Lecture 16: first convergence tests

Calculus II, section 3
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Let's take an aside for a bit today to talk about induction; we'll then come back and talk about some more convergence tests, and move towards the topic of power series.

Induction is a powerful tool you may or may not have seen before. The idea is this: suppose we have some statement which depends on a natural number n, e.g. "$2^n > n$ for every positive integer n." How could we prove this?

This is a fairly simple statement, so we could try to prove it directly, but let's try a different approach: first, we check that it is true for $n = 1$, since $2^1 = 2 > 1$; this is the base case. Next we *assume* that it is true for n, and try to prove it for $n + 1$: we have

$$2^{n+1} = 2 \cdot 2^n,$$

and by assumption $2^n > n$, so

$$2^{n+1} > 2n;$$

for $n \geq 1$, we have $2n \geq n + 1$, so for every $n \geq 1$ we have

$$2^{n+1} > n + 1$$

so long as we assume $2^n > n$; this is the induction step. We know that this is true for $n = 1$, so the induction step proves it must also be true for $n = 2$, and therefore also for $n = 3$, and so on.

We can generalize this method: if we want to prove some statement $P(n)$ for $n \geq n_0$, we can do it in two steps: first, check that $P(n_0)$ is true, and second, check that if $P(n)$ is true, so is $P(n + 1)$ for every $n \geq n_0$.

Note: the base case is very important here! For example, consider the statement $2^n \leq 0$ for $n \geq 0$. We can do the induction step: if $2^n \leq 0$, then

$$2^{n+1} = 2 \cdot 2^n \leq 0.$$

But for the base case $n = 0$, this is false, since $2^0 = 1 > 0$, and indeed this statement is *never* true!

This sort of thing is very useful for tricky sequences, especially recurrences since they give a direct relationship between a_{n+1} and a_n (or further values of a). It's also useful for computing sums. Here's a simple example: what is

$$\sum_{k=1}^{n} k$$

as a function of n?

We can compute the first few terms: $1, 1 + 2 = 3, 1 + 2 + 3 = 6, 1 + 2 + 3 + 4 = 10$, and so on. There are various clever methods of analyzing this sort of thing: for example, you can take the sequence $1, 2, 3, \ldots, n$, reverse it, and add it to itself to notice that every term becomes $n+1$, so since there are n terms the total is $n(n+1)$ and therefore the value of one sum is $\frac{n(n+1)}{2}$. We can also prove this by induction!

First, check that it's true in the base case $n = 1$: here the sum is just 1, and the formula gives $\frac{1\cdot(1+1)}{2} = 1$. Next, assume that it's true for n. Then

$$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + n + 1 = \frac{n(n+1)}{2} + n + 1 = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2},$$

which is the same formula after plugging in $n+1$ so the induction step works; and therefore the formula is correct for every n by induction.

More complicated calculations along these lines are also possible, and you'll have one on your homework. It's also just generally a very useful technique to keep in mind, and many proofs in all areas of math rely on induction.

Turning back to series and convergence: so far we have the limit test, the comparison test, the integral test, the alternating series test, and the computation of the geometric series. Today we'll introduce two new tests to formalize and generalize the idea of comparing to the geometric series: the root test and the ratio test.

Let's start with the root test. The nth term of the geometric series is $a_n = x^n$, and the geometric series converges if $|x| < 1$ and diverges if $|x| > 1$. If we didn't know that this was of the form x^n and just knew a_n (and n), we could recover x by taking nth roots: $x = (x^n)^{1/n}$, i.e. $a_n = x^n$ converges if $|a_n|^{1/n} < 1$ and diverges if $|a_n|^{1/n} > 1$.

Can we apply this to other series? Well, if a_n is not equal to x^n for some x, then $a_n^{1/n}$ won't be constant. We could hope, though, that it approaches a value. Thus we might guess the following: if $\lim_{n\to\infty} |a_n|^{1/n}$ converges (absolutely!) to a value less than 1, and diverges if it converges to a value greater than 1. This is called the root test. If the limit does not exist or is equal to 1 (as is unfortunately common), this test tells us nothing.

Is this true? Yes, by comparison: if $\lim_{n\to\infty} |a_n|^{1/n} = L < 1$, then choosing $L < M < 1$ for n large enough we'll always have $|a_n|^{1/n} < M$ and so $|a_n| < M^n$, so since $M < 1$ by the comparison test since

$$\sum_{n=0}^{\infty} M^n$$

converges. On the other hand, if the limit converges to $L > 1$ then we can again choose an intermediate $L > M > 1$ and then for n sufficiently large $|a_n| > M^n$, so since $M > 1$

$$\sum_{n=0}^{\infty} M^n$$

diverges and so by the comparison test so does $\sum_{n=0}^{\infty} a_n$.

For example, consider the sum

$$\sum_{n=0}^{\infty} e^{-n+1/n}.$$

This is not a geometric series due to the $\frac{1}{n}$ in the exponent, but we can apply the root test: $|e^{-n+1/n}|^{1/n} = e^{(-n+1/n)/n} = e^{-1+1/n^2}$, and in the limit $\frac{1}{n^2}$ tends to 0 and so this tends to $e^{-1} = \frac{1}{e} < 1$, so the series converges.

For most functions, though, this is a pain: taking nth roots of most functions is at least annoying and often difficult to evaluate in the limit. Can we use the same powerful idea to get something more convenient, at least in the cases where the root test doesn't work well?

