Siegel-Weil examples

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1. A two-dimensional case and Fermat's theorem

Consider the lattice $\Lambda := \mathbb{Z}^2 \subset \mathbb{R}^2$, with quadratic form $Q(x,y) = x^2+y^2$. As for any lattice, we can associate to it a theta function

$$\Theta_\Lambda(z) := \sum_{(x,y)\in\Lambda} q^{Q(x,y)} = \sum_{x,y\in\mathbb{Z}} q^{x^2+y^2}$$

where $q = e^{2\pi iz}$. Writing $r_2(n)$ for the number of pairs of integers (x,y) such that $x^2+y^2 = n$ for each $n \geq 0$, we can rewrite this as

$$\Theta_\Lambda(z) = \sum_{n\geq 0} r_2(n) q^n.$$

To ensure convergence we restrict z to the upper half-plane.

I claim that this is a modular form of weight 1. Indeed, we will prove the following stronger statement. Let $\Lambda \subset \mathbb{R}^n$ be an n-dimensional lattice with quadratic form Q, and suppose that Λ is unimodular, i.e. self-dual (or equivalently $\mathrm{vol}(\mathbb{R}^n/\Lambda) = 1$).

Proposition 1.1. *The theta function Θ_Λ of a unimodular lattice Λ is a modular form of weight $n/2$ and level 4, i.e. it has weight $n/2$ for the action of $\Gamma_1(4)$ on the upper half-plane.*

Here $\Gamma_1(4)$ is the standard congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant 1 such that $a \equiv d \equiv 1 \pmod 4$ and $c \equiv 0 \pmod 4$.

Proof. First, for z in the upper half-plane the growth condition on $\Theta_\Lambda(z)$ is immediate due to the rapid convergence of the series and the exponential dependence on z; therefore we will only concern ourselves with the transformation properties.

The appearance of the level 4 structure is nonobvious, and it is more natural to first ask if Θ_Λ transforms correctly under the action of the full modular group $\mathrm{SL}_2(\mathbb{Z})$, which is generated by $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. The first of these is clear: this is just the invariance under $z \mapsto z+1$, which follows from the fact that $\Theta_\Lambda(z)$ only depends on $q = e^{2\pi iz}$. The second corresponds to a relation between $\Theta_\Lambda(-1/z)$ and $\Theta_\Lambda(z)$. We can explicitly write out

$$\Theta_\Lambda\left(-\frac{1}{z}\right) = \sum_{x\in\Lambda} e^{-2\pi iQ(x)/z}.$$

Suppose first that $z = iy$ for $y > 0$; then this is

$$\Theta_\Lambda\left(-\frac{1}{z}\right) = \Theta_\Lambda\left(\frac{i}{y}\right) = \sum_{x\in\Lambda} e^{-2\pi Q(x)/y}.$$

Let $f_y(x) = e^{-2\pi Q(x)/y}$. Since Λ is unimodular, we can map it to $\mathbb{Z}^n$ without rescaling and thus apply Poisson summation to get

$$\Theta_\Lambda\left(\frac{i}{y}\right) = \sum_{x\in\mathbb{Z}^n} \hat{f}_y(x).$$

By standard Fourier analysis $\hat{f}_2 = f_2$, i.e. $f_2(x) = e^{-\pi Q(x)}$ is its own Fourier transform (since $Q(x)$ can be interpreted as the norm of $x \in \Lambda \subset \mathbb{R}^n$); since $Q(x)$ is a quadratic form we have $2Q(x)/y = Q((2/y)^{1/2}x)$ and so $f_y(x) = e^{-\pi Q((2/y)^{1/2}x)} = f_2((2/y)^{1/2}x)$ has Fourier transform

$$\hat{f}_y(x) = \left(\frac{y}{2}\right)^{n/2} f_2\left(\sqrt{\frac{y}{2}}x\right) = \left(\frac{y}{2}\right)^{n/2} e^{-\frac{1}{2}\pi Q(x)y}.$$

Therefore we have

$$\Theta_\Lambda\left(-\frac{1}{z}\right) = \Theta_\Lambda\left(\frac{i}{y}\right) = \left(\frac{y}{2}\right)^{n/2}\sum_{x\in\Lambda} e^{-\frac{1}{2}\pi Q(x)y} = \left(\frac{y}{2}\right)^{n/2}\Theta_\Lambda\left(\frac{iy}{4}\right) = \left(\frac{z}{2i}\right)^{n/2}\Theta_\Lambda\left(\frac{z}{4}\right).$$

Since Θ_Λ is holomorphic on the upper half-plane, this equation must hold on all of it, not just on the imaginary line.

This is *not*, in general, the correct weight $n/2$ transformation. However, it does give us something useful. The congruence subgroup $\Gamma_1(4)$ is generated by $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \\ -4 & 1 \end{pmatrix}$ (this can be checked explicitly, and is left as an exercise to the reader), and therefore it remains only to check that

$$\Theta_\Lambda\left(\frac{z}{1-4z}\right) = (1-4z)^{n/2}\Theta_\Lambda(z).$$

Applying the above relation to the left-hand side, we get

$$\Theta_\Lambda\left(\frac{z}{1-4z}\right) = \left(\frac{4-z^{-1}}{2i}\right)^{n/2}\Theta_\Lambda\left(1-\frac{1}{4z}\right).$$

Since we know that Θ_Λ is invariant under $z \mapsto z+1$, we can replace $1-\frac{1}{4z}$ by $-\frac{1}{4z}$ and apply the above relation again to get

$$\Theta_\Lambda\left(\frac{z}{1-4z}\right) = \left(\frac{4-z^{-1}}{2i}\right)^{n/2}(-2iz)^{n/2}\Theta_\Lambda(z) = (1-4z)^{n/2}\Theta_\Lambda(z)$$

as desired. Therefore we see that although Θ_Λ is not in general a modular form for $\mathrm{SL}_2(\mathbb{Z})$, it is one for $\Gamma_1(4)$. $\square$

In our case, $n = 2$ and $\Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2$ is unimodular, and so Θ_Λ is modular of weight 1 for $\Gamma_1(4)$. On the other hand, for any integers $k, N \geq 1$ and primitive odd Dirichlet character χ modulo N we have the Eisenstein series

$$E_{k,\chi}(z) = C\sum_{(x,y)\in\mathbb{Z}^2-\{(0,0)\}} \frac{\chi(y)}{(xNz+y)^k}$$

where C is some explicit normalizing constant (the specific value of which we will compute later).

