The Rankin-Selberg method*

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Our main tool to obtain extensions of L-functions of modular forms to $\mathbb{C}$ has been by exploiting their modularity, typically by representing the L-function as an integral and substituting suitably. Our goal is to demonstrate a related method: we pick a particular automorphic function, an Eisenstein series, and represent our L-function as an integral against this series; we can then use the automorphicity to obtain a functional equation.

We first define the Eisenstein series:

$$E(z,s)=\pi^{-s}\Gamma(s)\cdot\frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{\mathrm{Im}(z)^s}{|mz+n|^{2s}}.$$

This converges absolutely for $\mathrm{Re}(s)>1$, and is distinct from but related to our previous notion of Eisenstein series: we replace $(mz+n)^{-k}$ by $|mz+n|^{-2s}$ and multiply by some factors "at infinity." Its qualitative behavior is also related but different: for each fixed s, it is no longer holomorphic in z (due to the presence of the absolute value), and it transforms with weight 0, i.e. it is strictly invariant under the action of $\Gamma(1)$: $E(\gamma z,s)=E(z,s)$ for $\gamma\in\mathrm{SL}_2(\mathbb{Z})$. This can be seen in essentially the same way as for the holomorphic Eisenstein series E_k.

The main property of $E(z,s)$ is that it already, without passing to an L-series, satisfies a functional equation giving an analytic continuation to $\mathbb{C}$.

Theorem 1. *The Eisenstein series $E(z,s)$ has meromorphic continuation to $\mathbb{C}$, analytic except at $s=0$ or $s=1$, where it has simple poles. The residue at $s=1$ is the constant function $\mathrm{res}_{s=1}E(z,s)=\frac{1}{2}$. Further $E(z,s)$ satisfies the functional equation*

$$E(z,s)=E(z,1-s),$$

and it is bounded by $E(x+iy,s)=O(y^\sigma)$ as $y\to\infty$, where $\sigma=\max(\mathrm{Re}(s),1-\mathrm{Re}(s))$.

Proof. As for the L-function of E_k, our proof will proceed by explicitly calculating the Fourier coefficients, and then verify the functional equation on these coefficients.

Since $E(Tz,s)=E(z+1,s)=E(z,s)$ and is analytic in x and y separately where $z=x+iy$, in particular it has a Fourier expansion in x

$$E(x+iy,s)=\sum_{r=\infty}^{\infty}a_r(y,s)e^{2\pi irx},$$

where we can compute the coefficients by

$$a_r(y,s)=\int_0^1 E(x+iy,s)e^{-2\pi irx}\,dx.$$

*These notes are based on section 1.6 of [2].

By definition, we can expand this (for $\mathrm{Re}(s)$ sufficiently large) as

$$\frac{1}{2}\pi^{-s}\Gamma(s)y^s \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \int_0^1 \frac{1}{(mx+n)^2+m^2y^2)^s} e^{-2\pi irx}\,dx.$$

Since (m,n) and $(-m,-n)$ contribute the same term, we can restrict to terms of the form $(0,n)$ and (m,n) with $m \geq 1$, the latter with an extra factor of 2. The terms from $m=0$ are independent of x and so only contribute to $a_0(y,s)$; their contribution to $a_0(y,s)$ is equal for n and $-n$, and so is

$$\pi^{-s}\Gamma(s)y^s \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = \pi^{-s}\Gamma(s)\zeta(2s)y^s$$

(though this is not all of $a_0(y,s)$, as we will see).

Next, consider the contributions from the terms with $m \neq 0$,

$$\pi^{-s}\Gamma(s)y^s \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} \int_0^1 \frac{1}{((mx+n)^2+m^2y^2)^s} e^{-2\pi irx}\,dx.$$

Integrating over x from 0 to 1 and n over all integers is the same as integrating over all real x and n over representatives of congruence classes modulo m, i.e.

$$\pi^{-s}\Gamma(s)y^s \sum_{m=1}^{\infty}\sum_{n=0}^{m-1} \int_{-\infty}^{\infty} \frac{1}{((mx+n)^2+m^2y^2)^s} e^{-2\pi irx}\,dx.$$

Substituting $u = x + n/m$, the integral is

$$\frac{1}{m^{2s}} e^{2\pi irn/m} \int_{-\infty}^{\infty} \frac{1}{(u^2+y^2)^s} e^{-2\pi iru}\,du,$$

with the integral independent of m, and so the contribution from all $m \neq 0$ is

$$\pi^{-s}\Gamma(s)y^s \int_{-\infty}^{\infty} \frac{1}{(u^2+y^2)^s} e^{-2\pi iru}\,du \sum_{m=1}^{\infty} \frac{1}{m^{2s}} \sum_{n=0}^{m-1} e^{2\pi irn/m}.$$

