Lecture 8: surfaces of revolution and polar coordinates

Calculus II, section 3
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Today's first topic is *surfaces* of revolution. These are very similar to solids of revolution, except that instead of rotating a region (bounded by curves) about some axis, we just rotate the curves themselves. (Thus the boundary of a solid of revolution is an example of a surface of revolution.)

Just as for solids the natural question is to compute their volume, given a surface of revolution it is natural (and often useful) to ask for its surface area. How can we do this?

One idea is to do the same thing we do for Riemann integration: divide our curve into intervals, approximate it on each by a flat line, and rotate each of those lines around the axis in question to get an annulus or cylindrical shell, and add the areas. Unfortunately this turns out to give the wrong answer!

The reason is easier to explain in the one-dimensional case, where the analog of surface area is arc length. This method would suggest that, for example, to compute the arc length of the line $y = x$ between $x = 0$ and $x = N$, we could approximate it by a staircase function. We then have to decide if we're including the vertical steps—if so, the arc length of the staircase is $2N$ and if not it is N (regardless of the step size!). But by Pythagoras we know that the true length is $N\sqrt{2}$!

In fact, this is basically the essence of the problem: different paths between two points can have different lengths, and even when they tend to the same path in one sense (each point on one path approaches a point on the other) they may *still* have different lengths if the slopes continue to disagree. To ensure that the lengths are equal, we need to make sure that the approximation is also an approximation of slopes. (Looking at the formula for arc length, this is not surprising since it involves the derivative.)

Therefore what we do is instead what we do for arc lengths: we divide the curve into straight line segments, each approximating both the value and slope of the curve at that point, and rotate each segment about the appropriate axis. We add up the areas of the resulting bands, and get an approximation to the total surface area.

At each point on a band of width ds with the radius of the middle of the band r, the area of the band (after flattening it out) is just $2\pi r\, ds$. Therefore the total surface area is the integral of $2\pi r\, ds$ over a suitable region, where r is some function of x or y and the infinitesimal length ds is just the arc length, which we talked about last class.

Rather than state a formula, let's do an example: let's find (again) the surface area of a sphere of radius R. Take the curve $y = \sqrt{R^2 - x^2}$ between $x = -R$ and $x = R$, and rotate it around the x-axis to get a (hollow) sphere. At each point x, we get a band circling the x-axis with side length $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \frac{x^2}{R^2 - x^2}}$ and radius $r = y = \sqrt{R^2 - x^2}$. Therefore the total surface area is

$$\int_{-R}^{R} 2\pi\sqrt{R^2 - x^2}\sqrt{1 + \frac{x^2}{R^2 - x^2}}\, dx = \int_{-R}^{R} 2\pi\sqrt{R^2}\, dx = (2R) \cdot (2\pi R) = 4\pi R^2.$$

Another example for which the final answer may be less obvious is the surface area of a cone. To get a cone with height h and radius r, let's take the line $y = h - \frac{h}{r}x$, so that the x-intercept is at r, and rotate it about the y-axis. Then each band is about the y-axis with radius x and side length $ds = \sqrt{1 + \frac{h^2}{r^2}}\, dx$ and so the surface area is

$$\int_0^r 2\pi x \sqrt{1 + \frac{h^2}{r^2}}\, dx = \pi r^2 \sqrt{1 + \frac{h^2}{r^2}} = \pi r \sqrt{r^2 + h^2}.$$

Of course, usually in geometry we would also care about the base of the cone, which would add an extra πr^2, but for surfaces of revolution we will usually ignore such things.

We can do the same thing with parametric curves, just like with arc length. In this case $ds = \sqrt{x'(t)^2 + y'(t)^2}\, dx$, and so we multiply by $2\pi r$ (whatever r is) and integrate. For example, consider the circle $x = 2 + \cos t$, $y = \sin t$ for t between 0 and 2π. If we rotate this around the y-axis, we get a donut, or in slightly more mathematical terminology a torus. What is its surface area?

At time t, the radius with respect to the y-axis is $x = 2 + \cos t$, and $ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt = \sqrt{\sin^2 t + \cos^2 t}\, dt = dt$. Therefore the total surface area is just

$$\int_0^{2\pi} 2\pi(2 + \cos t)\, dt = 8\pi^2 + 4\pi \int_0^{2\pi} \cos t\ dt = 8\pi^2 + 4\pi(\sin(2\pi) - \sin(0)) = 8\pi^2.$$

Geometric aside: a torus can be thought of as the product of two circles, in this case one of radius 1 and one of radius 2. Therefore the surface area of the torus is the product of the circumferences of the circles, i.e. $2\pi \cdot 4\pi$. A fun exercise (though not on your homework) is to use a similar geometric method to compute the volume of a torus (which you can also do more directly by integration, as for general solids of revolution).