Proposition 1.2. *The Eisenstein series $E_{k,\chi}(z)$ is a modular form of weight k for $\Gamma_1(N)$.*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$, so that $a \equiv d \equiv 1 \pmod{N}$ and $c \equiv 0 \pmod{N}$. We have

$$\begin{aligned}
E_{k,\chi}(\gamma z) &= E_{k,\chi}\left(\frac{az+b}{cz+d}\right) \\
&= C \sum_{(x,y)\in\mathbb{Z}^2-\{(0,0)\}} \frac{\chi(y)}{(xN\frac{az+b}{cz+d}+y)^k} \\
&= (cz+d)^k C \sum_{(x,y)\in\mathbb{Z}^2-\{(0,0)\}} \frac{\chi(y)}{(xN(az+b)+y(cz+d))^k} \\
&= (cz+d)^k C \sum_{(x,y)\in\mathbb{Z}^2-\{(0,0)\}} \frac{\chi(y)}{((xa+yc/N)Nz+xNb+yd)^k} \\
&= (cz+d)^k C \sum_{(x',y')\in\mathbb{Z}^2-\{(0,0)\}} \frac{\chi(y')}{(x'Nz+y')^k} \\
&= (cz+d)^k E_{k,\chi}(z)
\end{aligned}$$

where $x' = xa + yc/N$ and $y' = xNb + yd$; the mapping $(x, y) \mapsto (x', y')$ is one-to-one (and integral) due to the conditions on a, c, d, and since χ is a character modulo N we have $\chi(y') = \chi(y)$ since $d \equiv 1 \pmod{N}$, which justifies the second-to-last equality. The convergence and growth conditions are less obvious than in the previous case (in fact some care is needed to make it converge), but we will not worry too much about these. □

In particular for $N = 4$ there is a unique odd Dirichlet character χ_4, defined by

$$\chi_4(n) = \begin{cases} 1 & n \equiv 1 \pmod{4} \\ -1 & n \equiv 3 \pmod{4} \\ 0 & n \equiv 0, 2 \pmod{4} \end{cases}.$$

Thus E_{1,χ_4} is also a modular form of weight 1 for $\Gamma_1(4)$; and it can be checked explicitly that the space of such forms is 1-dimensional, so that $\Theta_\Lambda = \alpha E_{1,\chi_4}$ for some nonzero scalar α since neither form is identically 0. In fact since E_{1,χ_4} is normalized and Θ_Λ has constant term $r_2(0) = 1$ since the only solution to $x^2 + y^2 = 0$ is $(0, 0)$, we have $\alpha = 1$ and so $\Theta_\Lambda = E_{1,\chi_4}$.

Proposition 1.3. *The normalized Eisenstein series $E_{1,\chi_4}(z)$ has Fourier expansion given by*

$$E_{1,\chi_4}(z) = 1 + 4\sum_{n\geq 1}\left(\sum_{d|n}\chi_4(d)\right)q^n.$$

Proof. The nth Fourier coefficient is given by

$$e^{2\pi nt}\int_0^1 e^{-2\pi ins}E_{1,\chi_4}(s+it)\,ds = Ce^{2\pi nt}\sum_{(x,y)\in\mathbb{Z}^2-\{(0,0)\}}\chi_4(y)\int_0^1 \frac{e^{-2\pi ins}}{4xs+4ixt+y}\,ds$$

for any $t > 0$. The sum over $y \neq 0$ for $x = 0$ is trivial for $n \geq 1$, since then the integral is 0, so we can assume $x \neq 0$; then the inner sum over y is given by

$$\sum_{y \in \mathbb{Z}} \frac{\chi_4(y)}{4x} \int_0^1 \frac{e^{-2\pi ins}}{s + it + \frac{y}{4x}} = \sum_{a \in \mathbb{Z}} \sum_{b=0}^{4|x|-1} \frac{\chi_4(4xa+b)}{4x} \int_0^1 \frac{e^{-2\pi ins}}{s + it + a + \frac{b}{4x}} \, ds$$

where $y = 4xa + b$. Set $s' = s + a$. Since χ_4 is a character modulo 4, our sum simplifies to

$$\sum_{a \in \mathbb{Z}} \sum_{b=0}^{4|x|-1} \frac{\chi_4(b)}{4x} \int_a^{a+1} \frac{e^{-2\pi ns'}}{s' + it + \frac{b}{4x}} \, ds' = \sum_{b=0}^{4|x|-1} \frac{\chi_4(b)}{4x} \int_{-\infty}^{\infty} \frac{e^{-2\pi ins'}}{s' + it + \frac{b}{4x}} \, ds'.$$

Setting $s'' = s' + \frac{b}{4x}$, this is

$$\frac{1}{4x} \sum_{b=0}^{4|x|-1} \chi_4(b) e^{\frac{1}{2}\pi inb/x} \int_{-\infty}^{\infty} \frac{e^{-2\pi ins''}}{s'' + it} \, ds'',$$

and the integral can be evaluated explicitly (e.g. via the residue theorem) to be $-2\pi i e^{-2\pi nt}$, which simplifies with multiplication by the leading factor $e^{2\pi nt}$. Therefore in all the nth Fourier coefficient is given by

$$-2C\pi i \sum_{x \in \mathbb{Z} - \{0\}} \frac{1}{4x} \sum_{b=0}^{4|x|-1} \chi_4(b) e^{\frac{1}{2}\pi inb/x}.$$

We can rewrite the inner sum as

$$\sum_{c=0}^{x-1} (e^{\frac{1}{2}\pi in(4c+1)/x} - e^{\frac{1}{2}\pi in(4c+3)/x}) = (e^{\frac{1}{2}\pi inx} - e^{\frac{3}{2}\pi inx}) \sum_{c=0}^{x-1} e^{2\pi inc/x}.$$

The sum is 0 unless $x|n$, in which case it is x, and the leading factor is $i^{nx} - (-i)^{nx}$, which is 0 if nx is even, $2i$ if it is 1 modulo 4, and $-2i$ if it is 3 modulo 4. Therefore for $n \geq 1$ the nth Fourier coefficient is

$$2C\pi \sum_{x|n} \chi_4(x),$$

where the additional factor of 2 comes from taking both positive and negative x (the sign change in x causes two sign changes in the sum, which cancel).