If $m|r$, the innermost sum is m; otherwise it is 0. If $r=0$, the former always holds and so the sums are just $\zeta(2s-1)$. In this case

$$\Gamma(s)y^s \int_{-\infty}^{\infty} \frac{1}{(u^2+y^2)^s}\,du = \int_{-\infty}^{\infty}\int_0^{\infty} e^{-t}\left(\frac{ty}{(u^2+y^2)}\right)^s \frac{dt}{t}\,du.$$

Setting $w = ty/(u^2+y^2)$, this is

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_0^{\infty} e^{-w(u^2+y^2)/y} w^s \frac{dw}{w}\,du &= \int_0^{\infty} w^{s-1} \int_{-\infty}^{\infty} e^{-w(u^2+y^2)/y}\,du\,dw \\
&= \sqrt{\pi y}\int_0^{\infty} w^{s-3/2} e^{-wy}\,dw \\
&= \sqrt{\pi} y^{1-s}\Gamma\left(s-\frac{1}{2}\right).
\end{aligned}$$

Therefore in total

$$a_0(y,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{\frac{1}{2}-s}y^{1-s}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1).$$

By the functional equation for $\zeta(s)$, this is the same thing as

$$a_0(y,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2(1-s))y^{1-s},$$

which is clearly symmetric under $s \leftrightarrow 1-s$.

For $r \neq 0$, the condition is nontrivial, and the total is

$$\pi^{-s}\Gamma(s)y^s \int_{-\infty}^{\infty} \frac{1}{(u^2+y^2)^s} e^{-2\pi i r u}\, du \sum_{m|r} m^{1-2s} = \pi^{-s}\Gamma(s)y^s \int_{-\infty}^{\infty} \frac{1}{(u^2+y^2)^s} e^{-2\pi i r u}\, du.$$

Similar arguments to above show that this is

$$2|r|^{s-\frac{1}{2}}\sqrt{y}K_{s-\frac{1}{2}}(2\pi y|r|),$$

where K_s is the Bessel function

$$K_s(y) = \frac{1}{2}\int_0^{\infty} e^{-y(t+t^{-1})/2}t^s\,\frac{dt}{t}.$$

If $y > 0$, the integrand decays rapidly and so this converges absolutely for all s. Since $\frac{y}{2}(t+t^{-1}) > \frac{y}{2} + t + t^{-1}$ for $y > 4$, since $ab > a+b$ for $a > 2$ and $b \geq 2$, this is bounded in absolute value by

$$\frac{1}{2}\int_0^{\infty} e^{-y/2}e^{(t+t^{-1})}|t^s|\,\frac{dt}{t} = e^{-y/2}K_{\mathrm{Re}(s)}(2)$$

for $y > 4$, so in particular $K_s(y)$ decays rapidly as $y \to \infty$ for any s. Since replacing (t,s) by $(t^{-1},-s)$ does not affect the integrand, we have $K_s(y) = K_{-s}(y)$.

Thus in all we have for $r \neq 0$

$$a_r(y,s) = 2|r|^{s-\frac{1}{2}}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi y|r|).$$

Thus

$$a_r(y,1-s) = 2|r|^{\frac{1}{2}-s}\sigma_{2s-1}(|r|)\sqrt{y}K_{\frac{1}{2}-s}(2\pi y|r|).$$

Since $K_{\frac{1}{2}-s} = K_{s-\frac{1}{2}}$, to get $a_r(y,s) = a_r(y,1-s)$ it suffices to show that

$$|r|^{s-\frac{1}{2}}\sigma_{1-2s}(|r|) = |r|^{\frac{1}{2}-s}\sigma_{2s-1}(|r|).$$

To simplify the notation we assume r is positive; the negative case is the same since we take absolute values. Then

$$\sum_{d_1 d_2 = r} d_1^s d_2^{-s} = \sum_{d|r} d^s(r/d)^{-s} = \sum_{d|r} d^{2s}r^{-s} = r^{-s}\sigma_{2s}(r);$$

on the other hand this is also equal to

$$\sum_{d|r}(r/d)^s d^{-s} = \sum_{d|r} d^{-2s} r^s = r^s \sigma_{-2s}(r).$$

Replacing s by $s - \frac{1}{2}$ gives

$$r^{s-\frac{1}{2}}\sigma_{1-2s}(r) = r^{\frac{1}{2}-s}\sigma_{2s-1}(r)$$

as desired.

