Lecture 9: more polar coordinates and complex numbers

Calculus II, section 3
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Let's talk a bit more about polar coordinates. Last time, we saw some graphs of functions in polar coordinates, and saw how to compute arc length using them; today, we'll expand to some other things we like to be able to do with calculus, such as finding the slope of curves and the area bounded by them.

As we did with calculus in rectangular coordinates, let's start with slope: given the graph of a function $r = f(\theta)$, what is the slope of this graph at a point (r, θ)?

As usual, the slope is given by $\frac{dy}{dx}$. Usually when we see something like this we want to think of y as a function of x, but in polar coordinates this may be very difficult or impossible. Instead, both y and x are functions of θ, given by $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, so writing $dy = y'(\theta)\,d\theta$ and $dx = x'(\theta)\,d\theta$ we conclude that $\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)}$. Using the product rule, this is

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

For example, take $r = f(\theta) = 1 - \sin\theta$, which is a cardioid. We have $f'(\theta) = -\cos\theta$, and so the slope is given by

$$\frac{dy}{dx} = \frac{(1-\sin\theta)\cos\theta - \cos\theta\sin\theta}{-\cos^2\theta + (1-\sin\theta)\sin\theta} = \frac{1-2\sin\theta}{\sin\theta - 1}\cos\theta.$$

For example, at $\theta = 0$ this gives $\frac{1}{-1}\cdot 1 = -1$.

For some purposes we don't need the full complexity of this formula. For example, suppose we want to find the highest points on this curve. These are the maxima of y. We could use this formula and set it equal to 0 to find critical points, but we could just as well treat y as a function of θ and set $y'(\theta) = 0$, which works out to be the same thing. In this case this gives $(1-2\sin\theta)\cos\theta = 0$, which holds if $\sin\theta = \frac{1}{2}$ or $\cos\theta = 0$, i.e. $\theta = \frac{\pi}{6}$, $\theta = \frac{5\pi}{6}$, $\theta = \frac{\pi}{2}$, or $\theta = \frac{3\pi}{2}$. Inspection of the graph shows that the maxima among these points are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, with corresponding values of $y = (1-\sin\theta)\sin\theta = \frac{1}{4}$ for both. If we don't want to bother drawing the graph, we could use the second derivative test: $y''(\theta) = \frac{d}{d\theta}(1-2\sin\theta)\cos\theta = -2\cos^2\theta - (1-2\sin\theta)\sin\theta = -\sin\theta - 2\cos(2\theta)$, and evaluating at these points gives $y''(\frac{\pi}{6}) = y''(\frac{5\pi}{6}) = -\frac{3}{2} < 0$, $y''(\frac{\pi}{2}) = 1 > 0$, and $y''(\frac{3\pi}{2}) = 3 > 0$, so the first two are maxima and the second two are minima.

Following the path of calculus 1, we can then turn to calculating area. For rectangular coordinates, we do this by breaking up our area into small rectangles and adding up their areas, and in the limit get an exact value for the area; similarly for polar coordinates we want to take small pieces of a circle, i.e. wedges. If, like for rectangular coordinates, we imagine that $r = f(\theta)$ is roughly constant as we change θ by a very small amount $d\theta$ (this assumption is okay since we are after area rather than length, and the difference in area is much smaller than the difference in length) we get a wedge of radius r and angle $d\theta$. If we were to go all the way around the circle, i.e. an angle of 2π, we would get an area of πr^2; since we go only $d\theta$, we get $\frac{d\theta}{2\pi}$ of that area, i.e. $\frac{d\theta}{2\pi}\pi r^2 = \frac{1}{2}f(\theta)^2\,d\theta$. Therefore we conclude that the area bounded by $r = f(\theta)$ between θ_1 and θ_2 is

$$\int_{\theta_1}^{\theta_2} \frac{1}{2}r^2\,d\theta = \int_{\theta_1}^{\theta_2} \frac{1}{2}f(\theta)^2\,d\theta.$$

We can take the same example as above: what is the area of the cardioid $r = 1 - \sin\theta$, between $\theta = 0$ and $\theta = 2\pi$?

If $r = 1 - \sin\theta$, then $r^2 = (1-\sin\theta)^2 = 1 - 2\sin\theta + \sin^2\theta$ and so

$$\int_0^{2\pi} \frac{1}{2}(1 - 2\sin\theta + \sin^2\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi} d\theta - \int_0^{2\pi} \sin\theta\,d\theta + \frac{1}{2}\int_0^{2\pi} \sin^2\theta\,d\theta.$$

The first integral is easy (π), the second is easy to evaluate directly or is zero by symmetry, and the third we know from your midterm, from replacing the integrand by $1 - \cos^2\theta$, or by using that since this is over a full period it is the same as the integral of $\sin^2\theta$ and therefore is half of $\frac{1}{2}\int_0^{2\pi} \sin^2\theta + \cos^2\theta\,d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \pi$, i.e. $\frac{\pi}{2}$. Therefore the total area is $\frac{3}{2}\pi$.

Finally let's briefly go back to arc length. Last time, we found a formula by converting to polar coordinates; this time let's stay in polar coordinates and use geometry, similar to the area model above. Just as in rectangular coordinates there are two directions, x and y, that we need to take into account to get $ds = \sqrt{dx^2 + dy^2}$, in polar coordinates we have two directions, θ and r. In the r direction the length is dr; in the θ direction the actual distance moved is $r\,d\theta$, and so the total distance is $ds = \sqrt{dr^2 + r^2\,d\theta^2}$. Using the formula $r = f(\theta)$, this is $ds = \sqrt{f'(\theta)^2 + f(\theta)^2}\,d\theta$, just as we found last time.