For $n = 0$, we can compute the Fourier coefficient by taking the limit as $q \to 0$, i.e. $z \to +i\infty$. For $z = it$, by Proposition 1.2 we have

$$E_{1,\chi}\left(\frac{z}{1-4z}\right) = E_{1,\chi}\left(\frac{it}{1-4it}\right) = (1-4it)E_{1,\chi}(it),$$

and so

$$\lim_{t \to \infty} E_{1,\chi}(it) = \lim_{t \to \infty} \frac{1}{1-4it} E_{1,\chi}\left(\frac{it}{1-4it}\right).$$

Since $\frac{it}{1-4it}$ tends to $-\frac{1}{4}$ as $t \to \infty$, we look at the behavior of $E_{1,\chi_4}(z) = C\sum_{(x,y)\in\mathbb{Z}^2-\{(0,0)\}} \frac{\chi_4(y)}{4xz+y}$ near $z = -\frac{1}{4}$, and we see that E_{1,χ_4} has a sum of simple poles coming from the diagonal $y = x$. The sum of the residues is

$$C\sum_{x\neq 0}\frac{\chi_4(x)}{4x} = \frac{1}{2}C\sum_{x\geq 1}\frac{\chi_4(x)}{x} = \frac{C}{2}L(\chi_4, 1)$$

where $L(\chi_4, s)$ is the L-function. Therefore

$$\lim_{t\to\infty} E_{1,\chi}(it) = \lim_{t\to\infty}\frac{1}{1-4it}E_{1,\chi}\left(\frac{it}{1-4it}\right) = \lim_{t\to\infty}\frac{1}{1-4it}\frac{C}{2}L(\chi_4,1)\frac{1}{\frac{it}{1-4it}+\frac{1}{4}} = 2CL(\chi_4,1).$$

This particular L-function value is well-known to be $\frac{\pi}{4}$, and so we conclude that the 0th Fourier coefficient is

$$\frac{\pi}{2}C.$$

But C is defined to be the scalar that makes this quantity 1, so $C = \frac{2}{\pi}$ and so the nth Fourier coefficient is given by

$$4\sum_{x|n}\chi_4(x)$$

as desired. $\square$

Corollary 1.4 (Fermat). *Let p be a prime. Then*

$$r_2(p) = \begin{cases} 8 & p \equiv 1 \pmod 4 \\ 0 & n \equiv 3 \pmod 4 \\ 4 & p = 2 \end{cases}.$$

In particular $p \geq 3$ can be written as the sum of two squares if and only if it is congruent to 1 modulo 4.

Proof. By the equality $\Theta_\Lambda = E_{1,\chi_4}$ and Proposition 1.3, comparing Fourier coefficients gives

$$r_2(n) = 4\sum_{d|n}\chi_4(d)$$

for each integer $n \geq 1$. For $n = p$ prime, the only divisors are 1 and p, and so $r_2(p) = 4(\chi_4(1) + \chi_4(p)) = 4(1 + \chi_4(p))$; the result follows from the definition of χ_4. $\square$

2. Reformulating

Now we want to reinterpret the identity $\Theta_\Lambda = E_{1,\chi}$ in more abstract language, in terms of a certain representation of $\mathrm{SL}_2(\mathbb{A})$, where $\mathbb{A}$ are the adeles. Let $V_\mathbb{A}$ be an n-dimensional vector space (free module of rank n) over the adeles $\mathbb{A}$, equipped with a rational-valued unimodular quadratic form $Q = \frac{1}{2}\langle\cdot,\cdot\rangle$, and $S = \mathcal{S}(V_\mathbb{A})$ be the space of Schwartz functions on $V_\mathbb{A}$ (so in our case above $n = 2$). Then we have an action of $\mathrm{SL}_2(\mathbb{A})$ on S, defined as follows. Fix an additive character ψ of $\mathbb{A}$ with real factor the exponential $a \mapsto e^{2\pi ia}$ which is trivial on $\mathbb{Q}$

and use it to define the Fourier transform $\hat{f}$ for $f \in S$. Then we define an action of $\mathrm{SL}_2(\mathbb{A})$ on S by

$$\left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \cdot f\right)(x) = \chi(a)|a|^{n/2} f(ax),$$

$$\left(\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \cdot f\right)(x) = \psi(aQ(x)) f(x),$$

and

$$\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \cdot f\right)(x) = \hat{f}(x)$$

for each $f \in S$, $x \in V_{\mathbb{A}}$, and $a \in \mathbb{A}$ where $|a| = \prod_v |a|_v$ is the adelic absolute value and $\chi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \{\pm 1\}$ is the idele character associated to the extension $\mathbb{Q}(\sqrt{(-1)^{n/2} \operatorname{disc} Q})/\mathbb{Q}$ by class field theory. By the Bruhat decomposition this suffices to define an action of all of $\mathrm{SL}_2(\mathbb{A})$ on S.

