \tilde{G}(\mathbb{R})) = \pi_0(Z(\mathbb{R})) \times \pi_0(\tilde{G}(\mathbb{R})) \to \pi_0(G(\mathbb{R})) \to H^1(G_{\mathbb{R}}, Z \cap G^{\mathrm{der}}).$$

Since Z has a finite index subgroup which is a quotient by a finite group of some product of copies of $\mathrm{U}(1)$ and $\mathbb{G}_{\mathrm{m}}$, $Z(\mathbb{R})$ must have finitely many connected components; and by a theorem of Cartan for a semisimple simply connected algebraic group $\tilde{G}$ over $\mathbb{R}$, $\tilde{G}(\mathbb{R})$ is simply connected. Therefore the left and right are finite and so so is the middle term. □

To complete the argument, we need a lemma:

Lemma 2.2. *Let $\varphi : G \to H$ be a surjection of algebraic groups over $\mathbb{R}$. Then the induced map $\varphi(\mathbb{R})^+ : G(\mathbb{R})^+ \to H(\mathbb{R})^+$ is surjective.*

This is not obvious: the analogous result for $\varphi(\mathbb{R}) : G(\mathbb{R}) \to H(\mathbb{R})$ is false, as can be seen by taking $G = H = \mathbb{G}_{\mathrm{m}}$ and φ the map sending $x \mapsto x^2$, which is surjective as a map of algebraic groups but not on $\mathbb{R}^\times$.

Proof. The map $\varphi(\mathbb{R})^+$ can be thought of as a smooth map of smooth manifolds, which is surjective on Lie algebras and so whose image contains a neighborhood of the identity. Since the image is a subgroup, it is open (since any point in it has an open neighborhood in the subgroup given by the translation of the subgroup of the identity), and therefore also closed, and so is all of $H(\mathbb{R})^+$ since the latter is connected. □

We can now define Shimura data.

Definition 2.3. A Shimura datum is a pair (G, X) for a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the following conditions:

(1) for every $h \in X$, the Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ defined by $\mathrm{Ad} \circ h$ is of type

$$\{(-1,1),(0,0),(1,-1)\};$$

(2) for every $h \in X$, $\mathrm{ad}(h(i))$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$;

(3) G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Notice the differences from the connected case: here we allow G to be reductive rather than semisimple, the homomorphisms h have target $G_{\mathbb{R}}$ rather than $G^{\mathrm{ad}}_{\mathbb{R}}$, and X is the full $G(\mathbb{R})$-conjugacy class rather than just $G(\mathbb{R})^+$.

Conditions (2) and (3) are the same as in Definition 1.6; let's take a moment to explain condition (1).

A Hodge structure on a real vector space V is a decomposition

$$V(\mathbb{C}) := V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$. The set of pairs (p,q) such that $V^{p,q}$ is nonzero is its type.

We can choose the isomorphism $\mathbb{S}(\mathbb{C}) \simeq \mathbb{C}^\times \times \mathbb{C}^\times$ such that the map $\mathbb{S}(\mathbb{R}) \simeq \mathbb{C}^\times \to \mathbb{S}(\mathbb{C}) \simeq \mathbb{C}^\times \times \mathbb{C}^\times$ induced by inclusion is $z \mapsto (z, \bar{z})$. Thus a representation of $\mathbb{S}$ on a real vector space V is a homomorphism $\mathbb{C}^\times \times \mathbb{C}^\times \to \mathrm{GL}(V(\mathbb{C}))$ compatible with this map $\mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times$; since $\mathbb{C}^\times \times \mathbb{C}^\times$ is abelian, this splits into the sum of characters, which are all of the form $(z_1, z_2) \mapsto z_1^r z_2^s$ for integers r, s, so a real representation is a sum of characters of the form $z \mapsto z^r \bar{z}^s$. Thus $X^*(\mathbb{S}) \simeq \mathbb{Z} \times \mathbb{Z}$ with the action of complex conjugation given by switching the coordinates, and so a representation of $\mathbb{S}$ on a real vector space V is a decomposition of $V(\mathbb{C})$ into complex vector spaces indexed by pairs of integers (p,q) satisfying $\overline{V^{p,q}} = V^{q,p}$, i.e. a Hodge structure. In particular, the allowed characters $z \mapsto \frac{z}{\bar{z}}, 1, \frac{\bar{z}}{z}$ of Definition 1.6 correspond to the Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ corresponding to $h : \mathbb{S} \to G_{\mathbb{R}} \to \mathrm{Aut}(\mathrm{Lie}(G_{\mathbb{R}}))$ having type $\{(-1,1),(0,0),(1,-1)\}$.

The terminology of connected Shimura data as opposed to Shimura data suggests that X should be, in suitable cases, a union of X^+ for Shimura data (G, X^+). We allow G to be more general than in the case of connected Shimura data, so we have to modify this a bit.

Proposition 2.4. *Let G be a reductive group over $\mathbb{R}$, and write $\bar{h} : \mathbb{S} \to G^{\mathrm{ad}}$ for the composition $\mathbb{S} \xrightarrow{h} G \to G^{\mathrm{ad}}$ and $\overline{X}$ be the $G^{\mathrm{ad}}(\mathbb{R})$-conjugacy class of homomorphisms $\bar{h} : \mathbb{S} \to G^{\mathrm{ad}}$ coming from $h : \mathbb{S} \to G$ in a $G(\mathbb{R})$-conjugacy class X.*

(a) The map $X \to \overline{X}$ sending $h \mapsto \bar{h}$ is injective, and its image is a union of connected components of $\overline{X}$.

(b) Let X^+ be a connected component of X, with image $\overline{X}^+$ in $\overline{X}$. If (G, X) is a Shimura datum, then so is $(G^{\mathrm{der}}, \overline{X}^+)$, and the stabilizer of X^+ in $G(\mathbb{R})$ is $G(\mathbb{R})_+$, the preimage of $G^{\mathrm{ad}}(\mathbb{R})^+$ in $G(\mathbb{R})$.