Since all of the Fourier coefficients have this symmetry and the series converges by the rapid decay of the Bessel function, so the sum $E(z,s)$ is also symmetric under $s \leftrightarrow 1-s$. Every $a_r(y,s)$ for $r \neq 0$ is analytic for all s, and a_0 has simple poles at $s = 0$ and $s = 1$ (the poles in the two terms at $\frac{1}{2}$ cancel), with residue at $s = 1$ given by $-\zeta(0) = \frac{1}{2}$ for all z. The bound on $E(x+iy,s)$ follows from the bound on $a_0(y,s)$ since the $a_r(y,s)$ for $r \neq 0$ decay rapidly as $y \to \infty$. □

We can also interpret the Eisenstein series as a sum over a group quotient, which is related to its adelic interpretation. Summing over all pairs (m,n) not both zero is almost the same as summing over pairs (c,d) of coprime integers, up to accounting for common factors which will change the total by a scalar; to any such pair (c,d) we can associate the coset in $\Gamma_\infty\backslash\Gamma(1)$ consisting of matrices with bottom row (c,d), where Γ_∞ is the subgroup generated by T. Note that if $\gamma \in \Gamma(1)$ has bottom row (c,d), then

$$\frac{\mathrm{Im}(z)^s}{|cz+d|^{2s}} = \mathrm{Im}(\gamma z)^s,$$

and so

$$\begin{aligned}
E(z,s) &= \pi^{-s}\Gamma(s)\cdot\frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}}\frac{\mathrm{Im}(z)^s}{|mz+n|^s} \\
&= \pi^{-s}\Gamma(s)\cdot\frac{1}{2}\sum_{N=1}^{\infty}\sum_{\substack{(c,d)\in\mathbb{Z}\backslash\{(0,0)\}\\ \gcd(c,d)=1}}\frac{\mathrm{Im}(z)^s}{|Ncz+Nd|^{2s}} \\
&= \pi^{-s}\Gamma(s)\zeta(2s)\cdot\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)}\mathrm{Im}(\gamma z)^s.
\end{aligned}$$

By viewing Γ_∞ and $\Gamma(1)$ as their quotients by scalars (by an abuse of notation), γ and $-\gamma$ give the same term and so we can drop the factor of $\frac{1}{2}$: interpreted in this sense,

$$E(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)}\mathrm{Im}(\gamma z)^s.$$

We can now discuss the Rankin-Selberg method. Let ϕ be an automorphic function on $\mathcal{H}$, i.e. a smooth (but not necessarily holomorphic) function invariant under the action of $\Gamma(1)$. Suppose that we have the boundedness condition $\phi(x+iy) = O(y^{-N})$ for all $N > 0$

as $y \to \infty$. Since ϕ is $\Gamma(1)$-invariant, it is in particular periodic in x with period 1 and so we have a Fourier expansion

$$\phi(x+iy) = \sum_{n=-\infty}^{\infty} \phi_n(y) e^{2\pi i n x},$$

where we can compute the $\phi_n(y)$ by

$$\phi_n(y) = \int_0^1 \phi(x+iy) e^{-2\pi i n x}\, dx.$$

We are particularly interested in the constant term ϕ_0. Since ϕ is bounded on the fundamental domain and $\Gamma(1)$-invariant, ϕ_0 is bounded as a function of y, and decays rapidly as $y \to \infty$. Therefore we can define its Mellin transform

$$M(s, \phi_0) = \int_0^\infty \phi_0(y) y^s \frac{dy}{y},$$

absolutely convergent for $\mathrm{Re}(s) > 0$. Set

$$\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s) M(s-1, \phi_0).$$

Proposition 2. *With notation as above,*

$$\Lambda(s) = \int_{\Gamma(1)\backslash\mathcal{H}} E(z,s)\phi(z)\, Dz,$$

which is valid since both $E(z,s)$ and $\phi(z)$ are $\Gamma(1)$-invariant in z. Here Dz is the usual $\Gamma(1)$-invariant metric, given explicitly by $Dz = \frac{dx\,dy}{y^2}$ for $z = x+iy$. This defines a meromorphic continuation of $\Lambda(s)$ to $\mathbb{C}$, with at most simple poles at $s=1$ and $s=0$, and

$$\mathrm{res}_{s=1} \Lambda(s) = \frac{1}{2} \int_{\Gamma(1)\backslash\mathcal{H}} \phi(z)\, Dz.$$

Proof. It suffices to show that the integral representation of $\Lambda(s)$ holds; then the analytic properties, including the residue, follow from the properties of $E(z,s)$ from Theorem 1.

