Lecture 13: integrating factors and the Laplace transform

Calculus II, section 3
March 21, 2022

First: recall from last time we discussed the second-order differential equation $x'' = -\frac{k}{m}x = -\alpha x$, which we solved to get $x = (A+B)\cos(\alpha t) + (A-B)i\sin(\alpha t)$. We then wanted to find the specific solution for given initial position x_0 and velocity v_0. I then made a mistake: when differentiating to get the equation for $v_0 = x'(0)$ in terms of A and B, I forgot to multiply by α: we should get

$$x'(t) = -(A+B)\alpha\sin(\alpha t) + (A-B)i\alpha\cos(\alpha t),$$

so in particular

$$v_0 = x'(0) = (A-B)i\alpha$$

so together with

$$x_0 = x(0) = A+B$$

we conclude that

$$x(t) = x_0\cos(\alpha t) + \frac{v_0}{\alpha}\sin(\alpha t).$$

Thanks to Augustine for catching that error.

Second: these pre-exam weeks continue to be a problem as far as homework, since the material will (potentially) be on the exam but the homework would otherwise be due on the day of the exam. The solution I want to try to this this time is: there will be *optional* homework posted by tomorrow covering today's material, which will be due Sunday night in the sense that if you want feedback on (any subset of) the problems, but it is not for a grade, only for practice.

Consider a differential equation like $y' + y = x$. It doesn't look like there's any way of separating parts, which is our only reliable way of solving first-order differential equations; we could try substituting a guess like $y = e^{rx}$, but doing this straightforwardly doesn't work. (We could do it by treating it as an inhomogeneous version of the homogeneous equation $y' + y = 0$, but that requires a lot of guessing and we'd like a better way if possible.)

What we can do instead is observe that if we multiply everything by e^x, we get

$$e^x y' + e^x y = xe^x.$$

The interesting thing about this formulation is that the left-hand side can be written as a derivative: using the product rule, $e^x y' + e^x y = e^x y' + (e^x)' y = \frac{d}{dx}(e^x y)$. Given the equation

$$\frac{d}{dx}(e^x y) = xe^x,$$

there's a straightforward thing to do, namely integrate:

$$e^x y = \int xe^x \, dx = xe^x - \int e^x \, dx = (x-1)e^x + C$$

by integration by parts. Dividing by e^x, we get

$$y = Ce^{-x} + x - 1.$$

Note that unlike with the method of undetermined coefficients, we know for sure that this is all the solutions, since we started with the equation and derived that the solution must look like this.

Now, there was a magical step in here: why did we multiply by e^x? Can we apply this more generally, and if so how do we know what to multiply by?

Suppose we have a differential equation like $y' + F(x)y = G(x)$ for some functions F and G, generalizing the above. The idea is to find some function $M(x)$, called the integrating factor, to multiply by in order to make the left-hand side the derivative of some product, i.e. we want $M(x)y' + M(x)F(x)y$ to look like the output of the product rule for $\frac{d}{dx}(M(x)y)$. In order for this to be true, we need $M(x)F(x) = M'(x)$, which is another differential equation! This one is easier to solve, though: we separate variables to get $\frac{dM}{M} = F(x)\,dx$ and so

$$\log M(x) = \int F(x)\,dx,$$

i.e.

$$M(x) = e^{\int F(x)\,dx}.$$

Note that the constant of integration isn't important here, since it just translates to multiplying the whole equation by a constant which does not affect the result.

In the case above, we had $F(x) = 1$, so $M(x) = e^x$. Let's try another example. Consider the equation $xy' + y = x$. This looks quite similar to our previous example, but to get it into the form above we need to divide by x: $y' + \frac{y}{x} = 1$. Therefore our integrating factor is going to be

$$M(x) = e^{\int \frac{1}{x}\,dx} = e^{\log x} = x,$$

so we're actually just putting it back how it was before, now with the observation that

$$xy' + y = \frac{d}{dx}(xy) = x$$

and so

$$xy = \frac{x^2}{2} + C,$$

and so

$$y = \frac{x}{2} + \frac{C}{x}.$$

It is also possible to use these sorts of methods for higher-order differential equations, but only in very special circumstances; one of the problems on the (optional) homework walks you through the setup for second-order equations.

For more general higher-order (or even first-order) equations, there is no general method, and indeed many equations have no easily expressible solution. There are however many techniques. One fun one is the Laplace transform.

There aren't actually very many equations the Laplace transform will allow us to solve that we don't know how to solve in other ways. However, it will allow us to remove the guesswork from solving (certain) inhomogeneous second-order equations, and there are some weird kinds of equations it is useful for understanding better. More generally, it's a fun change of perspective, and the first example of integral transforms, which are very common in certain areas of mathematics (Fourier transform, Mellin transform, Hilbert transform...).