In fact, this comes from a more general action (Proposition II.4.3 of [?]) of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by

$$f(x) \mapsto \chi(\alpha)|\alpha|^{n/2} \int_{\ker c \backslash V_{\mathbb{A}}} \psi\left(\frac{1}{2}\langle ax, bx\rangle + \langle bx, cy\rangle + \frac{1}{2}\langle cy, dy\rangle\right) f(ax + cy)\, dy$$

where $\alpha = c$ for c nonzero and a for $c = 0$, $\ker c$ is defined with respect to the scaling action of c on V, so that it is 0 if $c \neq 0$ and all of $V_{\mathbb{A}}$ if $c = 0$, and $\langle \cdot, \cdot \rangle$ is the pairing on $V_{\mathbb{A}}$. This version can be extended to higher symplectic groups beyond $\mathrm{SL}_2 = \mathrm{Sp}(1)$. Observe that for the first two cases above $c = 0$ and so this is just evaluation

$$f(x) \mapsto \chi(a)|a|^{n/2} \psi f(ax)$$

in the first case since $b = 0$ and

$$f(x) \mapsto \psi(aQ(x)) f(x)$$

in the second since $a = 1$; and finally if $a = d = 0$ and $-b = c = 1$ this gives

$$\int_{V_{\mathbb{A}}} \psi(-\langle x, y\rangle) f(y)\, dy = \hat{f}(x),$$

so this gives the same action as the above definition.

Now, write $V_{\mathbb{Q}} \subset V_{\mathbb{A}}$ for the $\mathbb{Q}$-valued points. These form a discrete lattice in $V_{\mathbb{A}}$, and for any test function $f \in S$ we can define the "theta function"

$$\sum_{x \in V_{\mathbb{Q}}} f(x).$$

We are interested in the action of $\mathrm{SL}_2(\mathbb{A})$ on S, and so for $g \in \mathrm{SL}_2(\mathbb{A})$ we define

$$\theta_f(g) = \sum_{x \in V_{\mathbb{Q}}} (g \cdot f)(x).$$

Note that since ψ and χ are trivial on $\mathbb{Q}$ the action of the parabolic $P(\mathbb{Q}) = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\}$ is by scaling the argument of f by a rational, which does not change the sum over all $x \in V_{\mathbb{Q}}$, so really this gives an action of $P(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})$.

There is also an action of the orthogonal group $\mathrm{O}(V_{\mathbb{A}}) = \mathrm{O}(Q, \mathbb{A})$ on S by (inverted) precomposition, and so for $(g, h) \in \mathrm{SL}_2(\mathbb{A}) \times \mathrm{O}(V_{\mathbb{A}})$ we define

$$\theta_f(g, h) = \sum_{x \in V_{\mathbb{Q}}} (g \cdot f)(h^{-1}x).$$

We want to remove the dependency on S, which we can do by making a canonical choice for f: specifically we want something which is self-dual under the adelic Fourier transform. We can do this by fixing a lattice $\Lambda \subset \mathbb{Q}^n$ and completing at each prime p to get a lattice $\Lambda_p \subset \mathbb{Q}_p^n$ and take ϕ_p to be the indicator function of Λ_p, and at infinity we set $\phi_\infty(x) = e^{-2\pi Q(x)}$. Then we have a canonical function $\Theta_\phi(g, h)$.

This depends on two variables, unlike our previous theta functions; but we can fix that. We saw above that the action of $g \in P(\mathbb{Q})$ is trivial, and the action of $h \in \mathrm{O}(V_{\mathbb{Q}})$ permutes the $x \in V_{\mathbb{Q}}$ and therefore does not affect the value of the sum. Therefore θ_f is a function on $P(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}) \times \mathrm{O}(V_{\mathbb{Q}})\backslash\mathrm{O}(V_{\mathbb{A}})$. In particular to eliminate the orthogonal action we can integrate over the compact quotient $\mathrm{O}(V_{\mathbb{Q}})\backslash\mathrm{O}(V_{\mathbb{A}})$ to get

$$\Theta_f(g) := \int_{\mathrm{O}(V_{\mathbb{Q}})\backslash\mathrm{O}(V_{\mathbb{A}})} \theta_f(g, h)\, dh$$

where dh is a Haar measure on $\mathrm{O}(V_{\mathbb{A}})$ normalized such that the stabilizer of Λ, i.e. the product $\mathrm{Stab}(\Lambda) = \prod_p \mathrm{Stab}(\Lambda_p) \times \mathrm{O}(V_{\mathbb{R}})$ of the stabilizers of Λ_p over all p, has volume 1. More generally for any automorphic form F for $\mathrm{O}(V_{\mathbb{A}})$ we can consider the theta integral

$$\Theta_\phi(F)(g) := \int_{\mathrm{O}(V_{\mathbb{Q}})\backslash\mathrm{O}(V_{\mathbb{A}})} F(h)\theta_\phi(g, h)\, dh.$$

How does this relate to our theta functions from the previous section? Well, $\mathrm{O}(V_{\mathbb{Q}})\backslash\mathrm{O}(V_{\mathbb{A}})$ decomposes as a finite union of $\mathrm{Stab}(\Lambda)$-cosets, each corresponding to a homothety class $a\Lambda$ of lattices in the $\mathrm{O}(V_{\mathbb{A}})$-orbit of Λ. For simplicity we consider the case where F is the indicator for one such coset $a\,\mathrm{Stab}(\Lambda)$ (we could also take $F = 1$ and take the sum over finitely many a). Then

$$\begin{aligned} \Theta_\phi(F)(g) &= \int_{a\,\mathrm{Stab}(\Lambda)} \theta_\phi(g, h)\, dh \\ &= \int_{\mathrm{Stab}(\Lambda)} \theta_\phi(g, ah)\, dh \\ &= \prod_v \int_{\mathrm{Stab}(\Lambda_v)} \sum_{x \in V_{\mathbb{Q}}} (g \cdot \phi)_v(h_v^{-1}a_v^{-1}x)\, dh_v. \end{aligned}$$

By strong approximation we can rescale g such that each component g_p is in $\mathrm{SL}_2(\mathbb{Z}_p)$ and therefore preserves the indicator function ϕ_p of Λ_p; and $\phi_p(h_p^{-1}a_p^{-1}x) = 1$ if $x \in a_p h_p \Lambda_p =$

