Lecture 19: Taylor series

Calculus II, section 3
April 20, 2022

Last time, we introduced Taylor series to represent (reasonably) arbitrary functions as power series, looked at some examples (around different points and with different radii of convergence), and as an application proved Euler's formula, which we used extensively to solve differential equations.

On the other hand, if we want to actually use these for practical (or at least non-mathematical) applications, we can't use infinitely many terms—we need to truncate the series at some point. This gives a finite sum of powers of x, which is a polynomial, called a Taylor polynomial, or specifically the Taylor polynomial of some degree d:

$$P_d(x) = \sum_{n=0}^{d} \frac{f^{(n)}(0)}{n!} x^n$$

for Maclaurin series, i.e. about the point $x = 0$, or more generally about $x = b$

$$P_d(x) = \sum_{n=0}^{d} \frac{f^{(n)}(b)}{n!} (x-b)^n.$$

For example, for $f(x) = e^x$, recall that every derivative is the same, and so $f^{(n)}(0)$ is always $e^0 = 1$; so the full Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and for example the degree 3 Taylor polynomial is

$$\sum_{n=0}^{3} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

For $f(x) = \sin x$, we looked at the Taylor series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ;$$

truncating at the first term gives the small-angle approximation, but we can keep going. Since the even terms vanish, $P_2(x) = P_1(x) = x$, but $P_3(x) = x - \frac{x^3}{6}$, $P_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$, etc.

The first-order approximation

$$P_1(x) = \sum_{n=0}^{1} \frac{f^{(n)}(b)}{n!} (x-b)^n = f(b) + f'(b)(x-b)$$

is the first-order approximation we started the class with. We talked about how we can view the derivative as defined by this approximation; similarly, it's possible to view the higher derivatives as defined by the higher-order Taylor polynomials, i.e. think of the higher-order derivatives as the coefficients (up to factorial terms) of the best polynomial approximation to $f(x)$ of a given degree.

For our first-order approximations, we only expected them to be "good" very close to the base point b, and we had no real way of quantifying how good an approximation they were. The whole point of Taylor polynomials is to get a better approximation, though, so we should have some way of saying how good they are.

Given a Taylor polynomial $P_d(x)$ for $f(x)$, this means we want to bound $|f(x) - P_d(x)|$. Let's start with the simplest case: $d = 0$. The zeroth-order approximation $P_0(x)$ to $f(x)$ at b is just $f(b)$, the constant term of the Taylor series, so we're looking to bound $|f(x) - f(b)|$. Since we know about first-order approximations, we expect that this looks something like $f'(b)(x - b)$. In fact, at b and x we can plot the value of f and draw the line between them; then at some (completely unknown!) point c between b and x, the derivative $f'(c)$ must be equal to this slope, which is $\frac{f(x)-f(b)}{x-b}$ (this is the mean value theorem, which you hopefully encountered in calculus 1; otherwise take my word for it), i.e. $f(x) - f(b) = f'(c)(x - b)$ for some c between b and x. Thus to bound $|f(x) - f(b)|$ it suffices to bound $|f'(c)|$ between b and x: if we could find some number M such that $|f'(c)| \leq M$ for every $b \leq c \leq x$, then we would have $|f(x) - f(b)| \leq M|x - b|$.

We can think of this process of bounding $f(x) - f(b)$ as bounding the integral of $f'(t)$ from b to x. In particular, if $|f'(t)| \leq M$ on this range, then the (absolute value of the) integral is at most $M \cdot |x - b|$.

The same idea works for higher-order approximations. By the degree $d + 1$th approximation, we expect $f(x) - P_d(x)$ to be roughly $\frac{f^{(d+1)}(b)}{(d+1)!}(x - b)^{d+1}$, and a more complicated application of the mean value theorem again tells us that indeed

$$\frac{f(x) - P_d(x)}{(x - b)^{d+1}}$$

must be equal to $\frac{f^{(d+1)}(c)}{(d+1)!}$ for *some* c between b and x. Therefore if we can find a bound M for $|f^{(d+1)}(c)|$ for c between b and x, then we conclude that

$$|f(x) - P_d(x)| \leq \frac{M}{(d + 1)!}|x - b|^{d+1}.$$

This bound on the right is called the remainder term $R_d(x)$. Again, we could think of it as integrating $f^{(d+1)}(t)$, but now to recover f we actually need to integrate $d + 1$ times! (Don't worry about this interpretation if it's confusing, it's just to give one more way of understanding it.)

For example, suppose we want to compute $e = e^1$ to within an error of at most 0.01, using the Taylor approximation at $x = 0$. (Note that 1 is fairly far from 0, so this might be tricky.) Above, we wrote down the third-order approximation to e^x at $x = 0$; is that good enough?

Well, to bound the error $|e^x - P_3(x)|$ using the remainder term $R_3(x)$, we need to know a bound M on the fourth derivative, which is again e^x, between 0 and 1. Since e^x is increasing, we can bound it by $e^1 = e$; sometimes this will be trickier and we might have to do actual optimization.