Fix a function $f(x)$, defined on the interval $[0, \infty)$. The idea is to produce a new function, which we'll write as $\mathcal{L}[f](s)$ in a new variable s. We define this by

$$\mathcal{L}[f](s) = \int_0^\infty f(x) e^{-sx}\, dx.$$

For example, if $f(x) = 1$, this is

$$\int_0^\infty e^{-sx}\, dx = \lim_{N\to\infty} -\frac{e^{-sx}}{s}\bigg|_0^\infty = \frac{1}{s}.$$

More generally, if $f(x) = x^n$, this is

$$\int_0^\infty x^n e^{-sx}\, dx,$$

which is reminiscent of the integral representation of $n!$; if we make the substitution $t = sx$, so $dx = \frac{dt}{s}$, this is

$$\int_0^\infty \left(\frac{t}{s}\right)^n e^{-t}\frac{dt}{s} = \frac{1}{s^{n+1}}\int_0^\infty t^n e^{-t}\, dt = \frac{n!}{s^{n+1}}.$$

We often want to go the other way, i.e. given a function $g(s)$ we want to find a function $f(x)$ such that $\mathcal{L}[f] = g$. In general this is pretty hard, but in some cases we can do it. For example, suppose we wanted to find a function whose Laplace transform was $\frac{1}{s-1}$. That means we want to replace s in the first example above by $s-1$; notice that $e^{-(s-1)x} = e^x e^{-sx}$, so we can use $f(x) = e^x$:

$$\int_0^\infty e^x e^{-sx}\, dx = \frac{1}{s-1}.$$

This can be generalized to a certain extent (see the homework).

The Laplace transform has a number of good properties. For example, it is linear: if f and g are suitable functions and a and b are constants, then $\mathcal{L}[af + bg] = a\mathcal{L}[f] + b\mathcal{L}[g]$, as follows from the linearity of the integral. (Note that it is *not* true that $\mathcal{L}[fg] = \mathcal{L}[f]\mathcal{L}[g]$. To deal with products, we need convolution, which gets tricky.)

One of the most interesting and useful properties of the Laplace transform is that it turns differentiation into an algebraic relation. In particular, fix a differentiable function $f(x)$ and consider

$$\mathcal{L}[f'](s) = \int_0^\infty f'(x)e^{-sx}\,dx.$$

By integrating by parts, this is

$$f(x)e^{-sx}\Big|_0^\infty + s\int_0^\infty f(x)e^{-sx}\,dx = -f(0) + s\mathcal{L}[f](s)$$

(where we assume that $\lim_{x\to\infty} f(x)e^{-sx} = 0$, since otherwise the integral diverges). Thus if we understand the Laplace transform of f, we also understand the Laplace transform of its derivatives.

We can repeat this process: for example,

$$\mathcal{L}[f''](s) = -f'(0) + s\mathcal{L}[f'](s) = -f'(0) - sf(0) + s^2\mathcal{L}[f](s).$$

This means we have a good tool for differential equations, since it turns differential equations into algebraic equations.

For example, consider an equation like $ay'' + by' + cy = G(x)$, where $y = f(x)$ and a, b, c are constants. We know how to solve this: solve the homogeneous equation (i.e. $G(x) = 0$) by guessing $y = e^{rx}$ and so on, and then guess a single solution. The Laplace transform gives us another method: take the Laplace transform of both sides to get

$$a(-f'(0) - sf(0) + s^2\mathcal{L}[f](s)) + b(-f(0) + s\mathcal{L}[f](s)) + c\mathcal{L}[f](s) = \mathcal{L}[G](s).$$

We can solve this algebraic equation for $\mathcal{L}[f](s)$ to get

$$\mathcal{L}[f](s) = \frac{a(f'(0) + sf(0)) + bf(0) + \mathcal{L}[G](s)}{as^2 + bs + c}.$$

This is still pretty horrible-looking, but actually if we understand $\mathcal{L}[G](s)$ and the whole thing ends up as something we can take the inverse Laplace transform of we're in a pretty good situation.

Let's take an example to clarify things. Suppose we have the equation $y'' - y' - 2y = 2xe^x$, with $y(0) = y'(0) = 0$. First, we compute $\mathcal{L}[G](s)$ with $G(x) = 2xe^x$: we get

$$\mathcal{L}[G](s) = \int_0^\infty 2xe^x e^{-sx}\,dx = 2\int_0^\infty xe^{-(s-1)x}\,dx = \frac{2}{(s-1)^2},$$

by either integration by parts or a substitution. Since $f(0) = f'(0) = 0$, we get

$$\mathcal{L}[f](s) = \frac{2}{(s-1)^2(s^2 - s - 2)}.$$

We can factor further to get

$$\frac{2}{(s-1)^2(s+1)(s-2)},$$

and use partial fractions to decompose this as

$$-\frac{1}{6}\left(\frac{3}{s-1}+\frac{6}{(s-1)^2}+\frac{1}{s+1}-\frac{4}{s-2}\right).$$

Since the Laplace transform is linear, we just need to find the inverse transform of all these terms, which adds together to give

$$-\frac{1}{6}\left(3e^{x}+6xe^{x}+e^{-x}-4e^{2x}\right).$$