Let's go back to the original framework, extracting x (or $|x|$) from $a_n = x^n$. From the formula x^n, the obvious method is via taking nth roots. There's another way to present this sequence, though, which is via a recursion: $a_{n+1} = xa_n$ and $a_0 = 1$, so that $a_1 = x \cdot 1 = x$, $a_2 = x \cdot x = x^2$, $a_3 = x \cdot x^2 = x^3$, etc. Thus if we're willing to take the data of both a_n and a_{n+1} we can recover x as the ratio $\frac{a_{n+1}}{a_n}$.

Again, for more general series this will not be constant, but we can take the limit. Thus our new guess is that $\sum_{n=0}^{\infty} a_n$ converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists and is less than 1 and diverges if it exists and is greater than 1. This is called the ratio test.

Is this one right? It is, by a similar argument: if the limit converges to $L < 1$, then for n sufficiently large the ratio is less than some M between L and 1, and so $a_{n+1} < Ma_n$ for all n greater than this threshold. In particular it follows by induction that $a_n < CM^n$ for all sufficiently large n: let n_0 be the first n for which $a_{n+1} < Ma_n$, and C be sufficiently large that $a_{n_0} < CM^{n_0}$, so the base case holds. Then assuming that $a_n < CM^n$, we have $a_{n+1} < Ma_n < M \cdot CM^n = CM^{n+1}$, so by induction this is true for all $n \geq n_0$. By the comparison test, the sum then converges. A similar argument works in the other direction.

For example, consider the series

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n}.$$

We have

$$\frac{(n+1)^3/3^{n+1}}{n^3/3^n} = \frac{1}{3} \cdot \left(\frac{n+1}{n}\right)^3 \to \frac{1}{3},$$

so by the ratio test the series converges.

For comparison, let's try the root test:

$$\left|\frac{n^3}{3^n}\right|^{1/n} = \frac{n^{3/n}}{3}.$$

This leaves us to evaluate $n^{3/n}$, which is a little tricky since on the one hand n is increasing to infinity while $3/n$ is decreasing to 0. We might expect this to agree with the $\frac{1}{3}$ above, since both are given by approximating our sequence by the same x^n, which would suggest $n^{3/n} \to 1$; the easiest way I know to see this uses a trick, which is to take the logarithm. Then $\log(n^{3/n}) = \frac{3}{n} \log n$ by the power rule, and we've seen before that this goes to 0 as $n \to \infty$ because $\frac{1}{n}$ decreases faster than $\log n$ increases, so $\log(n^{3/n}) \to 0$ and so $n^{3/n} \to e^0 = 1$.

This is a pretty good indication of the usual pattern: typically either will work (or both fail), but the ratio test is often easier unless the form of the sequence is particularly amenable to the root test; but it's really personal preference, and the logarithm trick we used here often makes the limits from the root test reasonable to evaluate. (Here's a fun example to work out: show that $n^{\frac{1}{\log n}}$ is always equal to e.)

Both of these tests work by guessing that our sequence a_n is "not too far" from x^n for some x, in the precise senses above. This suggests looking at series of the form

$$\sum_{n=0}^{\infty} a_n x^n,$$

where a_n is any sequence. If a_n does not grow or shrink too fast, then the convergence of this series will be the same as for the geometric series for x; in general, it may be different.

We can also think of these series as functions of our variable x. As such we can do various operations to them, like integration or differentiation; when the series are absolutely convergent, all of this is very well-behaved. When all but finitely many terms of a_n are zero, these power series are polynomials: for example, if $a_n = 1, 2, 1, 0, 0, 0, \ldots$, then

$$\sum_{n=0}^{\infty} a_n x^n = 1 + 2x + x^2 = (x+1)^2.$$

This, of course, converges for all values of x.

We've already seen that we can recover other functions from power series: for example,

$$\frac{x}{1-x} = \sum_{n=1}^{\infty} x^n = \sum_{n=0}^{\infty} x \cdot x^n.$$

You might then ask whether we can write other functions as power series; the (perhaps) surprising answer is that for a very large class of functions we can.

We'll talk more about this next week, but for now let's look at an example. Let $a_n = \frac{1}{n!}$, so our power series is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Since $\frac{1}{n!}$ decreases very fast, we might guess that this will converge even for large x; by the ratio test, we have

$$\frac{x^{n+1}/(n+1)!}{x^n/n!} = x \cdot \frac{n!}{(n+1)!} = \frac{x}{n+1}$$

by the definition of the factorial, which tends to 0 as $n \to \infty$ for any value of x. Therefore this series converges for all x.

I claim that in fact this series is equal to e^x. Why? Well, the characteristic property of e^x is that $\frac{d}{dx} e^x = e^x$, together with $e^0 = 1$; in other words, e^x is the unique solution to the differential equation $y' = y$, $y(0) = 1$. On the other hand

$$\frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

i.e. it satisfies the same differential equation, and plugging $x = 0$ into the series $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$ gives 1. Therefore they both satisfy the same first-order differential equation and initial condition, and so must be the same.

This is our first Taylor series; in practice, this is how e^x is often defined, and as we've seen this is equivalent to the differential equation definition. For example, this gives a new definition of e as

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

We'll see more Taylor series next week, as well as the full theory; in the meanwhile next time we'll concentrate on the theory of power series.