Lecture 3: integration by parts

Calculus II, section 3
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Last time we saw some different kinds of applications of integrals, and we'll see more later in the semester. However we've been fairly limited in our applications because after all we can only compute certain kinds of integrals: we don't have very many tools for integrating. Our goal today, and for the next couple weeks, is to learn more tools so we can integrate a wider array of functions.

Today's tool is a very broadly applicable one: integration by parts. Just like u-substitution is given by integrating the chain rule, integration by parts comes from integrating the product rule. Explicitly, let $f(x)$ and $g(x)$ be differentiable functions; the product rule states that

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x),$$

and integrating both sides gives

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Rearranging, we conclude that

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

One traditional way of writing this, which may be easier to remember, is as follows: if we write $u = f(x)$ and $v = g(x)$ (dropping the dependence on x from the notation), as for u-substitution we can write $g'(x)\,dx = dv$ and $f'(x)\,dx = du$, so that this equation becomes

$$\int u\,dv = uv - \int v\,du.$$

To apply this idea, we want a situation where we're integrating the product of two functions, one of which is easy to differentiate, which we'll call u, and one of which is easy to integrate, which we'll call v'. Then

$$\int uv'\,dx = \int u\,dv = uv - \int v\,du,$$

where we obtain du by differentiating u and v by integrating v'. Picking which function is which is not always obvious.

Let's do an example: finding the antiderivative of $x\cos x$. In this case, it's straightforward to differentiate x and it gives something simple, namely 1, so let $u = x$, and $\cos x$ is easy to integrate with integral $\sin x$ so we set $dv = \cos(x)\,dx$, so that $v = \sin x$. Then

$$\int x\cos(x)\,dx = \int u\,dv = uv - \int v\,du = x\sin x - \int \sin(x)\,dx.$$

This last integral is straightforward, and so we get

$$\int x\cos(x)\,dx = x\sin x + \cos x + C.$$

Notice that the choice of u and v was important: if we chose $u = \cos(x)$ and $dv = x\,dx$, so that $du = -\sin(x)\,dx$ and $v = \frac{x^2}{2}$, we get

$$\int x\cos(x)\,dx = \frac{1}{2}x^2\cos x + \frac{1}{2}\int x^2\sin(x)\,dx,$$

which is even harder than the integral we started with!

The basic idea here is pretty straightforward—integrate the product rule—but in practice it can be confusing, and it is among the most useful techniques, so we're going to spend the rest of the day doing examples.

In this class by now we have a tradition of finding a new way to antidifferentiate $\log x$ every class, so let's do it again. In this case the choice is simple if perhaps unexpected: take $u = \log x$, which is easy enough to differentiate, and $dv = dx$, i.e. $v = x$, since there is no other function to integrate. Then we have

$$\int \log x\,dx = \int u\,dv = uv - \int v\,du = x\log x - \int x\cdot\frac{1}{x}\,dx = x\log x - x + C.$$

We can also apply this method to definite integrals. Consider $\int_0^{2\pi}\cos^2 x\,dx$. Since we can view $\cos^2 x$ as composing x^2 with $\cos x$, we might be tempted to try a u-substitution, with $u = \cos x$; but there's no $\sin x$ in the picture, so this isn't going to make things any easier. Another possible route is to recall some trigonometric identity relating $\cos^2 x$ to $\cos(2x)$, but personally I can never remember those identities beyond $\sin^2 x + \cos^2 x = 1$ and in any case that's a lot of steps since we then have to substitute.

Instead let's view $\cos^2 x$ as the product $(\cos x)\cdot(\cos x)$ and apply integration by parts. In this case there is no choice of which is which, so we have to both integrate and differentiate $\cos x$ to get $du = -\sin x\,dx$ and $v = \sin x$ where $u = \cos x$ and $dv = \cos x\,dx$, so that

$$\int \cos^2 x\,dx = \int u\,dv = uv - \int v\,du = \sin(x)\cos(x) + \int \sin^2 x\,dx.$$

This integral on the right is just as bad as the original one we started with! However, now we can use a much simpler trigonometric identity, $\sin^2 x + \cos^2 x = 1$, i.e. $\sin^2 x = 1 - \cos^2 x$, so that this is

$$\int \cos^2 x\,dx = \sin(x)\cos(x) + \int 1 - \cos^2 x\,dx = \sin(x)\cos(x) + x - \int \cos^2 x\,dx.$$

Since the integral on the right is the same as the one on the left, we can add them together and divide by 2 to get

$$\int \cos^2 x\,dx = \frac{1}{2}(\sin x\cos x + x).$$

Evaluating at 0 and 2π, we get

$$\int_0^{2\pi} \cos^2 x\,dx = \frac{1}{2}(\sin(2\pi)\cos(2\pi) + 2\pi) - \frac{1}{2}(\sin(0)\cos(0) + 0) = \pi.$$

We could actually figure this out more easily, using the same trigonometric identity. In particular, integrating both sides of $\sin^2 x + \cos^2 x = 1$ gives

$$\int_0^{2\pi} \sin^2 x\,dx + \int_0^{2\pi} \cos^2 x\,dx = \int_0^{2\pi} 1\,dx = 2\pi,$$

and since sin and cos are the same function shifted by some amount when we integrate over their period we should get the same thing, so that this is

$$2\int_0^{2\pi} \cos^2 x\,dx = 2\pi.$$

Dividing by 2 gives the same result. Of course the integration by parts method is more powerful because it would let us evaluate the integral at any bounds, not just these convenient ones.

