Lecture 12: second-order differential equations

Calculus II, section 3
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Let's do some physics. The basic law of Newtonian mechanics is $F = ma$, i.e. given a certain force acting on a mass, we can find its acceleration from this law. Consider for example a mass m on a spring. This has a resting position; let's say at time t it is at distance x from the resting position, either compressing (x negative) or stretching (x positive) the spring. Hooke's law states that the force on our mass at position x is given by $F = -kx$ for some (positive) spring constant k, so the force is directed back towards the resting position no matter where the mass is.

If we think of x as a function of t, then the acceleration a is $x''(t)$, so the equation of motion is

$$x'' = a = \frac{F}{m} = -\frac{k}{m}x.$$

This is a second-order differential equation. How can we approach it?

There are a few ways, but the most naive is to just guess: if this was a simpler equation of the form $x' = cx$ for some constant c, then the solution would be of the form $x = Ae^{ct}$ for some constant A. What if we just try the same thing here? Write $x = e^{ct}$ (worry about the scalar factor later), and see what equation we get.

Well, if $x = e^{ct}$, then $x' = ce^{ct}$ and so $x'' = c^2e^{ct}$. Plugging this into our equation gives

$$c^2e^{ct} = -\frac{k}{m}e^{ct},$$

so canceling e^{ct} gives $c^2 = -\frac{k}{m}$ and so

$$c = \pm\sqrt{-\frac{k}{m}} = \pm i\sqrt{\frac{k}{m}}.$$

With this value of c, we can now verify that e^{ct} solves our differential equation!

Are we done? Well, not quite. For one thing, we can multiply by a scalar, just like for $y = y'$; that's easy enough. Another thing to be careful of is that we have two possible values for c, and either works, so we have two families of solutions:

$$x = Ae^{i\alpha t}, \qquad x = Be^{-i\alpha t},$$

where we write α for $\sqrt{k/m}$. This is a *linear* equation, so it has the special property that if we take two solutions and add them together, we get a third solution. Thus the full family of solutions is now

$$x = Ae^{i\alpha t} + Be^{-i\alpha t}.$$

This depends on two parameters, as we expect from a second-order equation.

Now, our original problem was a physical one, and so we'd expect x to at the very least be a real number. Let's write these complex exponentials out using Euler's formula: we get

$$x = A(\cos(\alpha t) + i\sin(\alpha t)) + B(\cos(\alpha t) - i\sin(\alpha t)) = (A+B)\cos(\alpha t) + (A-B)i\sin(\alpha t).$$

Note that A and B can be any scalars, real or complex, so it is not guaranteed that the real part will be cosine. For example, we could take $A = -i$, $B = i$ to get $x = 2\sin(\alpha t)$.

Let's try and interpret our constants A and B as initial conditions. Suppose we started our mass at the resting position with zero velocity. We'd expect it to stay there, i.e. we should get the trivial solution $x = 0$, with $A = B = 0$.

What if we start at position x_0 with zero velocity? Well, we have

$$x_0 = x(0) = (A+B)\cos(0) + (A-B)i\sin(0) = A + B$$

and

$$0 = x'(0) = -(A+B)\alpha\sin(0) + (A-B)i\alpha\cos(0) = (A-B)\alpha i,$$

so $A = B = \frac{x_0}{2}$ and therefore

$$x = x_0\cos(\alpha t).$$

Finally, let's start in full generality at position x_0 with velocity v_0, so again $x_0 = A + B$ and $v_0 = (A-B)\alpha i$ which we can solve to get $A = \frac{\alpha x_0 - iv_0}{2\alpha}$, $B = \frac{\alpha x_0 + iv}{2\alpha}$. Therefore

$$x(t) = x_0\cos(\alpha t) + \frac{v_0}{\alpha}\sin(\alpha t).$$

This is the general (real) solution to a harmonic oscillator. We could also talk about the damped or driven harmonic oscillator, some of which will be on your homework, but let's instead move back into the pure math realm. This was a second-order differential equation

$$x'' = -\alpha^2 x,$$

or equivalently

$$x'' + \alpha^2 x = 0.$$

We could generalize to something of the form

$$ax'' + bx' + cx = 0.$$

How can we solve something like this?

Well, we could try the same thing: guess that this is something of the form e^{rt} and solve for r. We get

$$ar^2 + br + c = 0$$

upon canceling e^{rt}, which we can solve by the quadratic formula to get

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The case above is where $b = 0$ and $b^2 - 4ac = -4ac < 0$. When these are not true, the behavior will be different.

For example, consider the differential equation

$$x'' - x' = 0.$$

This translates to $r^2 - r = 0$, which has solutions $r = 0$ and $r = 1$, so the full solution to our differential equation is

$$x = Ae^{0t} + Be^t = A + Be^t.$$

Now there are no complex numbers in sight, and everything is very simple!