Of course, we don't know e, so let's bound it in turn by 3. Thus we get that

$$|e^x - P_3(x)| \leq \frac{3}{4!}|1-0|^4 = \frac{1}{8} = 0.125.$$

This is pretty clearly not good enough, and even replacing 3 by a better upper bound on e wouldn't change things much.

Let's try the next term. Since the fifth derivative is also e^x, the bound is the same and so we get for free

$$|e^x - P_4(x)| \leq \frac{3}{5!}|1-0|^5 = \frac{1}{40} = 0.025,$$

which is much closer but still not quite there; replacing 3 by e again would improve things slightly but not enough. Let's go up one more, using the same bound:

$$|e^x - P_5(x)| \leq \frac{3}{6!}|1-0|^6 = \frac{1}{240} \approx 0.004167,$$

well below our cutoff. Thus to be sure of computing e in this way within an error of 0.01 we need to go up to the degree 5 approximation.

In fact, this turns out not to be necessary: the degree 4 approximation gives $P_4(1) \approx 2.708333$, which is within 0.00995 of e. However there was no way of knowing for sure in advance of actually calculating.

Let's give another example: the error in approximating the logarithm. Let's say we want to know the logarithm of 0.9. How can we compute it?

Well, we know the Taylor series for $\log x$ about $x = 1$:

$$\log x = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-1)^n.$$

Let's take the degree 3 approximation,

$$P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}.$$

For $x = 0.9$, this gives

$$P_3(0.9) = -.1 - \frac{.1^2}{2} - \frac{.1^3}{3} \approx -0.105333.$$

How close is this to the true value?

First we need to bound the 4th derivative of $\log x$ between 0.9 and 1. The derivatives are $\log x$, $\frac{1}{x}$, $-\frac{1}{x^2}$, $\frac{2}{x^3}$, $-\frac{6}{x^4}$, so we have to bound $-\frac{6}{x^4}$, or rather its absolute value $\frac{6}{x^4}$. This is decreasing, so its greatest value is at the endpoint 0.9, where it is $\frac{6}{.9^4} \approx 9.15$, so

$$|\log .9 - P_3(.9)| \leq \frac{6/.9^4}{4!}|.9-1|^4 \approx 0.0000381.$$

Indeed, we can check that the true value is $\log .9 \approx -0.1053605$, which differs from our estimate by about 0.000027, less than our bound.

This is all about practical applications to things other than pure math, where we might actually care about computing things. Perhaps surprisingly, it actually has an important application to pure mathematics as well: proving that functions are analytic!

Everything that we've been doing has required us to assume that our functions are analytic, i.e. can be represented by power series. Once we make that assumption, we can apply the Taylor series machine to figure out what the coefficients are, understand the radius of convergence, etc. However, if you were paying close attention above, we never actually needed that assumption for Taylor *polynomials*: we can still write down the (finite) approximations $P_d(x)$, no matter what $f(x)$ is, there's just no guarantee that this will be any kind of reasonable approximation to $f(x)$. The bound by the remainder still applies: $|f(x) - P_d(x)| \leq R_d(x)$. The question is just whether $R_d(x)$ eventually becomes small. Thus we can determine whether $f(x)$ is analytic: if $\lim_{d\to\infty} R_d(x) = 0$ for x sufficiently close to b, then $f(x)$ is analytic at b.

For example, let's prove that e^x is analytic. Fix some positive number r; we want to show that $R_d(x)$ converges for $|x| < r$. We can bound the $d + 1$th derivative of e^x on this range by e^r, so

$$R_d(x) = \frac{e^r}{(d+1)!}x^d.$$

Since e^r is constant and we know that $\frac{x^d}{(d+1)!}$ tends to 0 for any x, $\lim_{d\to\infty} R_d(x) = 0$ and so e^x is analytic around 0. Since this is true for any r, actually e^x is analytic on the whole real line.

Another calculus application is the 'true' L'Hopital's rule. We talked on the first day of class about how this is really a manifestation of first-order approximation: if $f(a) = g(a) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}.$$

This doesn't really explain what happens next: if $f'(a) = g'(a) = 0$, then usually in calculus 1 we would just repeat the process, but the above doesn't really justify it. The theory of Taylor series does, though: if we write f and g out as $f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\cdots$, for however many terms are zero we can just keep going as above. The conceptual way to think of this is that if the first n derivatives all vanish (for both functions, since otherwise evaluating is easy) then we're left with

$$\frac{\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \cdots}{\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \cdots},$$

i.e. we recover the ratio of the leading coefficients just like for polynomials.

This just about completes our regular material. Monday we'll review everything that we've done to sum up the class and make sure we haven't forgotten everything, and maybe go through some problems just for fun; Wednesday we'll start presentations.