Our second topic for today is complex numbers. The easiest way to think of the complex numbers are as numbers of the form $a + bi$, where a and b are real numbers and i is some symbol such that $i^2 = -1$. We can think of these as a plane, where one axis is the real line and the other axis is called the imaginary axis, and consists of numbers xi for real numbers x.

Algebraically, there is a more canonical definition: the complex numbers are the way we need to extend the real numbers in order to be able to solve all algebraic equations. It's pretty easy to see that these are equivalent: as we've discussed, every polynomial over the real numbers factors as a product of linear or quadratic polynomials; linear polynomials always have a real root, and quadratic equations $ax^2 + bx + c = 0$ can be solved with the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac$ is positive, this has solutions over the real numbers; if it is negative, we can factor out the positive part so this reduces to whether or not we can take the square root of -1. A simple example is $x^3 - 1 = (x-1)(x^2 + x + 1)$ has solutions $x = 1$, $x = \frac{-1\pm\sqrt{-3}}{2} = \frac{-1\pm i\sqrt{3}}{2}$.

We can add, multiply, and subtract complex numbers exactly as you might think: $(a + bi) + (c + di) = (a+c) + (b+d)i$ and $(a+bi)(c+di) = ac + bci + adi + bdi^2 = ac - bd + (bc + ad)i$.

If we want to be able to divide complex numbers and get something in this standard form, we need to be able to somehow clear denominators. We do this using the conjugate.

Given a complex number $z = a + bi$, we say that its conjugate $\bar{z}$ is $a - bi$. Note that algebraically speaking, these are kind of the same: the defining property $i^2 = -1$ is also true of $-i$, since $(-i)^2 = i^2 = -1$, so it is natural to switch i and -1.

The product of a complex number z with its complex conjugate $\bar{z}$ is always a real number: $(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2$. This is extremely useful: for example, if we have a fraction like $\frac{1}{a+bi}$ we can use the complex conjugate to get i out of the denominator: $\frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$. We can do the same thing to divide complex numbers and get back into standard form.

Other than this form, there are a few other ways we can write complex numbers. The most important of them is polar form, giving the connection to today's earlier topic. The idea is exactly the same: instead of giving the "rectangular" real and imaginary coordinates a and b, we can give "polar" coordinates given by the distance $r = \sqrt{a^2 + b^2}$ from the origin together with an angle θ off of the real line. Just like for usual polar and rectangular coordinates we can convert back by setting $a = r\cos\theta$ and $b = r\sin\theta$, i.e. $z = r\cos\theta + ri\sin\theta$.

The way this is usually written out is $z = re^{i\theta}$. This is because of Euler's formula, which is just the equality of this form with the expression above: $e^{i\theta} = \cos\theta + i\sin\theta$, so this follows by multiplying by r. Note that for example if $\theta = \pi$ we get $e^{\pi i} = -1$, which is a famous and aesthetically pleasing identity. Perhaps more concerningly, if $\theta = 2\pi$ it implies that $e^{2\pi i} = e^0 = 1$; in other words, the exponential function, once extended to the complex numbers, is no longer one-to-one, but instead is periodic with period $2\pi i$—indeed, $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$. This corresponds to the logarithm being badly behaved over the complex numbers—it is multivalued, e.g. $\log 1$ could be said to be $2\pi i$ (or $4\pi i$, $-10\pi i$, etc.) just as well as 0.

We'll derive Euler's formula near the end of the semester; it follows quickly from the theory of Taylor series. For now, observe that it gives an easy way of deriving trigonometric identities. For example, $\cos(\alpha+\beta)$ is the real part of $e^{(\alpha+\beta)i}$; on the other hand this is $e^{\alpha i+\beta i} = e^{\alpha i}e^{\beta i} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + (\cos\alpha\sin\beta + \sin\alpha\cos\beta)i$, so the real part is just $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ as usual.

It is also possible to do calculus with complex numbers, much the same as you would do with real numbers: functions can be complex-valued, and can be differentiated or integrated as usual. There are a couple of things to be careful of, though. One is that limits existing is a much stronger condition, since the limit has to converge from every direction, not just two. Thus in particular being differentiable is a much stronger condition; we'll come back to this at the end of the semester too.

The second issue is about integration. Over real numbers, we only have the option of integrating over intervals (though via limits we can extend to infinite intervals, as we've seen). Over complex numbers, though, we have many more options: we have the whole complex plane to integrate in, and so we can take very complicated paths. The catch here is that integration is not necessarily path-independent, which means that the fundamental theorem of calculus no longer holds!

For example, consider the function $f(z) = \frac{1}{z}$, and take the path wrapping around the origin from 1 back to 1, parametrized by $z = e^{i\theta}$ with θ going from 0 to 2π, so $dz = ie^{i\theta}\, d\theta$. This is

$$\oint \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} \cdot ie^{i\theta}\, d\theta = \int_0^{2\pi} i\, d\theta = 2\pi i.$$

On the other hand by the fundamental theorem of calculus we'd expect that since the starting point and the ending point are the same, the integral should be zero! This is closely related to the fact that the logarithm is multivalued on complex numbers; more generally it is the beginning of the field of complex analysis, which is a deep and beautiful theory which unfortunately is beyond the scope of this class but which I encourage you to learn more about.