$a_p\Lambda_p = (a\Lambda)_p$, since $h \in \mathrm{Stab}(\Lambda)$, and 0 otherwise. Therefore the factor at p is just

$$\int_{\mathrm{Stab}(\Lambda_p)} dh_p$$

and so by our choice of measure and the fact that ϕ_∞ is $\mathcal{O}(V_{\mathbb{R}})$-invariant we have

$$\Theta_\phi(F)(g) = \sum_{x \in a\Lambda} (g \cdot \phi)_\infty(x).$$

In particular the dependence on g is only via its component at infinity $g_\infty \in \mathrm{SL}_2(\mathbb{R})$. Since ϕ_∞ is its own Fourier transform (up to simple terms), we can restrict attention to g_∞ of the form

$$g_\infty(z) = \begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & \\ & 1/\sqrt{\beta} \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & \frac{\alpha}{\sqrt{\beta}} \\ & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

for $z = \alpha + i\beta$ in the upper half-plane; note that

$$g_\infty(z) \cdot i = \alpha + \beta i = z$$

and so this is a natural way of viewing a function on the upper half-plane as one on $\mathrm{SL}_2(\mathbb{R})$. The action of $g_\infty(z)$ is then given by

$$(g_\infty(z) \cdot \phi)_\infty(x) = \psi_\infty(\alpha Q(x))\beta^{n/4} e^{-\pi Q(\sqrt{\beta}x)}.$$

Since at infinity ψ is given by the exponential this is just

$$\beta^{n/4} e^{2\pi i \alpha Q(x)} e^{-2\pi\beta Q(x)} = \beta^{n/4} e^{2\pi i Q(x) z}.$$

Therefore

$$\Theta_\phi(F)(g) = \Theta_\phi(F)(g_\infty(z)) = \beta^{n/4} \sum_{x \in a\Lambda} e^{2\pi i Q(x) z}.$$

The sum is the theta series $\Theta_{a\Lambda}(z)$ from the previous section; the extra factor of $\beta^{n/4}$ corresponds to the fact that the theta series is a modular form of weight $n/2$, and for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = g_\infty(z)$ we have $(cz+d)^{n/2} = \beta^{-n/4}$, so that $\Theta_\phi(F)$ lifts $\Theta_{a\Lambda}$ via evaluation at i. In particular for our case $n = 2$ and $\Lambda = \mathbb{Z}^2$ as in section 1 there is only one homothety class of lattices in the orbit of Λ and so $\Theta_\phi(1)(g_\infty(z)) = \sqrt{\beta}\Theta_\Lambda(z)$.

This generalizes one side of the correspondence. We still have to deal with Eisenstein series. For a fixed Schwartz function f, consider the function on $\mathrm{SL}_2(\mathbb{A})$ given by

$$g \mapsto (g \cdot f)(0).$$

For

$$p = \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & ab \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \in P(\mathbb{A}) \subset \mathrm{SL}_2(\mathbb{A}),$$

we have

$$(p \cdot f)(0) = \chi(a)|a|^{n/2} f(0),$$

i.e. the unipotent radical acts trivially and the Levi subgroup acts by $\chi(\cdot)|\cdot|^{n/2}$. Again the action of $P(\mathbb{Q})$ is trivial. Define

$$E_f(g) = \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})} (\gamma g \cdot f)(0),$$

and to remove the dependence on a Schwartz function f choose our self-dual function $f = \phi$ as above.

The map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [c:d]$ gives a bijection $P(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q}) \to \mathbb{P}^1(\mathbb{Q})$, and in the same way gives a bijection $P(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{P}^1(\mathbb{Z})$; since $\mathbb{P}^1(\mathbb{Q})$ and $\mathbb{P}^1(\mathbb{Z})$ can be naturally identified it follows that $P(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})$ and $P(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})$ are in bijection.

For the component at infinity, $\phi_\infty(x) = e^{-2\pi Q(x)}$, we have the decomposition

$$\mathrm{SL}_2(\mathbb{R}) = P(\mathbb{R})\,\mathrm{SO}(2,\mathbb{R}),$$

and for

$$g = \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \in P(\mathbb{R})$$

and

$$h_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2,\mathbb{R})$$

we have

We can compute that $h_\theta \cdot \phi_\infty = e^{i\theta n/2}\phi_\infty$. It is not hard to see from the above formulas and the fact that ϕ is its own Fourier transform that the action of h_θ on ϕ must give some multiple of ϕ, possibly rescaled by some factor a depending on θ. But in fact this factor must be 1: for θ any rational multiple of 2π, by iterating the action of h_θ we must recover the original ϕ, which is not possible for ϕ_∞ unless $a = 1$ for those values of θ; and since the action is continuous it follows that $a = 1$ for all θ. Therefore it remains only to compute the multiple of ϕ, or equivalently the value of $h_\theta \cdot \phi$ at 0.

This is most easily done using the general formula introduced above: for $\sin\theta \neq 0$, the integral is

$$\int_{V_\mathbb{A}} \psi(-\sin\theta\cos\theta Q(x) - \sin^2\theta\,\langle x, y\rangle + \sin\theta\cos\theta Q(y))\phi(x\cos\theta + y\sin\theta)\,dy,$$

which at $x = 0$ is just

$$\int_{V_\mathbb{A}} \psi(\sin\theta\cos\theta Q(y))\phi(y\sin\theta)\,dy.$$

(If $\sin\theta = 0$, then $h_\theta = \pm 1$ and the claim is trivial.) Since all components are trivial away from infinity, we can restrict to the component at infinity where this becomes

$$\int_{\mathbb{R}^n} e^{2\pi i \sin\theta\cos\theta Q(y)) - 2\pi Q(y\sin\theta)}\,dy = \int_{\mathbb{R}^n} e^{-2\pi i e^{-i\theta}\sin\theta Q(y)}\,dy.$$

Since $2Q = \langle\cdot,\cdot\rangle$ is unimodular, standard methods give the integral as

$$\frac{1}{(ie^{-i\theta}\sin\theta)^{n/2}}.$$

The leading terms of the action contribute the remaining factors to give a total value of

$$(h_\theta \cdot \phi)(0) = e^{i\theta n/2},$$

which by the previous remark concludes the computation.