Proof. (a) Suppose that two homomorphisms $h, h' : \mathbb{S} \to G$ have the same projections to T and G^{ad}. It follows that h and h' differ by some map $\mathbb{S} \to Z \cap G^{\text{der}}$; as in the proof of Proposition 2.1, the image is finite, so since $\mathbb{S}$ is connected this map must be trivial. Thus every $h : \mathbb{S} \to G$ can be recovered from its projections to T and G^{ad}.

Every $h \in X$ is $G(\mathbb{R})$-conjugate, and so every h has the same projection to T since T is abelian. Therefore every $h \in X$ can be recovered from its projection $\bar{h}$ to G^{ad} alone, i.e. the map $X \to \overline{X}$ is injective. Since $G^{\text{ad}}(\mathbb{R})^+$ acts transitively on the connected components of $\overline{X}$ and $G(\mathbb{R})^+ \to G^{\text{ad}}(\mathbb{R})^+$ is surjective (see Lemma 2.2), the image must be a union of connected components.

(b) It is straightforward to check that the axioms (1), (2), and (3) of Definition 2.3 are satisfied for $(G^{\text{der}}, \overline{X}^+)$ if they are for (G, X). The stabilizer of $\overline{X}^+$ in $G^{\text{ad}}(\mathbb{R})$ is $G^{\text{ad}}(\mathbb{R})^+$, so the stabilizer in $G(\mathbb{R})$ is its preimage. □

In particular, if we take a connected component X^+ of X for a Shimura datum (G, X), by part (a) it maps isomorphically to a connected component of $\overline{X}$ stable under the (transitive) action of $G(\mathbb{R})^+$, and so we can think of X^+ as a $G(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S} \to G^{\text{ad}}_{\mathbb{R}}$.

Corollary 2.5. *If (G, X) is a Shimura datum and X^+ is a connected component of X, viewed as a $G(\mathbb{R})^+$-conjugacy class, then (G^{der}, X^+) is a connected Shimura datum. Thus X is a finite disjoint union of Hermitian symmetric domains.*

Proof. The claim follows from the above discussion together with part (b) of Proposition 2.4 and the equivalence of the conditions of Definition 2.3 and Definition 1.6. □

For a Shimura datum (G, X), we restrict the action of $\mathbb{S}$ on $\text{Lie}(G)_{\mathbb{C}}$ via $\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \to \text{GL}(\text{Lie}(G)_{\mathbb{C}})$ to be by the characters $z/\bar{z}$, 1, or $\bar{z}/z$. In particular real numbers (as contained in $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$) act trivially on $\text{Lie}(G)_{\mathbb{C}}$. The adjoint action of G on its Lie algebra factors through $G^{\text{ad}} = G/Z$, and the adjoint representation $G^{\text{ad}} \to \text{GL}(\text{Lie}(G))$ is injective, so the triviality of the action of $r \in \mathbb{R}^\times \subset \mathbb{S}(\mathbb{R})$ is equivalent to the image of r being trivial in G^{ad}, i.e. $h(r) \in Z(\mathbb{R})$. Since all $h \in X$ are $G(\mathbb{R})$-conjugate, for $h, h' \in X$ for some $g \in G(\mathbb{R})$ we have $h'(r) = gh(r)g^{-1} = h(r)$ since $h(r) \in Z(\mathbb{R})$. Thus the restriction of any h to $\mathbb{G}_{\text{m}} \subset \mathbb{S}$ gives a map $\mathbb{G}_{\text{m}} \to G_{\mathbb{R}}$ depending on X (but not on h), and in particular we can find some homomorphism $w_X : \mathbb{G}_{\text{m}} \to G_{\mathbb{R}}$ such that $w_X \cdot h : \mathbb{G}_{\text{m}} \to G_{\mathbb{R}}$ is trivial for every $h \in X$; we call this the weight homomorphism.

Any representation $\rho : G_{\mathbb{R}} \to \text{GL}(V)$ gives a composition $\rho \circ w_X : \mathbb{G}_{\text{m}} \to \text{GL}(V)$, i.e. a representation of $\mathbb{G}_{\text{m}}$, which decomposes as the sum of characters indexed by integers $V = \bigoplus V_n$. This is the weight decomposition of the Hodge structure corresponding to $\rho \circ h : \mathbb{S} \to \text{GL}(V)$ for any $h \in X$.

Proposition 2.6. *Let (G, X) be a Shimura datum. Then X has a unique complex manifold structure such that for every representation $\rho : G_{\mathbb{R}} \to \text{GL}(V)$, letting h range over X the pair $(V, \rho \circ h)$ gives a holomorphic family of Hodge structures. With respect to this complex structure this family is a variation of Hodge structures, and X is a finite disjoint union of Hermitian symmetric domains.*

This follows from axiom (1) of Definition 2.3 and properties of Hodge structures.

For a Shimura datum (G, X) and a compact open subgroup K of $G(\mathbb{A}_f)$, we can now consider the double coset space

$$\mathrm{Sh}_K(G, X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

as in the previous section, with $G(\mathbb{Q})$ acting on X (by conjugation) and $G(\mathbb{A}_f)$ (by multiplication) on the left and K acting on $G(\mathbb{A}_f)$ (by multiplication) on the right. An inclusion of open compact subgroups $K' \subset K$ induces a regular map $\mathrm{Sh}_{K'}(G, X) \to \mathrm{Sh}_K(G, X)$, so varying K we get an inverse system $\mathrm{Sh}(G, X)$ on which $G(\mathbb{A}_f)$ acts naturally on the right.