Assume $\mathrm{Re}(s) > 1$, so that we can substitute

$$E(z,s) = \pi^{-s}\Gamma(s)\zeta(2s) \sum_{\gamma \in \Gamma_\infty\backslash\Gamma(1)} \mathrm{Im}(\gamma z)^s.$$

Using the fact that $\phi(\gamma z) = \phi(z)$, this gives for the right-hand side

$$\pi^{-s}\Gamma(s)\zeta(2s) \sum_{\gamma \in \Gamma_\infty\backslash\Gamma(1)} \int_{\Gamma(1)\backslash\mathcal{H}} \mathrm{Im}(\gamma z)^s \phi(\gamma z)\, Dz.$$

Combining the sum and the integral and using the invariance of the measure, this is

$$\pi^{-s}\Gamma(s)\zeta(2s) \int_{\Gamma_\infty\backslash\mathcal{H}} \mathrm{Im}(z)^s \phi(z)\, Dz.$$

A fundamental domain for the action of Γ_∞ on $\mathcal{H}$ is given by the strip $0 < x < 1$, $y > 0$, so this is

$$\pi^{-s}\Gamma(s)\zeta(2s)\int_0^\infty y^{s-1}\int_0^1 \phi(x+iy)\,dx\,\frac{dy}{y} = \pi^{-s}\Gamma(s)\zeta(2s)\int_0^\infty \phi_0(s)y^{s-1}\,\frac{dy}{y}.$$

But this is just $\pi^{-s}\Gamma(s)\zeta(2s)M(s-1,\phi_0) = \Lambda(s)$. □

We can now turn to the original application of the Rankin-Selberg method. Let $f(z) = \sum_n a_n q^n$ and $g(z) = \sum_n b_n q^n$ be modular forms. Each has an L-function, which we know is well-behaved (has an analytic continuation, a functional equation, etc.); we could also consider the "product" L-function $\sum_n a_n b_n n^{-s}$.

To get this in the language of Proposition 2, take $\phi(z) = f(z)\overline{g(z)}y^k$, which we know is $\Gamma(1)$-invariant. To get the required bounds, we assume that at least one of f and g is cuspidal. We have

$$\phi_0(y) = \int_0^1 f(x+iy)\overline{g(x+iy)}y^k\,dx = \sum_{m=0}^\infty\sum_{n=0}^\infty\int_0^1 a_m\overline{b_m}e^{2\pi i(m-n)x}e^{-2\pi(m+n)y}y^k\,dx.$$

Since the integral of $e^{2\pi i(m-n)x}$ vanishes unless $m = n$, this is

$$\phi_0(y) = \sum_{n=0}^\infty a_n\overline{b_n}e^{-4\pi ny}y^k.$$

If we assume further that f and g are Hecke eigenforms, then in particular the b_n are eigenvalues of the self-adjoint Hecke operators and therefore real, so $\overline{b_n} = b_n$ and so

$$\phi_0(y) = \sum_{n=0}^\infty a_n b_n e^{-4\pi ny}y^k.$$

Thus

$$M(s,\phi_0) = \sum_{n=0}^\infty a_n b_n\int_0^\infty e^{-4\pi ny}y^{s+k}\,\frac{dy}{y} = (4\pi)^{-s-k}\Gamma(s+k)\sum_{n=0}^\infty a_n b_n n^{-s-k},$$

so

$$\Lambda(s) = 4^{-s-k}\pi^{-2s-k+1}\Gamma(s)\Gamma(s+k-1)\zeta(2s)\Gamma(s+k-1)\sum_{n=0}^\infty a_n b_n n^{-s-k+1}.$$

It is convenient to adjust this definition slightly: set

$$L(s, f\times g) = \zeta(2(s-k+1))\sum_{n=1}^\infty a_n b_n n^{-s}$$

and

$$\Lambda(s, f\times g) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+1)L(s,f\times g) = \pi^{1-k}\Lambda(s-k+1).$$

Theorem 3. *With notation as above, $\Lambda(s, f \times g)$ has meromorphic continuation to all of $\mathbb{C}$, holomorphic except for at most simple poles at $s = k$ and $s = k - 1$ and satisfying the functional equation*

$$\Lambda(s, f \times g) = \Lambda(2k - 1 - s, f \times g).$$

The residue of $\Lambda(s, f \times g)$ at $s = k$ is $\frac{1}{2}\pi^{1-k} \langle f, g \rangle$, where $\langle \cdot, \cdot \rangle$ is the Petersson inner product.