Let $z = \alpha + \beta i$ and $g_\infty(z)$ be such that $g_\infty(z) \cdot i = z$ as above for the usual action on the upper half-plane. We can use the decomposition $\mathrm{SL}_2(\mathbb{R}) = P(\mathbb{R})\,\mathrm{SO}(2,\mathbb{R})$ to write $\gamma g_\infty(z) = g' h_\theta$ for some $g' \in P(\mathbb{R})$ and $h_\theta \in \mathrm{SO}(2,\mathbb{R})$ as above; set $z' = g' \cdot i$. Then

$$(\gamma g_\infty(z) \cdot \phi_\infty)(0) = (g' h_\theta \cdot \phi_\infty)(0) = e^{i\theta n/2}(g' \cdot \phi_\infty)(0).$$

Since $g' \in P(\mathbb{R})$ and $g' \cdot i = z'$ we can write $g' = g_\infty(z')$, and from above we know that

$$(g_\infty(z') \cdot \phi_\infty)(x) = (\beta')^{n/4} e^{2\pi Q(x) z'}$$

where we write $z' = \alpha' + \beta' i$. Since we want to evaluate at $x = 0$, the remaining thing is to compute β'.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have

$$\gamma g_\infty(z) \cdot i = \gamma \cdot z = \frac{az+b}{cz+d}.$$

On the other hand

$$g' h_\theta \cdot i = g' \cdot \frac{i\cos\theta - \sin\theta}{i\sin\theta + \cos\theta} = g' \cdot \frac{ie^{i\theta}}{e^{i\theta}} = g' \cdot i = z',$$

so we can compute the imaginary part of z' explicitly in terms of z to get

$$\beta' = \frac{\beta}{|cz+d|^2}.$$

Since

$$g' = g_\infty(z') = \begin{pmatrix} \sqrt{\beta'} & \frac{\alpha'}{\sqrt{\beta'}} \\ & \frac{1}{\sqrt{\beta'}} \end{pmatrix}$$

we can compute

$$h_\theta = {g'}^{-1}\gamma g_\infty(z) = \begin{pmatrix} \sqrt{\beta'} & \frac{\alpha'}{\sqrt{\beta'}} \\ & \frac{1}{\sqrt{\beta'}} \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & \frac{\alpha}{\sqrt{\beta}} \\ & \frac{1}{\sqrt{\beta}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta}{\beta'}}(a - \alpha c) & \frac{\alpha a + b - \alpha^2 c - \alpha d}{\sqrt{\beta\beta'}} \\ c\sqrt{\beta\beta'} & \sqrt{\frac{\beta'}{\beta}}(\alpha c + d) \end{pmatrix}$$

and therefore

$$e^{i\theta} = \cos\theta + i\sin\theta = \sqrt{\frac{\beta'}{\beta}}(\alpha c + d) - ic\sqrt{\beta\beta'} = \sqrt{\frac{\beta'}{\beta}}(c\bar{z} + d).$$

Therefore

$$(\gamma g_\infty(z) \cdot \phi_\infty)(0) = e^{i\theta n/2}(\beta')^{n/4} = \left(\frac{\beta'}{\beta}\right)^{n/4}(c\bar{z}+d)^{n/2}\frac{\beta^{n/4}}{|cz+d|^{n/2}} = \frac{\beta^{n/4}(c\bar{z}+d)^{n/2}}{|cz+d|^n}.$$

Since $|cz+d| = \sqrt{(cz+d)(c\bar{z}+d)}$, this is just

$$\frac{\beta^{n/4}}{(cz+d)^{n/2}}.$$

This gives the component at infinity of $(\gamma g_\infty(z)\cdot\phi)(0)$. To get the remaining components, observe that at each finite prime p since $g_\infty(z)_p$ is the identity this is just $(\gamma\cdot\phi)(0)$. Let N be the conductor of the character χ. Since χ is defined by the extension $\mathbb{Q}(\sqrt{(-1)^{n/2}\operatorname{disc} Q})/\mathbb{Q}$, by the definition of ϕ_p we have $(\gamma\cdot\phi)(0)$ nonzero if and only if $N|c$, so that rescaling by c preserves Λ. If this holds, the only effect of the action of γ is to multiply $\phi(0) = 1$ by $\chi_p(d)$. Thus the total contribution at γ is $\frac{\chi(d)\beta^{n/4}}{(cz+d)^{n/2}}$ for $N|c$; from now on we replace c by cN to ensure that this holds. Since the sum over γ can be taken over $\mathbb{P}^1(\mathbb{Z})$, it suffices to sum over all coprime pairs (c,d) (the difference between $\gcd(c,d) = 1$ and $\gcd(cN,d) = 1$ doesn't matter, since if $\gcd(N,d) > 1$ we have $\chi(d) = 0$ in any case), where say c is restricted to be nonnegative. Therefore we have

$$E_\phi(g_\infty(z)) = \beta^{n/4}\sum_{\substack{\gcd(c,d)=1\\ c\geq 0}}\frac{\chi(d)}{(cNz+d)^{n/2}}.$$