To get a decomposition of $\mathrm{Sh}_K(G, X)$ into finitely many pieces which look something like connected Shimura varieties, we need two lemmas.

Lemma 2.7. *Let (G, X) be a Shimura datum. For every connected component X^+ of X, the natural map*

$$G(\mathbb{Q})_+\backslash X^+ \times G(\mathbb{A}_f) \to G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)$$

is a bijection.

Lemma 2.8. *For every open subgroup K of $G(\mathbb{A}_f)$, the set $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$ is finite.*

Combining these, we can hope to decompose the double quotient $\mathrm{Sh}_K(G, X)$ into components indexed by the finite set of Lemma 2.8, and then use Lemma 2.7 to give each component as the quotient of some connected component of X by an arithmetic subgroup.

To prove Lemma 2.7, we need real approximation. This is the theorem that for a connected algebraic group G over $\mathbb{Q}$, $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$; according to Deligne, this is proven by reducing to the case of tori.

Proof of Lemma 2.7. By real approximation, $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$, which in turn acts transitively on X, we can write any $x \in X$ as hx^+ for $h \in G(\mathbb{Q})$ and $x^+ \in X^+$. Therefore the map is surjective: for any $[x, g] \in G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)$, write this as $[hx^+, g] = [x^+, h^{-1}g]$, which is in the image of this map.

Let (x, g) and (x', g') be elements of $X^+ \times G(\mathbb{A}_f)$. If they have the same image in $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)$, then there exists some $h \in G(\mathbb{Q})$ such that $x' = hx$ and $g' = hg$. Since x and x' are in X^+, there is a unique element of $G(\mathbb{R})$ taking x to x', which (by Proposition 2.4(b)) must be in the stabilizer $G(\mathbb{R})_+$ of X^+. Therefore $h \in G(\mathbb{R})_+ \cap G(\mathbb{Q}) = G(\mathbb{Q})_+$, and so (x, g) and (x', g') also have the same image modulo $G(\mathbb{Q})_+$, i.e. the map is injective. □

Proof of Lemma 2.8. The map $G(\mathbb{Q})_+\backslash G(\mathbb{Q}) = G(\mathbb{Q}) \cap G(\mathbb{R})_+\backslash G(\mathbb{Q}) \simeq G(\mathbb{R})_+\backslash G(\mathbb{Q})G(\mathbb{R})_+ \to G^{\mathrm{ad}}(\mathbb{R})^+\backslash G^{\mathrm{ad}}(\mathbb{R})$ is injective, since $G(\mathbb{R})_+$ is just the preimage of $G^{\mathrm{ad}}(\mathbb{R})^+$, and the latter quotient is $\pi_0(G^{\mathrm{ad}})$, which is finite by Proposition 2.1. Therefore we can replace $G(\mathbb{Q})_+$ by $G(\mathbb{Q})$ without changing the result, i.e. it suffices to show that $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ is finite.

If we could apply strong approximation, we'd be done, but G is not necessarily either semisimple or simply connected. However, G has a finite cover by $Z \times G^{\mathrm{der}}$, as in the proof of Proposition 2.1, with G^{der} semisimple, and

$$Z(\mathbb{Q})\backslash Z(\mathbb{A}_f)/Z \cap K$$

is finite since Z is (a finite extension of a finite quotient of) a product of copies of U(1) and $\mathbb{G}_{\mathrm{m}}$, both of which give variants of the class group of $\mathbb{Q}$ of some level K, which are finite. (Note we are omitting some details.) Therefore we can replace G by G^{der}. In the case where G^{der} is simply connected, we are done: by strong approximation, we can then write $G^{\mathrm{der}}(\mathbb{A}_f) = G^{\mathrm{der}}(\mathbb{Q}) \cdot K$ and conclude that the double quotient is trivial, and so the whole thing is finite. If not, we can replace it by finite covers by a similar argument, and in the limit by its universal cover $\tilde{G}$ and then apply strong approximation again. □

We can finally prove our decomposition.

Proposition 2.9. *Let (G, X) be a Shimura datum, and let X^+ be a connected component of X. Then there is a homeomorphism*

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \simeq \bigsqcup_g \Gamma_g\backslash X^+,$$

where g ranges over a set of representatives for $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$ and $\Gamma_g = gKg^{-1} \cap G(\mathbb{Q})_+ \subset G(\mathbb{Q})_+$.

Note that by Lemma 2.8 this is a decomposition into finitely many pieces.

Proof. Fix some such g and consider the map $\Gamma_g\backslash X^+ \to G(\mathbb{Q})_+\backslash X^+ \times G(\mathbb{A}_f)/K$ sending $[x] \mapsto [x, g]$. By Lemma 2.7, the right-hand side is naturally in bijection with $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$, so it suffices to show that this map is injective and that taking the disjoint union of the images of these maps over varying g gives the whole space. Once we have a bijection, upgrading to a homeomorphism follows in the same way as in the proof of Proposition 1.3.

To see that this map is injective for each fixed g, suppose that $[x, g] = [x', g]$. Then we can write $x' = hx$ and $g = hgk$ for some $h \in G(\mathbb{Q})_+$ and $k \in K$, so $h = gk^{-1}g^{-1} \in gKg^{-1} = \Gamma_g$, and so $[x'] = [hx] = [x]$.

