Partial fractions with more than two terms

Calculus II, section 3
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Since we didn't cover this in class, let's try an example of partial fractions where there are more than two terms: for example,

$$\int \frac{x+2}{x^3-9x}\,dx.$$

We can factor the denominator as $x(x+3)(x-3)$, so we're looking to find constants A, B, and C such that $\frac{x+2}{x^3-9x} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-3}$.

We can cross-multiply and solve a system of equations, but the most convenient thing is to only multiply by one factor at a time. This is because we can then get rid of two terms at once! To see how this works, first try multiplying both sides by x; we then have

$$\frac{x+2}{x^2-9} = A + \frac{Bx}{x+3} + \frac{Cx}{x-3}.$$

Setting $x = 0$ then means that the second two terms vanish and we're left with just $A = -\frac{2}{9}$.

Next, go back to the original equation and now multiply through by $x+3$; this gives

$$\frac{x+2}{x(x-3)} = \frac{A(x+3)}{x} + B + \frac{C(x+3)}{x-3}.$$

Setting $x = -3$ then kills the first and third term on the right, so we're left with $B = -\frac{1}{18}$.

Finally, we multiply the original equation by $x-3$ to get

$$\frac{x+2}{x(x+3)} = \frac{A(x-3)}{x} + \frac{B(x-3)}{x+3} + C,$$

and setting $x = 3$ gives just $C = \frac{5}{18}$. Thus in total we have

$$\frac{x+2}{x^3-9x} = \frac{1}{18}\left(-\frac{4}{x} - \frac{1}{x+3} + \frac{5}{x-3}\right).$$

Integrating is now straightforward:

$$\begin{aligned}\int \frac{x+2}{x^3-9x}\,dx &= -\frac{2}{9}\int \frac{1}{x}\,dx - \frac{1}{18}\int \frac{1}{x+3}\,dx + \frac{5}{18}\int \frac{1}{x-3}\,dx \\ &= -\frac{2}{9}\log x - \frac{1}{18}\log(x+3) + \frac{5}{18}\log(x-3) + C.\end{aligned}$$

(Insert absolute value signs as needed.)

Notice that we can also do this method of only multiplying through by one factor at a time in the case of only two terms (or arbitrarily many terms); it has the advantage that it immediately gives the constants A, B, etc. without having to solve for them, but the disadvantage that we have to work with several different versions of the equation instead of just one. Feel free to use whichever version you prefer; they are equivalent.