Like the theta series, this differs from our previous definition by a factor of $\beta^{n/4}$; it also differs in that the sum is now over only coprime pairs of integers. This is easily rectified:

$$\begin{aligned}
\beta^{-n/4}E_\phi(g_\infty(z)) &= \sum_{\substack{\gcd(c,d)=1\\ c\geq 0}}\frac{\chi(d)}{(cNz+d)^{n/2}}\\
&= \sum_{\substack{(c,d)\neq(0,0)\\ c\geq 0}}\sum_{k|\gcd(c,d)}\mu(k)\frac{\chi(d)}{(cNz+d)^{n/2}}\\
&= \sum_{k\geq 1}\mu(k)\sum_{\substack{(c,d)\neq(0,0)\\ c\geq 0}}\frac{\chi(kd)}{(kcNz+kd)^{n/2}}\\
&= \frac{1}{2}\sum_{k\geq 1}\frac{\chi(k)\mu(k)}{k^{n/2}}E_{n/2,\chi}(z)\\
&= \frac{1}{2CL(\chi,n/2)}E_{n/2,\chi}(z)
\end{aligned}$$

where C is the normalizing constant; the factor of $\frac{1}{2}$ comes from adding the $c < 0$ terms (note that the $c = 0$ terms cancel since $\chi(-d) = -\chi(d)$). In our particular case where $n = 2$ and $N = 4$, we have $L(\chi,1) = \frac{\pi}{4}$ and $C = \frac{2}{\pi}$ so this gives

$$E_\phi(g_\infty(z)) = \sqrt{\beta}E_{1,\chi}(z).$$

Thus this is actually a cleaner expression as far as the normalization.

Combining this with the equality above

$$\Theta_\phi(1)(g_\infty(z)) = \beta^{n/4}\Theta_\Lambda(z),$$

we can view the result $\Theta_\Lambda(z) = E_{1,\chi}(z)$ from section 1 as an instance of the Siegel-Weil formula stating that

$$\Theta_\phi(1) = E_\phi.$$

3. The E_8 lattice

Our next example is Λ equal to the E_8 lattice, which is the unique rank 8 positive-definite unimodular even lattice. One way to define it explicitly as a lattice in $\mathbb{R}^8$ is as the set of vectors whose entries are either all integers or all half integers and whose sum is an even integer; an example basis is

$$\begin{aligned}
e_1 &= (2,0,0,0,0,0,0,0),\\
e_2 &= (-1,1,0,0,0,0,0,0),\\
e_3 &= (0,-1,1,0,0,0,0,0),\\
e_4 &= (0,0,-1,1,0,0,0,0),\\
e_5 &= (0,0,0,-1,1,0,0,0),\\
e_6 &= (0,0,0,0,-1,1,0,0),\\
e_7 &= (0,0,0,0,0,-1,1,0),\\
e_8 &= \frac{1}{2}(1,1,1,1,1,1,1,1).
\end{aligned}$$

Notice that this has quadratic form

$$Q(x_1,\dots,x_8) = \frac{1}{2}(x_1^2+\cdots+x_8^2)$$

given in this basis by

$$\begin{aligned}
8Q(x_1e_1+\cdots+x_8e_8) = (4x_1-2x_2+x_8)^2 + (2x_2-2x_3+x_8)^2 + \cdots + (2x_6-2x_7+x_8)^2 \\
+ (2x_7+x_8)^2 + x_8^2.
\end{aligned}$$

As this is always even, we normalize it by again dividing by 2.

We can associate to Λ a theta function

$$\Theta_\Lambda(z) = \sum_{x\in\Lambda} q^{Q(x)}.$$

By Proposition 1.1, Θ_Λ is a modular form of weight 4 and level 4. Observe that the same arguments as in section 2 show that Θ_Λ can be reinterpreted as $\beta^{-2}\Theta_\phi(F)(g_\infty(z))$.

On the other side of the Siegel-Weil formula, we have the (abstract) Eisenstein series

$$E_\phi(g) = \sum_{\gamma\in P(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})} (\gamma g\cdot\phi)(0).$$

Since Λ is positive-definite and $4|8$, the discriminant of Λ is 1 and so the associated character χ is trivial; therefore

$$E_\phi(g_\infty(z)) = \beta^2 \sum_{\substack{\gcd(c,d)=1 \\ c\geq 0}} \frac{1}{(cz+d)^4} = \frac{\beta^2}{2\zeta(4)} \sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(cz+d)^4}.$$

Thus from the Siegel-Weil formula we expect

$$\Theta_\Lambda(z) = \frac{1}{2\zeta(4)} \sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(cz+d)^4}.$$

The following proposition tells us that the right-hand side is the normalized Eisenstein series, so that as in section 1 we have an equality of a theta series and an Eisenstein series.

Proposition 3.1. *We have*

$$E_{4,1}(z) = \frac{1}{2\zeta(4)} \sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(cz+d)^4},$$

i.e. the normalizing constant of $E_{4,1}(z)$ is $\frac{1}{2\zeta(4)}$, and $E_{4,1}(z)$ has Fourier expansion

$$E_{4,1}(z) = 1 + 240 \sum_{n\geq 1} \left(\sum_{d|n} d^3 \right) q^n.$$

Proof. It is clear that the right-hand side is a constant multiple of the left-hand side, so in particular by Proposition 1.2 both sides are modular forms of weight 4 for $\Gamma_1(1) = \mathrm{SL}_2(\mathbb{Z})$. As in the proof of Proposition 1.3, we compute the constant term by taking the limit as $q \to 0$, i.e. as $z \to +i\infty$. Letting $z = it$, we have

$$E_{4,1}\left(\frac{i}{t}\right) = E_{4,1}\left(-\frac{1}{z}\right) = E_{4,1}(z) = E_{4,1}(it)$$

since $E_{4,1}$ is a modular form of weight 4 for $\mathrm{SL}_2(\mathbb{Z})$, so

$$\lim_{t\to\infty} E_{4,1}(it) = \lim_{t\to\infty} E_{4,1}\left(\frac{i}{t}\right) = \lim_{t\to 0} E_{4,1}(it) = C \lim_{t\to 0} \sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(ict+d)^4}.$$

For $t > 0$ this converges absolutely, and so we can pair the terms at (c,d) with those at $(-c,d)$; then taking $t = 0$ the summation over c nonzero cancels and we are left with

$$C \sum_{d\in\mathbb{Z}-\{0\}} \frac{1}{d^4} = 2C\zeta(4).$$

Since this is normalized to be 1, we conclude that $C = \frac{1}{2\zeta(4)}$, which proves the first claim.