To see that varying g gives the whole space, let (x, j) be any element of $X^+ \times G(\mathbb{A}_f)$. Let g be the representative in $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$ of j, i.e. $j = hgk$ for $h \in G(\mathbb{Q})_+$ and $k \in K$. Then $[x, j] = [x, jgk] = [h^{-1}x, g]$, which is in the image of $\Gamma_g\backslash X^+$, so these maps are jointly surjective. It remains only to show that they are jointly injective, i.e. $[x, g] = [x', g']$ implies $g = g'$ (since we already know that they are injective for each g). If $[x, g] = [x', g']$, then we have $x' = hx$ and $g' = hgk$ for $h \in G(\mathbb{Q})_+$, $k \in K$, so since g, g' are part of a set of unique representatives of $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$ the second equation implies that $g' = g$. □

Thus $\mathrm{Sh}_K(G, X)$ is the disjoint union of quotients of X^+ by arithmetic subgroups, i.e. arithmetic locally symmetric varieties (at least for K sufficiently small), and so $\mathrm{Sh}(G, X)$ is an inverse system of algebraic varieties with a right action of $G(\mathbb{A}_f)$.

Now that we have our objects (Shimura data and varieties), we would like to have a notion of morphisms.

Definition 2.10. A morphism of Shimura data $(G, X) \to (G', X')$ is a homomorphism $G \to G'$ of algebraic groups sending X into X'. A morphism of Shimura varieties $\mathrm{Sh}(G, X) \to \mathrm{Sh}(G', X')$ is an inverse system of regular maps of algebraic varieties compatible with the right action of $G(\mathbb{A}_f)$.

A nonobvious fact is that every morphism of Shimura data $(G, X) \to (G', X')$ defines a morphism of Shimura varieties $\mathrm{Sh}(G, X) \to \mathrm{Sh}(G', X')$, which is a closed immersion if $G \to G'$, i.e. for every sufficiently small compact open $K' \subset G'(\mathbb{A}_f)$ there is a compact open $K \subset G(\mathbb{A}_f)$ and a closed immersion $\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}(G', X')$ in the inverse system $\mathrm{Sh}(G, X) \to \mathrm{Sh}(G', X')$.

We would like to understand the structure of Shimura varieties, i.e. the set of connected components and the structure of each component. In light of Proposition 2.9, we might hope that each connected component is a connected Shimura variety and the set of connected components is a finite "0-dimensional Shimura variety"; in general things are more complicated, but under the (mild) assumption that G^{der} is simply connected this s in fact the case.

Denote by ν the surjection $G \to T$, and write $T(\mathbb{R})^\dagger$ for the image of $Z(\mathbb{R})$ under $\nu(\mathbb{R})$ and $T(\mathbb{Q})^\dagger$ for $T(\mathbb{Q}) \cap T(\mathbb{R})^\dagger$. Since $G^{\mathrm{ad}} = G/Z$ is semisimple, it has no nontrivial abelian quotients, and so in particular $G/ZG^{\mathrm{der}} = T/\nu(Z)$ is abelian and so trivial, i.e. $\nu(Z) = T$, the composition $Z \hookrightarrow G \xrightarrow{\nu} T$ is surjective. Therefore by Lemma 2.2 restricting to $Z(\mathbb{R})^+$ we get a surjection $\nu(\mathbb{R})^+ : Z(\mathbb{R})^+ \to T(\mathbb{R})^+$, and so $T(\mathbb{R})^+$ is in the image $T(\mathbb{R})^\dagger$ of $Z(\mathbb{R})$. Indeed, by Proposition 2.1 $T(\mathbb{R})$ has finitely many connected components, and so $T(\mathbb{R})^+$ is a finite index subgroup of $T(\mathbb{R})$; since $T(\mathbb{R})^\dagger$ is between these two groups, it is also finite index in $T(\mathbb{R})$, and the same applies to $T(\mathbb{Q})^\dagger \subset T(\mathbb{Q})$. For example, if $G = \mathrm{GL}_2$ then $T(\mathbb{R})^\dagger = T(\mathbb{R})^+ = \mathbb{R}_{>0} \subset T(\mathbb{Q}) = \mathbb{R}^\times$, and similarly for $\mathbb{Q}$.

Theorem 2.11. *Let (G, X) be a Shimura datum with G^{der} simply connected. For any sufficiently small compact open $K \subset G(\mathbb{A}_f)$, the double quotient $T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(K)$ is finite, and the natural map*

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \to T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(K)$$

identifies the image with $\pi_0(\mathrm{Sh}_K(G, X))$, with the fiber over $[1]$ canonically isomorphic to $\Gamma\backslash X^+$ for some congruence subgroup $\Gamma \subset G^{\mathrm{der}}(\mathbb{Q})$ containing $K \cap G^{\mathrm{der}}(\mathbb{Q})$ and X^+ a connected component of X.

We'll reduce the proof of this theorem to a series of lemmas; let's proceed through the proof, stating the lemmas where we need them, and defer their proofs until the end. In all lemmas, (G, X) is a Shimura datum with G^{der} simply connected, as in the theorem.

In order to even construct this natural map, we need our first lemma:

Lemma 2.12. *Each $t \in T(\mathbb{Q})$ is in $T(\mathbb{Q})^\dagger$ if and only if it lifts to an element of $G(\mathbb{Q})_+$.*

In particular, this shows that $\nu(G(\mathbb{Q})_+) \subset T(\mathbb{Q})^\dagger$. Therefore applying Lemma 2.7 we get a map

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \simeq G(\mathbb{Q})_+\backslash X^+ \times G(\mathbb{A}_f)/K \xrightarrow{[x,g]\mapsto\nu(g)} T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(K).$$

Indeed, it gives the stronger statement

$$T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(K) = \nu(G(\mathbb{Q})_+)\backslash T(\mathbb{A}_f)/\nu(K).$$