Proof. Using the discussion above, the functional equation is equivalent to

$$\Lambda(s - k + 1) = \Lambda(k - s);$$

replacing s by $k - s$ gives

$$\Lambda(s) = \Lambda(1 - s),$$

so if we can show this for all s the original functional equation follows. The expression for $\Lambda(s)$ from Proposition 2 is manifestly symmetric in $s \leftrightarrow 1 - s$ since $E(z, s)$ is, so the functional equation holds. By Proposition 2, $\Lambda(s)$ has at most simple poles at $s = 1$ and $s = 0$, so $\Lambda(s, f \times g) = \pi^{1-k}\Lambda(s - k + 1)$ has at most simple poles where $s - k + 1$ is equal to 0 or 1, i.e. at $k - 1$ and k; and the residue where $s - k + 1 = 1$, i.e. at $s = k$, is

$$\pi^{1-k} \cdot \frac{1}{2} \int_{\Gamma(1)\backslash\mathcal{H}} f(z)\overline{g(z)} y^k \, Dz = \frac{1}{2}\pi^{1-k} \langle f, g \rangle$$

by Proposition 2. □

Since we assume f and g Hecke eigenforms, their L-functions have Euler products; we might hope that so does $L(s, f \times g)$. Factor the Hecke polynomials as $1 - a_p x + p^{k-1}x^2 = (1 - \alpha_1(p)x)(1 - \alpha_2(p)x)$ and $1 - b_p x + p^{k-1}x^2 = (1 - \beta_1(p)x)(1 - \beta_2(p)x)$. Then we have the following.

Theorem 4. *With notation as above, we have*

$$L(s, f \times g) = \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \frac{1}{(1 - \alpha_i(p)\beta_j(p)p^{-s}}.$$

This is an instance of a more general algebraic fact:

Lemma 5. *Suppose that*

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{(1 - \alpha_1 x)(1 - \alpha_2 x)}, \qquad \sum_{n=0}^{\infty} b_n x^n = \frac{1}{(1 - \beta_1 x)(1 - \beta_2 x)}.$$

Then

$$\sum_{n=0}^{\infty} a_n b_n x^n = (1 - \alpha_1\alpha_2\beta_1\beta_2 x^2) \prod_{i=1}^{2} \prod_{j=1}^{2} \frac{1}{(1 - \alpha_i\beta_j(p)x}.$$

Indeed, applying this lemma with $x = p^{-s}$ on each Euler factor gives the desired expansion.

Proof. Suppose $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$. Then we can compute a_n and b_n from the theory of power series:

$$a_n = \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}, \qquad b_n = \frac{\beta_1^{n+1} - \beta_2^{n+1}}{\beta_1 - \beta_2}.$$

Therefore

$$a_n b_n = \frac{(\alpha_1\beta_1)^{n+1} - (\alpha_1\beta_2)^{n+1} - (\alpha_2\beta_1)^{n+1} + (\alpha_2\beta_2)^{n+1}}{(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)}.$$

Multiplying by x^n and summing, we get a sum of geometric series

$$\frac{1}{(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)} \cdot \left(\frac{\alpha_1\beta_1}{1 - \alpha_1\beta_1 x} - \frac{\alpha_1\beta_2}{1 - \alpha_1\beta_2 x} - \frac{\alpha_2\beta_1}{1 - \alpha_2\beta_1 x} + \frac{\alpha_2\beta_2}{1 - \alpha_2\beta_2 x} \right)$$

which simplifies to the claimed value. (Bump's proof uses complex analysis instead.) □

Since $L(s, f \times g)$ has an analytic continuation, a functional equation, and an Euler product, one might suspect that via converse theorems it should itself come from a modular form. This is at least morally true: if f and g correspond to automorphic representations π_f and π_g, then $f \times g$ in this sense corresponds to the tensor product representation $\pi_f \otimes \pi_g$ on GL(4).

References

[1] Jeremy Booher. Viewing modular forms as automorphic representations, 2015.

[2] Daniel Bump. *Automorphic forms and representations*. Number 55. Cambridge University Press, 1998.