Like for quadratic equations, there are three possibilities. Either

(a) $b^2 - 4ac > 0$, like here, in which case there are two real solutions r_1, r_2 for r and so the final answer is simply $x = Ae^{r_1 t} + Be^{r_2 t}$; or

(b) $b^2 - 4ac < 0$, like above, in which case there are two complex solutions $a_1 + b_1 i$, $a_2 + b_2 i$ and so the final answer is

$$x = Ae^{a_1 t + b_1 it} + Be^{a_2 t + b_2 it} = Ae^{a_1 t}\cos(b_1 t) + Be^{a_2 t}\cos(b_2 t) + i(Ae^{a_1 t}\sin(b_1 t) + Be^{a_2 t}\sin(b_2 t))$$

which can be massaged into a more convenient form depending on initial conditions; or

(c) $b^2 - 4ac = 0$, in which case there is a single real solution for r, namely $-\frac{b}{2a}$.

What happens in this last case? Well, in principle, it should be the same: we just get one term now, so

$$x = Ae^{rt}.$$

This does satisfy the equation. However, the fact that all the other solutions depend on two parameters and this depends only on one suggests that there might be another solution out there, and indeed there is: $x = te^{rt}$ works as well!

To verify this, we can plug it in: $x' = e^{rt} + rte^{rt}$, so $x'' = 2re^{rt} + r^2te^{rt}$. Therefore

$$ax'' + bx' + cx = a(2re^{rt} + r^2te^{rt}) + b(e^{rt} + rte^{rt}) + cte^{rt} = (2ar + ar^2t + b + brt + ct)e^{rt}.$$

Since $r = -\frac{b}{2a}$, this is

$$\left(-b + \frac{b^2}{4a}t + b - \frac{b^2}{2a}t + ct\right)e^{rt} = -\frac{b^2 - 4ac}{4a}te^{rt} = 0$$

since $b^2 - 4ac = 0$.

Thus we again have two terms:

$$x = Ae^{rt} + Bte^{rt}.$$

Since r is real, everything is again pleasantly straightforward.

All this is when a, b, and c are constants, and in general there is no direct dependence on t and indeed every term has some factor of (a derivative of) x. What happens if we put in some function of t, e.g.

$$ax'' + bx' + cx = f(t)\ ?$$

Here $f(t)$ could be a constant, so independent of t but still adding a new term, or it could genuinely depend on t.

This is called an inhomogeneous equation, compared to the homogeneous version where $f(t) = 0$ (since there if we multiply x by some constant nothing changes, which is no longer true). How can we hope to solve this?

Well, let's start by solving the homogeneous version: suppose that $y(t)$ is some function such that

$$ay'' + by' + cy = 0.$$

Let's also suppose that we know *one* function $x_0(t)$ such that

$$ax_0'' + bx_0' + cx_0 = f,$$

i.e. we know one solution to the inhomogeneous equation. Then we can use this to find more: by linearity, $x_0 + y$ is also a solution to the inhomogeneous equation! This means that if we can find a single solution to the inhomogeneous equation, we can find a whole family of them by finding all the solutions to the homogeneous equation and adding them. (We don't know for sure, but we'd expect this to be all the solutions since we get a two-parameter family.) We could get it in a more definite form using initial conditions.

How do we find our one solution x_0? Well, by guessing again. Say $f(t)$ is some polynomial. Then a naive guess is that something whose sums of derivatives gives a polynomial might also be a polynomial, probably of the same degree since we have the cx-term. For example, say we have the equation

$$x'' + 2x = t - 1.$$

We can guess $x_0(t) = at + b$ and plug this in to get

$$2(at + b) = 2at + 2b = t - 1,$$

so $a = \frac{1}{2}$ and $b = -\frac{1}{2}$ to get $x_0 = \frac{t-1}{2}$. On the other hand the general (real) solution to the homogeneous equation would be

$$x = A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t),$$

so the general solution to the inhomogeneous equation is

$$x = A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t) + \frac{t-1}{2}.$$

This is the method of undetermined coefficients: we know roughly what the solution should look like, but don't know the coefficients, so we use the differential equation (and, if necessary, initial conditions) to solve for them.

Similarly, if $f(t)$ is exponential, we might guess that our initial solution $x_0(t)$ should be exponential; if it is trigonometric, we could guess $x_0(t)$ should be trigonometric.

There is a method to find $x_0(t)$ without guessing like this, but although you have the background to do it (which just requires integrating) to see where it comes from requires linear algebra, so I'll leave that to future differential equations classes you may take in the future.