Using Lemma 2.7, we can focus on the map

$$G(\mathbb{Q})_+\backslash X^+ \times G(\mathbb{A}_f)/K \xrightarrow{[x,g]\mapsto\nu(g)} \nu(G(\mathbb{Q})_+)\backslash T(\mathbb{A}_f)/\nu(K),$$

and in particular we look at its fiber over [1]. Suppose that $g \in G(\mathbb{A}_f)$ satisfies $[\nu(g)] = [1]$, so that $\nu(g) = \nu(h)\nu(k)$ for some $h \in G(\mathbb{Q})_+$, $k \in K$. Since ν is a homomorphism we therefore have $\nu(h)^{-1}\nu(g)\nu(k)^{-1} = \nu(h^{-1}gk^{-1}) = 1$, so $h^{-1}gk^{-1} \in \ker\nu(\mathbb{A}_f) = G^{\mathrm{der}}(\mathbb{A}_f)$ and so $g \in G(\mathbb{Q})_+G^{\mathrm{der}}(\mathbb{A}_f)K$. In particular, for every $[x, g]$ on the left-hand side over [1] we can assume that $g \in G^{\mathrm{der}}(\mathbb{A}_f)$. By strong approximation, $G^{\mathrm{der}}(\mathbb{A}_f) = G^{\mathrm{der}}(\mathbb{Q}) \cdot (K \cap G^{\mathrm{der}}(\mathbb{A}_f))$. Since $G^{\mathrm{der}}(\mathbb{Q})$ consists of the $g \in G(\mathbb{Q})$ whose image in $T(\mathbb{Q})$ is trivial, the image is certainly contained in $T(\mathbb{Q})^\dagger$ and so $G^{\mathrm{der}}(\mathbb{Q}) \subset G(\mathbb{Q})_+$, so via the action of $G(\mathbb{Q})_+$ and $K \cap G^{\mathrm{der}}(\mathbb{A}_f)$ we can represent any element of the fiber over [1] as $[x, 1]$ for some $x \in X^+$, i.e. the fiber is a quotient of X^+. Explicitly, it is the quotient of X^+ by $K \cap G(\mathbb{Q})_+$, or more precisely by the image Γ of $K \cap G(\mathbb{Q})_+$ in $G^{\mathrm{ad}}(\mathbb{Q})^+$. This is a congruence subgroup containing the image of $K \cap G^{\mathrm{der}}(\mathbb{Q})$ as desired, and by choosing K sufficiently small we can get Γ arbitrarily small, and so the inverse system of fibers over [1] indexed by compact open subgroups $K \subset G(\mathbb{A}_f)$ is equivalent to the inverse system $\mathrm{Sh}^\circ(G^{\mathrm{der}}, X^+) = \lim_\Gamma \Gamma\backslash X^+$, as we expect from Corollary 2.5.

Once we know that the fiber over each $[t]$ is nonempty, a similar description follows in the same way. This follows from our next lemma:

Lemma 2.13. *The map $\nu : G(\mathbb{A}_f) \to T(\mathbb{A}_f)$ is surjective and sends compact open subgroups to compact open subgroups.*

If we can show that $T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_f)/\nu(K)$ is finite (and thus discrete), then we are done: we have identified the connected component, and correspondingly all the other components, with the description of the connected components of Proposition 2.9, though the indexing is different. Since $T(\mathbb{Q})^\dagger$ has finite index in $T(\mathbb{Q})$, we can replace $T(\mathbb{Q})^\dagger$ by $T(\mathbb{Q})$, and since we only need to show the result for K sufficiently small we can assume $\nu(K) \subset T(\widehat{\mathbb{Z}})$, which is defined in the obvious way as $\prod_\ell T(\mathbb{Z}_\ell)$ with $T(\mathbb{Z}_\ell) \subset T(\mathbb{Q}_\ell)$ the subset with $\chi(t)$ integral for every character $\chi \in X^*(T)$. Since $T(\widehat{\mathbb{Z}})$ is compact and $\nu(K)$ is an open subgroup, it is finite index, so we can replace $\nu(K)$ by $T(\widehat{\mathbb{Z}})$, i.e. we just need to show that

$$T(\mathbb{Q})\backslash T(\mathbb{A}_f)/T(\widehat{\mathbb{Z}})$$

is finite. This turns out to be easier to prove for a larger set

$$H(T) := T(\mathbb{Q})\backslash T(\mathbb{R}) \times T(\mathbb{A}_f)/(T(\mathbb{R}) \times T(\widehat{\mathbb{Z}}),$$

of which our set is a quotient and so is finite if this is. We call $H(T)$ the class group of T. Thus to complete our proof of Theorem 2.11 we need only the following final lemma.

Lemma 2.14. *The class group of every torus T over $\mathbb{Q}$ is finite.*

Finally, let's look at the proofs of our lemmas.

Proof of Lemma 2.12. Consider the commutative diagram of long exact sequences from Galois cohomology

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G^{\mathrm{der}}(\mathbb{Q}) & \longrightarrow & G(\mathbb{Q}) & \xrightarrow{\nu} & T(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q}, G^{\mathrm{der}}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & G^{\mathrm{der}}(\mathbb{R}) & \longrightarrow & G(\mathbb{R}) & \xrightarrow{\nu} & T(\mathbb{R}) & \longrightarrow & H^1(\mathbb{R}, G^{\mathrm{der}})
\end{array}.$$

A general fact is that for any simply connected algebraic group H over $\mathbb{Q}$, the map $H^1(\mathbb{Q}, H) \to \prod_{\ell \leq \infty} H^1(\mathbb{Q}_\ell, H)$ is injective (the Hasse principle), and in fact for ℓ finite $H^1(\mathbb{Q}_\ell, H)$ is trivial, so $H^1(\mathbb{Q}, G^{\mathrm{der}}) \to H^1(\mathbb{R}, G^{\mathrm{der}})$ is injective.

