Fargues-Scholze's construction of local Langlands parameters

Avi Zeff

My goal here is to (informally) write down the local Langlands mapping $\pi \mapsto \varphi_\pi$ defined by Fargues-Scholze [1] from smooth representations π of $G(E)$ to an L-parameter $\varphi_\pi : W \to \widehat{G}(L) \rtimes W$ for an algebraically closed field L, taking all of the "good behavior"-type results of [1] for granted. (Here $W = W_E$ is the Weil group of E.) (In fact the action of W factors through some finite quotient Q, but we will ignore such subtleties; we can get away with this since we're not proving anything.) We will also ignore the categorical upgrading that Fargues and Scholze construct, and indeed will forget all the structure we can get away with in order to be as concrete as possible.

Let's briefly introduce some of the key objects. Fix a $\mathbb{Z}_\ell$-algebra Λ, which we will often take to be an algebraically closed field L over $\mathbb{Z}_\ell$. On the "automorphic" side we have $\mathcal{D}_{\text{lis}}(\text{Bun}_G, \Lambda)$, which includes the category of smooth representations of $G(E)$ (and also has a great deal of other structure which we will ignore). On the "Galois" side we have a scheme $Z^1(W, \widehat{G})$ over $\mathbb{Z}_\ell$ whose Λ-points classify L-parameters $W \to \widehat{G}(\Lambda) \rtimes W$. There is an action of $\widehat{G}$ on $Z^1(W, \widehat{G})$, and the points of the (coarse) quotient $Z^1(W, \widehat{G}) /\!\!/ \widehat{G}$ correspond to L-parameters up to $\widehat{G}$-conjugation.

There is an action of representations of $(\widehat{G} \rtimes W)^I$ on $\mathcal{D}_{\text{lis}}(\text{Bun}_G, \Lambda)$ by Hecke operators, for each finite set I. We'll abstract this out for now: suppose we have a suitable category $\mathcal{C}$ equipped with an action of $(\widehat{G} \rtimes W)^I$, i.e. $(\widehat{G} \rtimes W)^I \ni V \mapsto T_V \in \text{End}(\mathcal{C})$.

Define an *excursion datum* to be a tuple $D = (I, V, \alpha, \beta, (\gamma_i)_{i \in I})$ where I is a finite set, V is a representation of $(\widehat{G} \rtimes W)^I$ (which can also be viewed as a $\widehat{G}$-representation by restriction to the diagonal embedding $\widehat{G} \subset \widehat{G}^I \subset (\widehat{G} \rtimes W)^I$, which we denote $V|_{\widehat{G}}$), $\alpha : 1 \to V|_{\widehat{G}}$ and $\beta : V|_{\widehat{G}} \to 1$ are morphisms of representations where 1 is the trivial $\widehat{G}$-representation, and $(\gamma_i)_{i \in I}$ is a tuple of elements of W.

Given an excursion datum D, we can define an endomorphism of the identify functor id of $\mathcal{C}$ by

$$S_D : \text{id} = T_1 \xrightarrow{T_\alpha} T_V \xrightarrow{(\gamma_i)_{i \in I}} T_V \xrightarrow{T_\beta} T_1 = \text{id}\,.$$

Allowing the γ_i to vary, this gives for each $(\gamma_i)_{i \in I}$ an endomorphism of id and thus a group homomorphism

$$S_{(I,V,\alpha,\beta)} : W^I \to \text{Aut}(\text{id}) \subseteq \text{End}(\text{id}).$$

Now for each (V, α, β) and tuple $(g_i)_{i \in I} \in (\widehat{G} \rtimes W)^I$ we get an automorphism of representations

$$1 \xrightarrow{\alpha} V \xrightarrow{(g_i)_{i \in I}} V \xrightarrow{\beta} 1,$$

and since the only endomorphisms of the trivial representation are the scalars this gives a function $f = f(V, \alpha, \beta)$ on $(\widehat{G} \rtimes W)^I$ for each choice (V, α, β), which turns out to be bi-invariant under the action of $\widehat{G}$. In fact it can be checked that $S_{I,V,\alpha,\beta}$ depends on (V, α, β) only through f, and thus fixing I we get a map

$$\Theta^I : \mathcal{O}(\widehat{G} \backslash (\widehat{G} \rtimes W)^I / \widehat{G}) \to \text{Hom}(W^I, \text{End}(\text{id}))$$

sending f to $S_{I,f} : W^I \to \mathrm{End}(\mathrm{id})$ with the obvious notation. It turns out that Θ^I is a map of algebras and is functorial in I, and via the identification

$$\mathcal{O}(\widehat{G} \backslash (\widehat{G} \rtimes W)^I / \widehat{G}) \otimes_{\mathcal{O}(W^{\{0,\dots,n\}})} \mathcal{O}(W^{\{1,\dots,n\}}) \simeq \mathcal{O}((\widehat{G} \rtimes W)^I /\!\!/ \widehat{G})$$

we can view this as a system of maps

$$\Theta_n : \mathcal{O}((\widehat{G} \rtimes W)^n /\!\!/ \widehat{G}) \to \mathrm{Hom}(W^n, \mathrm{End}(\mathrm{id})).$$