In this last example, by integrating by parts we got not a simpler integral, but one we could relate to the original one to get an equation that we could solve. This is not an uncommon method for integration by parts, especially for integrals involving exponential or trigonometric functions where the derivatives and integrals have close relationships to the original functions. Let's see another example of this. Consider the integral

$$\int e^x \sin(x)\,dx.$$

Each of e^x and $\sin x$ is pretty easy to either integrate or differentiate, so our choice of u and v doesn't much matter; let's take $u = e^x$ and $dv = \sin(x)\,dx$, so that $du = e^x\,dx$ and $v = -\cos(x)$. Then we have

$$\int e^x \sin(x)\,dx = \int u\,dv = uv - \int v\,du = -e^x\cos(x) + \int e^x \cos(x)\,dx.$$

This doesn't help much: just like last time, the integral on the right is just as hard as the one we started with, and this time there's no convenient trigonometric identity to help us out. Let's try just integrating by parts again, in the same direction (if we go the reverse direction we just get back where we started): $u = e^x$, $dv = \cos(x)\,dx$ so that $v = \sin(x)$. Then we get

$$\int e^x \cos(x)\,dx = e^x\sin(x) - \int e^x \sin(x)\,dx,$$

so combined with the above

$$\int e^x \sin(x)\,dx = -e^x\cos(x) + e^x\sin(x) - \int e^x \sin(x)\,dx.$$

Adding the integral to both sides and dividing by 2 gives

$$\int e^x \sin(x)\, dx = \frac{1}{2} e^x (\sin x - \cos x)$$

up to an additive constant.

We can also combine integration by parts with u-substitution, for a more powerful method. For example, let's find the integral

$$\int_0^{\pi^2} \cos(\sqrt{x})\, dx.$$

First, this is a composition of functions so we substitute $u = \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}}\, dx$ and so $dx = 2\sqrt{x}\, du = 2u\, du$. Therefore this integral becomes (leaving evaluation at the bounds until the end)

$$\int 2u \cos(u)\, du.$$

Now, if we differentiate u we get something simpler and integrating $\cos u$ is easy, so we integrate by parts with the first function u (sorry for the collision of notation!) and the second function $\cos u$, pulling out the factor of 2, to get

$$\int 2u \cos(u)\, du = 2u \sin u - 2 \int \sin u\, du = 2u \sin u + 2 \cos u.$$

Plugging back in $u = \sqrt{x}$ gives

$$\int \cos(\sqrt{x})\, dx = 2(\sqrt{x} \sin \sqrt{x} + \cos \sqrt{x}),$$

and evaluating at π^2 and 0 gives

$$\int_0^{\pi^2} \cos(\sqrt{x})\, dx = 2(\pi \sin \pi + \cos \pi) - 2(0 \cdot \sin 0 + \cos 0) = -2 - 2 = -4.$$

Another application, though technically this uses ideas we haven't talked about yet, is to the factorial. Recall from the optional problem on homework 1 that the factorial is defined by

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

One could then ask: can we extend this to non-integer arguments? For example, what should (2.5)! be?

It turns out that we can do this, via an integral. Consider the indefinite integral

$$I(n) = \int x^n e^{-x}\, dx.$$

Integrating by parts, this is

$$-x^n e^{-x} + n \int x^{n-1} e^{-x}\, dx = -x^n e^{-x} + nI(n-1),$$

so there's some relationship between $I(n)$ and $I(n-1)$. To make this more usable, we need to deal with this extra term $-x^n e^{-x}$; we do this by making our integral into a definite one. If we evaluate $-x^n e^{-x}$ at 0, it vanishes, which is good; ideally, we'd like to choose the other bound such that it vanishes there too. Unfortunately there is no nonzero real number x such that $x^n e^{-x} = 0$. However, if we take $x \to \infty$ then $e^{-x} = \frac{1}{e^x} \to 0$, so we can think of the upper bound as ∞; this doesn't make sense from what we've done so far, but we'll see in a few classes that it actually can be formalized by taking a limit. If we are willing to take our integral from 0 to ∞, what we get is

$$I(n) = \int_0^\infty x^n e^{-x}\, dx = n \int_0^\infty x^{n-1} e^{-x}\, dx = nI(n-1),$$

so iterating this property we get

$$\begin{aligned} I(n) &= nI(n-1) \\ &= n \cdot (n-1) I(n-2) \\ &= n \cdot (n-1) \cdot (n-2) I(n-3) \\ &= \\ &\vdots \\ &= n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \cdot I(0) \\ &= n! I(0), \end{aligned}$$

so it suffices to compute $I(0)$. But this is just

$$\int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = 0 - (-1) = 1,$$

so we conclude that $n! = I(n)$ for every positive integer n. But now $I(n)$ is actually defined even for non-integer values of n: it turns out that this integral makes sense for every non-negative real number, and always has this property that $I(x) = xI(x-1)$ (which as it turns out lets us define it for every complex number x other than negative integers!). Thus for example if we want to compute $(2.5)!$, this is $2.5 \cdot (1.5)! = 2.5 \cdot 1.5 \cdot (\frac{1}{2})!$; it turns out that $(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$, although this integral is quite difficult to compute, and so $(2.5)! = \frac{15}{8}\sqrt{\pi}$.