For $n > 0$, the nth Fourier coefficient is given by

$$e^{2\pi nt}\int_0^1 e^{-2\pi ins}E_{4,1}(s+it)\,ds = \frac{e^{2\pi nt}}{2\zeta(4)}\sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}}\int_0^1 \frac{e^{-2\pi ins}}{(cs+ict+d)^4}\,ds$$

for any $t > 0$. We can proceed just as in the proof of Proposition 1.3, adjusting for level and weight, to get

$$\frac{1}{2\zeta(4)}e^{2\pi nt}\sum_{c\in\mathbb{Z}-\{0\}}\frac{1}{c^4}\sum_{b=0}^{|c|-1}e^{2\pi inb/c}\int_{-\infty}^{\infty}\frac{e^{-2\pi ins''}}{(s''+it)^4}\,ds''.$$

The integral is

$$\frac{8\pi^4n^3}{3}e^{-2\pi nt}$$

and the inner sum is 0 unless c divides n, in which case it is $|c|$, so in all (doubling and restricting to $c > 0$, since $c < 0$ gives the same term) this is

$$\frac{8\pi^4n^3}{3\zeta(4)}\sum_{c|n}\frac{1}{c^3} = \frac{8\pi^4}{3\zeta(4)}\sum_{c|n}\left(\frac{n}{c}\right)^3.$$

Since $\zeta(4) = \frac{\pi^4}{90}$ and replacing c by $\frac{n}{c}$ only permutes the divisors, this is

$$240\sum_{c|n}c^3,$$

which gives the desired formula. $\square$

Since the space of modular forms of weight 4 and level $\Gamma_1(4)$ has dimension 3 we can now conclude that $\Theta_\Lambda = E_{4,1}$ by computing the first three Fourier coefficients of Θ_Λ, i.e. the number of vectors in Λ with length 0 (one), length 1 (240 - recall that we've normalized by dividing Q by 2), and length 2 (2160), agreeing with the first three coefficients of $E_{4,1}$.

Notice that it follows that Θ_Λ is in fact a modular form of weight 4 for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, rather than just for $\Gamma_1(4)$ as from Proposition 1.1. This can also be checked directly. Since the space of modular forms of weight 4 for $\mathrm{SL}_2(\mathbb{Z})$ is one-dimensional, we can conclude immediately that $\Theta_\Lambda = E_{4,1}$ immediately from the normalization, without computing further Fourier coefficients.

4. The rank 16 and 24 cases

There are two positive-definite unimodular even lattices in rank 16: one is just two copies of the E_8 lattice, and the other is given by generalizing its definition in the obvious way to rank 16. We are concerned with the latter, which we will call Λ as usual, or the E_{16} lattice.

As in the case $n = 8$, our Eisenstein series is

$$E_\phi(g_\infty(z)) = \frac{\beta^4}{2\zeta(8)}\sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}}\frac{1}{(cz+d)^8},$$

and by the same method as in the proof of Proposition 3.1 (replacing everywhere $4 = \frac{8}{2}$ by $8 = \frac{16}{2}$) we see that this is equal to $\beta^4 E_{8,1}(z)$, and that we have a Fourier expansion

$$E_{8,1}(z) = 1 + \frac{1}{\zeta(8)} \cdot \frac{16\pi^8}{315} \sum_{n \geq 1} \left(\sum_{d|n} d^7 \right) q^n = 1 + 480 \sum_{n \geq 1} \left(\sum_{d|n} d^7 \right) q^n.$$

This is a modular form of weight 8 for the full modular group $\mathrm{SL}_2(\mathbb{Z})$; the space of such modular forms is one-dimensional, and again Θ_Λ is also such a modular form since $8|16$. Since it is also normalized we conclude that again $\Theta_\Lambda = E_{8,1}$.

In fact we can generalize this: for any positive integer k, the corresponding integral is

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i n s''}}{(s'' + it)^{2k}} \, ds'' = -\frac{4k\zeta(2k)}{B_{2k}}$$

where B_{2k} is the Bernoulli number, and so

$$E_{2k,1}(z) = 1 - \frac{1}{\zeta(2k)} \cdot \frac{4k\zeta(2k)}{B_{2k}} \sum_{n \geq 1} \left(\sum_{d|n} d^{2k-1} \right) q^n = 1 - \frac{4k}{B_{2k}} \sum_{n \geq 1} \left(\sum_{d|n} d^{2k-1} \right) q^n.$$

(Note that since B_{2k} is always negative for even k, the leading sign will be positive for our cases of interest $4|k$.)

However, it is *not* in general true that $\Theta_\Lambda = E_{2k,1}$ for Λ the generalization of the E_8 and E_{16} lattices to rank $4k$. One easy way to see this is that the Fourier coefficients of Θ_Λ must all be integers, but e.g. for $k = 6$ we have

$$E_{12,1}(z) = 1 + \frac{65520}{691} \sum_{n \geq 1} \left(\sum_{d|n} d^{11} \right) q^n.$$

This is due to the failure of the equation $\Theta_\phi(1) = \beta^6 \Theta_\Lambda$, which is due to the presence of additional homothety classes in the orbit of Λ, so that we need to average over all of them, weighted by the size of their automorphism groups. In other words we have

$$\Theta_\phi(1) = \frac{\sum_{\Lambda'} \frac{1}{\mathrm{Aut}\,\Lambda'} \Theta_{\Lambda'}}{\sum_{\Lambda'} \frac{1}{\mathrm{Aut}\,\Lambda'}}$$

where the sum is over all lattices in the genus of Λ, i.e. all unimodular positive-definite even lattices of rank 24. There are 24 of these, and since the space of modular forms of weight 12 and level 1 is 2-dimensional it suffices to compute the first coefficient of each theta series, as well as the (very large) automorphism groups, to verify the claim.