We first want to show that $t \in T(\mathbb{Q})^\dagger$ lifts to $G(\mathbb{Q})_+$. Let's first construct its lift to $G(\mathbb{Q})$. Its image in $T(\mathbb{R})$ is certainly in $T(\mathbb{R})^\dagger$, and so there is some $z \in Z(\mathbb{R})$ whose image in $T(\mathbb{R})$ is the image of t in $T(\mathbb{R})$. Viewing z as an element of $G(\mathbb{R})$, the exactness of the lower row shows that its image in $H^1(\mathbb{R}, G^{\mathrm{der}})$ is trivial; the commutativity of the rightmost square together with the injectivity of the rightmost vertical map shows that the image of $t \in T(\mathbb{Q})^\dagger$ in $H^1(\mathbb{Q}, G^{\mathrm{der}})$ is also trivial. By the exactness of the upper row, this means that t lifts to some $g \in G(\mathbb{Q})$. The commutativity of the corresponding square implies that the image of g in $G(\mathbb{R})$ agrees with $z \in G(\mathbb{R})$ up to an element of $G^{\mathrm{der}}(\mathbb{R})$, i.e. the image of g is in $G^{\mathrm{der}}(\mathbb{R}) \cdot z \subset G^{\mathrm{der}}(\mathbb{R})Z(\mathbb{R})$. Since G^{der} is simply connected, $G^{\mathrm{der}}(\mathbb{R})$ is connected and so its image in $G(\mathbb{R})^{\mathrm{ad}}$ is also connected and therefore a subgroup of $G^{\mathrm{ad}}(\mathbb{R})^+$, and since $Z(\mathbb{R})$ is trivial in $G^{\mathrm{ad}}(\mathbb{R})$ the same is true for $G^{\mathrm{der}}(\mathbb{R})Z(\mathbb{R})$, i.e. $G^{\mathrm{der}}(\mathbb{R})Z(\mathbb{R}) \subseteq G(\mathbb{R})_+$. Therefore the image of g in $G(\mathbb{R})$ is in $G(\mathbb{R})_+$ and so g is in $G(\mathbb{Q})_+$ as desired.

To go the other way, we need a stronger version of this statement: in fact, $G^{\mathrm{der}}(\mathbb{R})Z(\mathbb{R})$ is equal to $G(\mathbb{R})_+$. Since we've proven one direction above, this amounts to the inclusion $G(\mathbb{R})_+ \subseteq G^{\mathrm{der}}(\mathbb{R})Z(\mathbb{R})$. To see this, consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Z(\mathbb{R}) \cap G^{\mathrm{der}}(\mathbb{R}) & \xrightarrow{z \mapsto (z^{-1}, z)} & Z(\mathbb{R}) \times G^{\mathrm{der}}(\mathbb{R}) & \xrightarrow{(z,g) \mapsto zg} & G(\mathbb{R}) & \longrightarrow & H^1(\mathbb{R}, Z \cap G^{\mathrm{der}}) \\
 & & \| & & \downarrow{\scriptstyle (z,g) \mapsto g} & & \downarrow & & \| \\
1 & \longrightarrow & Z(\mathbb{R}) \cap G^{\mathrm{der}}(\mathbb{R}) & \longrightarrow & G^{\mathrm{der}}(\mathbb{R}) & \longrightarrow & G^{\mathrm{ad}}(\mathbb{R}) & \longrightarrow & H^1(\mathbb{R}, Z \cap G^{\mathrm{der}})
\end{array}.$$

Since every element g of G^{ad} lifts to zg with $z \in Z$, and Z already maps surjectively to T so we can choose z such that zg has trivial image in T, i.e. is in G^{der}, the map $G^{\mathrm{der}} \to G^{\mathrm{ad}}$ is surjective; therefore by Lemma 2.2 the induced map $G^{\mathrm{der}}(\mathbb{R})^+ = G^{\mathrm{der}}(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})^+$ is surjective. Therefore $g \in G(\mathbb{R})$ is in $G(\mathbb{R})_+$ if and only if its image in $G^{\mathrm{ad}}(\mathbb{R})$ lifts to $G^{\mathrm{der}}(\mathbb{R})$, since if it is in $G^{\mathrm{ad}}(\mathbb{R})^+$ it will lift to $G^{\mathrm{der}}(\mathbb{R})$. By the exactness of the lower sequence, this is true if and only if its image in $H^1(\mathbb{R}, Z \cap G^{\mathrm{der}})$ is trivial. By the commutativity of the last square and the exactness of the top sequence, this is the same as g lifting to an element of $Z(\mathbb{R}) \times G^{\mathrm{der}}(\mathbb{R})$, i.e. $g \in G(\mathbb{R})_+$ if and only if we can write it as zg' for $z \in Z(\mathbb{R})$ and $g' \in G^{\mathrm{der}}(\mathbb{R})$, so $G(\mathbb{R})_+ = G^{\mathrm{der}}(\mathbb{R})Z(\mathbb{R})$.

We can now complete our proof: if $t \in T(\mathbb{Q})$ lifts to $G(\mathbb{Q})_+$, we want to show that in fact $t \in T(\mathbb{Q})^\dagger$. This is now straightforward: if t lifts to $G(\mathbb{Q})_+$, its image in $G(\mathbb{R})_+$ can be written as gz for $g \in G^{\mathrm{der}}(\mathbb{R})$ and $z \in Z(\mathbb{R})$. The image of g in $T(\mathbb{R})$ is trivial by definition, so the image of z in $T(\mathbb{R})$ is the image of t in $T(\mathbb{R})$, i.e. both are in $T(\mathbb{R})^\dagger$ and therefore $t \in T(\mathbb{Q})^\dagger$. □

Proof of Lemma 2.13. For surjectivity, we can reduce to proving the claim one place at a time: $\nu : G(\mathbb{Q}_\ell) \to T(\mathbb{Q}_\ell)$ is surjective for all (finite) ℓ, and $\nu : G(\mathbb{Z}_\ell) \to T(\mathbb{Z}_\ell)$ is surjective for almost all ℓ. The statement on compact open subgroups then follows, since at almost every place these are given by $G(\mathbb{Z}_\ell)$ and $T(\mathbb{Z}_\ell)$ respectively.