In fact, this is related to our next concept: for some purposes, especially anything related to circles but also more general applications, it is useful to think of points not in terms of (x, y) coordinates but instead by starting at the origin and specifying a distance and a direction. Explicitly, we give two coordinates, r and θ (called polar coordinates), where r is the distance from the origin and θ is the angle above the x-axis. We can define these formally in terms of Cartesian (i.e. x-y) coordinates by the relation $x = r\cos\theta$, $y = r\sin\theta$; it is also possible to go the other direction, though slightly more annoying, via $r = \sqrt{x^2 + y^2}$ (by Pythagoras, or directly from the formulae for x and y) and $\theta = \tan^{-1}(\frac{y}{x})$ (by geometry or again from those formulae). Like how in Cartesian coordinates we typically are concerned with (graphs of) functions $y = f(x)$, in polar coordinates we usually write $r = f(\theta)$, though similarly we could also use parametric equations or other types as desired.

The simplest example is of course something like $r = 1$, a circle of radius 1, which is of course much simpler in these coordinates. We could also easily define a spiral $r = \theta$ (say for $\theta > 0$), which is simple in polar coordinates but quite complicated in Cartesian coordinates: there it becomes $\sqrt{x^2 + y^2} = \tan^{-1}(\frac{y}{x})$, which certainly does not give y as a function of x and doesn't seem like it simplifies to a nice relation.

On the other hand, there are many functions to which Cartesian coordinates are well-suited but which are complicated in polar coordinates. Consider for example the graph of

$y = x^2+1$. In polar coordinates, this is $r\sin\theta = r^2\cos^2\theta+1$, which is only defined for certain θ; we can solve for r by the quadratic formula to get $r = \frac{1}{2\cos^2\theta}\left(\sin\theta \pm \sqrt{\sin^2\theta - 4\cos^2\theta}\right)$.

A better-behaved but perhaps surprising example is $r = \cos\theta$. It is not obvious what this curve will look like, but by plotting some points we can guess that it will be a circle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$. Why should this be?

If we convert to Cartesian coordinates, we get that this is $\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}}$, i.e. $x^2+y^2 = x$, which can be manipulated into the equation $(x-\frac{1}{2})^2+y^2 = \frac{1}{4}$. We could also do this by writing $x = r\cos\theta = \cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos(2\theta)$ and $y = r\sin\theta = \sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$, so viewing this as a *parametric* equation with parameter θ we see that going from 0 to π should give us the expected circle. This is an important point which is not obvious from the first method, or the initial presentation: to go around the circle exactly once, we go from $\theta = 0$ to $\theta = \pi$, not 2π.

There's a lot to be said about polar coordinates, but for now let's look at arc length. We could try to do some complicated thing using small circular segments, but it's much easier to use our formula for arc length for Cartesian coordinates and view $x = r\cos\theta$ and $y = r\sin\theta$ as parametric equations in θ, where we substitute $r = f(\theta)$. We then get, by the product rule, $x'(\theta) = f'(\theta)\cos\theta - f(\theta)\sin\theta$ and $y'(\theta) = f'(\theta)\sin\theta + f(\theta)\cos\theta$, so

$$\begin{aligned}
ds &= \sqrt{x'(\theta)^2 + y'(\theta)^2}\,d\theta \\
&= \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\,d\theta \\
&= \sqrt{f'(\theta)^2\cos^2\theta + f(\theta)^2\sin^2\theta + f'(\theta)^2\sin^2\theta + f(\theta)^2\cos^2\theta}\,d\theta \\
&= \sqrt{f'(\theta)^2 + f(\theta)^2}\,d\theta.
\end{aligned}$$

For example, let's take our equation $r = f(\theta) = \cos\theta$. Then $f'(\theta) = -\sin\theta$ and so we get

$$ds = \sqrt{\sin^2\theta + \cos^2\theta}\,d\theta = d\theta,$$

and so the arc length is just

$$\int_0^\pi d\theta = \pi$$

as expected.

We could also ask about surfaces of revolution in polar coordinates (though we need to introduce Cartesian coordinates to have axes to rotate around). Let's take $r = \cos\theta$ again, and rotate it around the y-axis; this gives us an almost-torus, i.e. a donut which just barely does not have a hole. The radius at θ is $x = r\cos\theta = \cos^2\theta$, and (as we've just seen) $ds = d\theta$, so this is

$$\int_0^\pi 2\pi\cos^2\theta\,d\theta = \pi(\sin\theta\cos\theta + \theta)\Big|_0^\pi = \pi^2.$$

This corresponds to the product of the circumferences of two circles, each of radius $\frac{1}{2}$.