By definition an endomorphism of id consists of maps $X \to X$ for each object $X \in \mathcal{C}$ functorially in X. Thus for each $X \in \mathcal{C}$, $f \in \mathcal{O}((\widehat{G} \rtimes W)^n /\!\!/ \widehat{G})$, and $\gamma_1, \dots, \gamma_n$ we get an element $\Theta_n(f)(\gamma_1, \dots, \gamma_n)_X$ of $\mathrm{End}(X)$, and thus a system of maps

$$\Theta_n(X) : \mathcal{O}((\widehat{G} \rtimes W)^n /\!\!/ \widehat{G}) \to \mathrm{Hom}(W^n, \mathrm{End}(X)).$$

It turns out that systems of maps

$$\Theta_n : \mathcal{O}((\widehat{G} \rtimes W)^n /\!\!/ \widehat{G}) \to \mathrm{Hom}(W^n, L)$$

for an algebraically closed field L over $\mathbb{Z}_\ell$ are in bijection with L-points of $Z^1(W, \widehat{G})$, or equivalently L-parameters $W \to \widehat{G}(L) \rtimes W$. This can be seen using two properties of the *excursion algebra*

$$\mathrm{Exc}(W, \widehat{G}) := \operatorname*{colim}_{(n, F_n \to W)} \mathcal{O}(Z^1(F_n, \widehat{G}))^{\widehat{G}},$$

where the colimit is over maps $F_n \to W$ where F_n is the free group on n generators and n is an integer, and $Z^1(F_n, \widehat{G})$ is $\widehat{G}^n$ with a certain twisted action of $\widehat{G}$. The first property is that $\mathrm{Exc}(W, \widehat{G})$ is the universal $\mathbb{Z}_\ell$-algebra A equipped with a system of maps $\Theta_n : \mathcal{O}((\widehat{G} \rtimes W)^n /\!\!/ \widehat{G}) \to \mathrm{Hom}(W^n, A)$ with functorial properties as above. Since everything is functorial and all the F_n are equipped with maps to W, taking the colimit we see that $\mathrm{Exc}(W, \widehat{G})$ is equipped with a map to $\mathcal{O}(Z^1(W, \widehat{G}))^{\widehat{G}} = \mathcal{O}(Z^1(W, \widehat{G}) /\!\!/ \widehat{G})$, and the second key property is that this map is in fact an isomorphism.

Combining these facts, we see that for each system of maps $\Theta_n : \mathcal{O}((\widehat{G} \rtimes W)^n /\!\!/ \widehat{G}) \to \mathrm{Hom}(W^n, L)$, by universality we have a map $\mathrm{Exc}(W, \widehat{G}) \to L$, and by the description of $\mathrm{Exc}(W, \widehat{G})$ this is equivalently an L-point of $Z^1(W, \widehat{G}) /\!\!/ \widehat{G}$. But this is just a conjugacy class of L-parameters $W \to \widehat{G}(L) \rtimes W$.

Now specialize to the case $\mathcal{C} = \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, L)$. Suppose that $X \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, L)$ is Schur-irreducible, i.e. $\mathrm{End}(X) \simeq L$. Then the above construction gives a system of maps $\Theta_n(X) : \mathcal{O}((\widehat{G} \rtimes W)^n /\!\!/ \widehat{G}) \to \mathrm{Hom}(W^n, \mathrm{End}(X)) = \mathrm{Hom}(W^n, L)$ and therefore an L-point of $Z^1(W, \widehat{G}) /\!\!/ \widehat{G}$, i.e. a conjugacy class of L-parameters $\varphi_X : W \to \widehat{G}(L) \rtimes W$. In particular in the case that X is a smooth representation π of $G(E)$ we get (up to conjugacy) an L-parameter φ_π, which we can check satisfies various good properties. In particular for $G = \mathrm{GL}_n$ this construction agrees with the classical local Langlands parametrization.

References

[1] Laurent Fargues and Peter Scholze. Geometrization of the local Langlands correspondence. *arXiv preprint arXiv:2102.13459*, 2021.