For each finite ℓ, since $H^1(\mathbb{Q}_\ell, G^{\mathrm{der}})$ is trivial as in the proof of Lemma 2.12 we have a short exact sequence

$$1 \to G^{\mathrm{der}}(\mathbb{Q}_\ell) \to G(\mathbb{Q}_\ell) \xrightarrow{\nu} T(\mathbb{Q}_\ell) \to 0,$$

so the first claim is immediate.

The situation for $\mathbb{Z}_\ell$ is more complicated, but essentially similar. Choose models $\mathcal{G}$, $\mathcal{T}$ over $\mathbb{Z}[1/N]$ which are group schemes with generic fibers G and T respectively, and extend ν to a homomorphism of group schemes $\mathcal{G} \to \mathcal{T}$. For N sufficiently large, this is a smooth morphism of group schemes with kernel $\mathcal{G}'$ having smooth connected fibers. For $\ell \nmid N$, we can base change to $\mathbb{Z}_\ell$ to get an exact sequence of group schemes over $\mathbb{Z}_\ell$

$$0 \to \mathcal{G}'_\ell \to \mathcal{G}_\ell \xrightarrow{\nu} \mathcal{T}_\ell \to 0$$

with $\mathfrak{n}$ smooth and $\mathcal{G}'_\ell$ having special fiber $(\mathcal{G}'_\ell)_{\mathbb{F}_\ell}$ smooth and connected. Lang's lemma states that then $H^1(\mathbb{F}_\ell, (\mathcal{G}'_\ell)_{\mathbb{F}_\ell}) = 0$ and so

$$\mathcal{G}_\ell(\mathbb{F}_\ell) \to \mathcal{T}_\ell(\mathbb{F}_\ell)$$

is surjective. Any fiber of $\nu : \mathcal{G}_\ell \to \mathcal{T}_\ell$ is smooth and connected, and so a variant of Newton's lemma shows that an $\mathbb{F}_\ell$-point lifts to a $\mathbb{Z}_\ell$-point; thus any $\mathbb{Z}_\ell$-point of $\mathcal{T}_\ell$ lifts to $\mathcal{G}_\ell$, and so on generic fibers we get the desired surjectivity of $\nu : G(\mathbb{Z}_\ell) \to T(\mathbb{Z}_\ell)$. □

Proof sketch of Lemma 2.14. We will not prove Lemma 2.14 in its full strength, but note that in the simple case $T = (\mathbb{G}_{\mathrm{m}})_{F/\mathbb{Q}}$ for any number field we have $H(T) = \mathrm{Cl}(F)$, which is finite by algebraic number theory. □

We've seen that each connected component of $\mathrm{Sh}_K(G, X)$ looks like a connected Shimura variety, and Theorem 2.11 gives the set of connected components as the double quotient $T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(K)$, which we hope to interpret as a "zero-dimensional Shimura variety."

Well, according to Definition 2.3, a Shimura datum for a torus T over $\mathbb{Q}$ consists of $(T, \{h\})$ for a single homomorphism $h : \mathbb{S} \to T_{\mathbb{R}}$ since conjugacy classes under the action of T are singletons. This h is determined by the map on real points $h : \mathbb{C}^\times \to T(\mathbb{R})$, and in this case the conditions (1), (2), (3) are trivially true since $T^{\mathrm{ad}} = T/Z$ is trivial. This gives rise to a Shimura variety $\mathrm{Sh}(T, \{h\})$, which at each compact open subgroup $K \subset T(\mathbb{A}_f)$ (given by $\nu(K)$ above) has points

$$\mathrm{Sh}_K(T, \{h\}) = T(\mathbb{Q})\backslash\{h\} \times T(\mathbb{A}_f)/K \simeq T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K.$$

This is a finite discrete set, as we'd expect. However it is a fairly rigid notion: for one thing, we've unintentionally arrived back at things which have to be connected, i.e. the singleton $\{h\}$, and moreover this type of Shimura variety doesn't include our quotient $T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(K)$.

To fix both of these issues, we make the obvious generalization: replace $\{h\}$ by some finite set Y on which $\pi_0(T(\mathbb{R}))$ acts transitively, and define

$$\mathrm{Sh}(T,Y) = \varprojlim_K \mathrm{Sh}_K(T,Y) = \varprojlim_K T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/K,$$

with K running over compact open subgroups of $T(\mathbb{A}_f)$. We call this a zero-dimensional Shimura variety.

If we are in the setting of Theorem 2.11, where (G,X) is a Shimura datum, G^{der} is simply connected, and $T = G/G^{\mathrm{der}}$, then setting $Y = T(\mathbb{R})/T(\mathbb{R})^\dagger$ by real approximation we have $Y \simeq T(\mathbb{Q})/T(\mathbb{Q})^\dagger$ and so

$$\mathrm{Sh}_{\nu(K)}(T,Y) = T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/\nu(K) \simeq T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_f)/\nu(K),$$

which is canonically in bijection with $\pi_0(\mathrm{Sh}_K(G,X))$ by Theorem 2.11. Thus the connected components of the Shimura variety form a zero-dimensional Shimura variety as desired.

Let's examine the case of tori a little more. Let T be a torus over $\mathbb{Q}$, and let $T(\mathbb{Z})$ be some arithmetic subgroup of $T(\mathbb{Q})$, e.g. $T(\mathbb{Z}) = \mathrm{Hom}(X^*(T), \mathcal{O}_L^\times)^{\mathrm{Gal}(L/\mathbb{Q})}$ where L is a Galois splitting field of T. It is known (due to Serre) that every subgroup of $T(\mathbb{Z})$ of finite index contains a congruence subgroup, and so the topology of $T(\mathbb{Q})$ induced by the topology on $T(\mathbb{A}_f)$ can be described by setting $T(\mathbb{Z})$ open with induced topology the profinite topology. In particular, since $T(\mathbb{Z})$ is open compact in $T(\mathbb{Q})$, $T(\mathbb{Q})$ is discrete if and only if $T(\mathbb{Z})$ is, which is true if and only if $T(\mathbb{Z})$ is finite.

