Notes on the Gross-Zagier formula: deformation theory sketch

This is an attempt to understand an alternate proof of Proposition 2.3 of [2] using deformation theory rather than explicit arguments involving the defining equations for elliptic curves. The essence of the argument appears in section 4 of [1], though I have expanded some arguments and omitted others.

Fix elliptic curves E_1, E_2 over a complete DVR W with complex multiplication by orders of imaginary quadratic fields $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Let π be a uniformizer of W and v be the valuation, and assume that E_1 and E_2 are isomorphic over W/π.

Let X be the modular curve, base changed to W, and Δ be the diagonal. We have the uniformization $j : X \xrightarrow{\sim} \mathbb{P}^1_W$ and thus $j \times j : X \times X \xrightarrow{\sim} \mathbb{P}^1_W \times \mathbb{P}^1_W$, with the diagonal cut out by $j_1 - j_2$ where j_i denotes j on the ith component; thus the intersection number of Δ with the point $E_1 \times E_2 \in X \times X$ is given by the valuation $v(j(E_1) - j(E_2))$. By reduction to the diagonal this is just $E_1 \cdot E_2$.

Let E be the elliptic curve defined over W/π such that mod π we have $E_1 \simeq E \simeq E_2$. Thus both E_1 and E_2 are deformations of E to W. Letting R be the universal deformation ring of E with universal deformation $\tilde{E}/R$, it follows that there are maps $\varphi_i : R \to W$ such that base changing along them and then reducing modulo π sends $\tilde{E} \mapsto E_i \mapsto E$. As an abstract ring, R is isomorphic to $W[[T]]$; note that the (completed) local ring $\widehat{\mathcal{O}}_{X,E}$ is also isomorphic to $W[[T]]$, since $X \simeq \mathbb{P}^1_W$. In fact $\widehat{\mathcal{O}}_{X,E}$ consists of the (isomorphism classes of) functions on a neighborhood of $\tilde{E}$ in $X(W)$ which descend to a neighborhood of E in $X(W/\pi)$, i.e. those that are $\mathrm{Aut}(E)$-invariant; but $\widehat{\mathcal{O}}_{X,\tilde{E}}$ is just R. Since every elliptic curve carries an involution given by inversion, this involution acts trivially on R, so setting $G = \mathrm{Aut}(E)/\{\pm 1\}$ to be the quotient by this involution we get $\widehat{\mathcal{O}}_{X,E} = R^G \subseteq R$.

As above we can think of the curves E_i as deformations of E to W, and therefore we can identify them with maps $\phi_i : R \to W$. Fix generators T_i of $\ker \phi_i$. To get invariant elements, set

$$t_i = \prod_{g \in G} g(T_i);$$

then $\widehat{\mathcal{O}}_{X,E} = R^G \simeq W[[t_i]]$, and the restriction of ϕ_i to R^G has kernel generated by t_i. Thus we have maps $\alpha_i : \widehat{\mathcal{O}}_{X,E} \to R$ which are the identity on W and send $t_i \mapsto T_i$, which have degree $|G|$.

We've found that the prime ideals corresponding to the points E_1, E_2 over the geometric point E on X are given by (t_1) and (t_2), so we have

$$E_1 \cdot E_2 = \mathrm{length}(\widehat{\mathcal{O}}_{X,E}/(t_1, t_2)) = \frac{1}{|G|} \mathrm{length}\left(R \Big/ \left(\prod_{g_1 \in G} g_1(T_1), \prod_{g_2 \in G} g_2(T_2) \right) \right)$$

since the map $\widehat{\mathcal{O}}_{X,E} \to R$ is of degree $|G|$ (making some niceness assumptions). This can be rewritten as

$$\frac{1}{|G|} \sum_{g_1, g_2 \in G} \mathrm{length}(R/(g_1(T_1), g_2(T_2))).$$

Now, the choices of T_1 and T_2 correspond to the deformations E_1 and E_2 of E respectively. Given $g \in G$, we can generate new ones as follows: we have a reduction map $E_i \to E$, which we can compose with g as an automorphism (after choosing a representative in $\mathrm{Aut}(E)$). This gives a new reduction map of elliptic curves $E_i \to E$, and thus a new deformation of E to W (though as an elliptic curve over W it is the same one). Call this new deformation $g(E_i)$. Then we get reduction maps $g_1(E_1) \leftarrow \tilde{E} \to g_2(E_2)$ for each g_1, g_2.

Quotienting by the ideal $(g_1(T_1), g_2(T_2))$, both of these maps become isomorphisms, yielding an isomorphism $g_1(E_1) \simeq g_2(E_2) \pmod{(g_1(T_1), g_2(T_2))}$; this is over the ring $R/(g_1(T_1), g_2(T_2))$, which has finite length and therefore is of the form W/π^n. On the other hand for any $k > n$ at least one of the maps $\tilde{E} \to g_i(E_i)$ is no longer an isomorphism, and by canonicity if $g_1(E_1)$ and $g_2(E_2)$ were isomorphic modulo π^k then these isomorphisms would have to come via $\tilde{E}$. Thus $\mathrm{length}(R/(g_1(T_1), g_2(T_2)))$ is precisely the largest n such that the isomorphism $g_1(E_1) \simeq g_2(E_2) \pmod{\pi}$ lifts to W/π^n. Writing $n = n(g_1, g_2)$, it follows that

$$E_1 \cdot E_2 = \frac{1}{|G|} \sum_{g_1, g_2 \in G} n(g_1, g_2).$$

If $\varphi : g_1(E_1) \xrightarrow{\sim} g_2(E_2) \pmod{\pi^n}$, then $g_1^{-1}\varphi$ gives an isomorphism $E_1 \xrightarrow{\sim} g_1^{-1} g_2(E_2) \pmod{\pi^n}$, so we can assume without loss of generality that $g_1 = 1$ at the cost of canceling the factor of $\frac{1}{|G|}$. For each g_2, think of $n(1, g_2)$ as a column of height $n(1, g_2)$; put all these together. The total number of boxes in the bottom row is the same thing as the number of isomorphisms $E_1 \xrightarrow{\sim} E_2 \pmod{\pi}$, since we're ranging over all g_2; and we divide by 2 since we quotiented out by the inversion involution, so we really only allow half the isomorphisms. The total number of boxes in the second row up is (half) the number of isomorphisms which lift to W/π^2; and so on. Thus all in all we conclude

$$v(j(E_1) - j(E_2)) = E_1 \cdot E_2 = \frac{1}{2} \sum_{n \geq 1} \# \, \mathrm{Iso}_{W/\pi^n}(E_1, E_2).$$

This is Proposition 2.3 of [2].

References

[1] Brian Conrad. Gross-Zagier revisited. *Heegner points and Rankin L-series*, 49:67–163, 2004.

[2] Benedict H Gross and Don B Zagier. On singular moduli. *Journal für die reine und angewandte Mathematik*, 355:191–220, 1984.