Proposition 2.15. *Let T be a torus over $\mathbb{Q}$, and let T^a be the intersection of the kernels of all characters $T \to \mathbb{G}_m$ rational over $\mathbb{Q}$. Then $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$ if and only if $T^a(\mathbb{R})$ is compact.*

Proof. By a theorem of Ono, $T(\mathbb{Z}) \cap T^a(\mathbb{Q})$ is of finite index in $T(\mathbb{Z})$ and is cocompact in $T^a(\mathbb{R})$. It is an arithmetic subgroup of $T^a(\mathbb{Q})$ and therefore discrete in $T^a(\mathbb{R})$; therefore if $T^a(\mathbb{R})$ is compact $T(\mathbb{Z}) \cap T^a(\mathbb{Q})$ is discrete and compact and therefore finite, so by the above equivalence $T^a(\mathbb{Q})$ is discrete, and since it is finite index in $T(\mathbb{Q})$ the latter is also discrete in $T(\mathbb{A}_f)$. Conversely if $T(\mathbb{Q})$ is discrete so is $T^a(\mathbb{Q})$ and so $T(\mathbb{Z}) \cap T^a(\mathbb{Q})$ is finite, so since it is cocompact in $T^a(\mathbb{R})$ the latter must be compact. ☐

Generally speaking, we say that a torus T over a field k is anisotropic if there are no characters $T \to \mathbb{G}_{\mathrm{m}}$ defined over k, so that in this case we have defined T^a to be the largest anisotropic subtorus of T. In the case $k = \mathbb{R}$, a real torus T is anisotropic if and only if $T(\mathbb{R})$ is compact, so Proposition 2.15 is saying that $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$ if and only if the largest anisotropic subtorus of T remains anisotropic over $\mathbb{R}$.

Finally, let's look at the passage to the limit, as in Proposition 1.4.

Proposition 2.16. *Let (G,X) be a Shimura datum, and write $Z(\mathbb{Q})^-$ for the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}_f)$, where Z is the center of G. Then*

$$\varprojlim_K \mathrm{Sh}_K(G,X) = (G(\mathbb{Q})/Z(\mathbb{Q}))\backslash X \times (G(\mathbb{A}_f)/Z(\mathbb{Q})^-).$$

This is proven by mimicking the proof of Proposition 1.4, together with the observation

$$\begin{aligned}\mathrm{Sh}_K(G,X) &= G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \\ &\simeq (G(\mathbb{Q})/Z(\mathbb{Q}))\backslash X \times (G(\mathbb{A}_f)/Z(\mathbb{Q})K) \\ &\simeq (G(\mathbb{Q})/Z(\mathbb{Q}))\backslash X \times (G(\mathbb{A}_f)/Z(\mathbb{Q})^{-}K).\end{aligned}$$

This is a much less nice answer than we get in the case of Proposition 1.4, though. To fix this, it is convenient to introduce some extra axioms for Shimura varieties, which will not always be true but will hold in many interesting cases and which will generally make our lives easier.

The extra conditions we may ask for (frequently we will assume some but not all) given a Shimura datum (G,X):

(2*) for every $h \in X$, $\mathrm{ad}(h(i))$ is a Cartan involution on $G_{\mathbb{R}}/w_X(\mathbb{G}_{\mathrm{m}})$ (rather than on $G_{\mathbb{R}}^{\mathrm{ad}} = G_{\mathbb{R}}/Z_{\mathbb{R}}$);

(4) the weight homomorphism $w_X : \mathbb{G}_{\mathrm{m}} \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$ (in which case we say that the weight is rational);

(5) the group $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$;

(6) the torus Z° splits over a CM-field, where Z° is the connected component of Z.

Let's explain these briefly. As we've seen, a (rational) representation of G on some vector space V defines a Hodge structure on $V(\mathbb{R})$ for every $h \in X$; the axiom (4) means that these are rational Hodge structures, i.e. coming from $\mathbb{Q}$-vector spaces. Axiom (5) improves the moduli structure of Shimura varieties; this will be discussed more later, and we will shortly see a direct application. Axiom (6) is useful to make some statements more natural, but not essential (Deligne omits it); it guarantees that the weight is defined over some totally real field, and that the field of definition of the Shimura variety is either totally real or CM. Finally axiom 2 is a strengthening of (5), which we will not need in practice; it is included for historical reasons, namely that it was used by Deligne in place of our axiom (5).

We can now improve Proposition 2.16.

Proposition 2.16′. *Let (G,X) be a Shimura datum additionally satisfying axiom (5). Then*

$$\varprojlim_K \mathrm{Sh}_K(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f).$$

Proof. For $Z(\mathbb{Q})$ discrete in $Z(\mathbb{A}_f)$, the closure $Z(\mathbb{Q})^-$ is just $Z(\mathbb{Q})$ again, and so the claim follows immediately from the original Proposition 2.16. □

We can think of this inverse limit $\mathrm{Sh}(G,X)$ as a scheme over $\mathbb{C}$. It is not of finite type, but it is regular and locally noetherian. It carries a right action of $G(\mathbb{A}_f)$, and for compact open subgroups K we can take the quotient to get $\mathrm{Sh}(G,X)/K = \mathrm{Sh}_K(G,X)$, as follows from Proposition 2.16′.

References

[1] James S Milne. Introduction to Shimura varieties. *Harmonic analysis, the trace formula, and Shimura varieties*, 4:265–378, 2005.