Lecture 11: first-order differential equations

Calculus II, section 3
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We now get to start differential equations. A differential equation is any equation of *functions* in which one of the operations we can do to functions is differentiation. If y is a function of x, a typical differential equation might be something like

$$y' + y^2 = 1 - x.$$

Like for integrals, in general there's no reason to expect that a differential equation has a simple solution, and many do not, including this one; but by the end of this unit we'll be able to solve many special kinds of differential equations, including many that come up in practice.

One very simple kind of differential equation is something like $y' = f(x)$, i.e. specifying only what the derivative of y is as a function of x. This is really what we've been doing all along with integrals: finding y is the same thing as finding the antiderivative $\int f(x)\,dx$! Thus we can already solve many differential equations of this simple form. (If we only had y instead of y', we wouldn't even need to integrate.)

Where things become more complicated is when the equation involves both y and its derivatives. For example, consider the equation $y = y'$. How could we solve such a thing?

Well, this is asking for a function which is its own derivative. We know one like that: namely $y = e^x$, which satisfies $y' = \frac{dy}{dx} = \frac{d}{dx}e^x = e^x$. Therefore we've found a solution to our equation.

Is that the only solution? No: for example, $y = 0$ gives a (rather trivial) solution. More generally, $y = Ce^x$ gives a solution for any constant C. Are these the only solutions?

Suppose that $y = f(x)$ satisfies $y' = f'(x) = y = f(x)$. Consider the ratio $u(x) = \frac{f(x)}{e^x} = f(x)e^{-x}$. By the product rule, we have $u'(x) = f'(x)e^{-x} - f(x)e^{-x}$. But by assumption $f' = f$, so this is 0 and so $u(x)$ is some constant C, i.e. $f(x) = u(x)e^x = Ce^x$ for a constant C. Thus we've found all of the possible solutions.

This is about what we expect to happen for first-order equations, i.e. equations involving only one derivative:[1] we get essentially one solution, up to some constant, which may be additive, multiplicative, or other. Indeed, this is what already happens in our simple differential equations $y' = f(x)$: we end up with $y = \int f(x)\,dx$ up to some additive constant C.

We could generalize this to something like

$$y' = yf(x)$$

for some function $f(x)$. We've actually seen something like this before too: if we rewrite it as $\frac{y'}{y} = f(x)$, we can solve by noticing that the left-hand side is the derivative of $\log y$, by

[1] Compare for example $y'' = 1$ as a second-order equation, since we differentiate y twice. To solve for y, we can integrate twice: $y' = \int 1\,dx = C_1$, so $y = \int C_1\,dx = C_1x + C_2$ for some constants C_1 and C_2.

the chain rule. Then we have

$$\frac{d}{dx}\log y = f(x)$$

and so integrating

$$\log y = \int f(x)\,dx + C$$

and so

$$y = e^{\int f(x)\,dx + C}.$$

We can even turn this $+C$ in the exponent into multiplying the whole thing by a (different) constant $c = e^C$, to get a general solution

$$y = ce^{\int f(x)\,dx}.$$

We could do something similar to the above to see that this gives all solutions. (Note that the case $f(x) = 1$ gives the above as a special case.)

This is the method we want to generalize. For a (reasonably well-behaved) first-order equation, we can use algebra to get it into a form like

$$y' = F(x, y)$$

for some function F. This is potentially very complicated and hard to solve. However, sometimes we can split it as a product

$$y' = u(x)v(y),$$

with u depending only on x and v depending only on y. For example, above we had $u(x) = f(x)$ and $v(y) = y$. This is called separation of variables: we can separate the x-terms and the y-terms into a product.

If we can do this, just like above we then bring all the y-terms to the same side to separate even further:

$$\frac{y'}{v(y)} = u(x).$$

The trick is this. Rewrite y' as $\frac{dy}{dx}$ and think of these as separate infinitesimals dy and dx,

$$\frac{1}{v(y)} \cdot \frac{dy}{dx} = u(x).$$

Then we can multiply both sides by dx to get

$$\frac{1}{v(y)}\,dy = u(x)\,dx.$$

We now have something only depending on y on the left, with a dy, and only depending on x on the right, with a dx, so if we attach an integral sign to both sides we get

$$\int \frac{1}{v(y)}\,dy = \int u(x)\,dx.$$

Note that on the left we're integrating with respect to y, not to x. If $\frac{1}{v(y)}$ and $u(x)$ are things we can integrate, we're in business: if we have functions $U(x)$, $V(y)$ such that $U'(x) = u(x)$ and $V'(y) = \frac{1}{v(y)}$, then this gives

$$V(y) = U(x) + C.$$

If V is invertible, we can then solve for y:

$$y = V^{-1}(U(x) + C).$$

This is all very abstract, but we've actually already done it. In the case above, we just had $v(y) = y$, so $V(y) = \int \frac{1}{v(y)}\,dy = \int \frac{1}{y}\,dy = \log y$. (We can neglect the constant because we have it on the other side.) Therefore we have

$$\log y = U(x) + C = \int f(x)\,dx + C$$

and so

$$y = e^{\int f(x)\,dx + C} = ce^{\int f(x)\,dx}.$$

The constant C (or c) can be thought of as an "initial condition." Many differential equations can be thought of as something evolving in time; the classic example for something like $y' = cy$ where y' is proportionate to y is compounding interest, where we know how much money we have at some given time and want to know how this will change in the future. Specifying this initial amount of money determines the constant C. For example, if $y' = y$, so $y = Ce^x$, and we have the initial condition that at $x = 0$ we have $y = 4$, then $4 = C \cdot e^0 = C$.

Now, this thing we did where we broke up $\frac{dy}{dx}$ into dy and dx and moved them around independently is sketchy, and is really more of an intuition than anything rigorous. However, if we're skeptical we can double-check by taking our solution, which in general is just any function y of x satisfying $V(y) = U(x) + C$, and check that it satisfies $y' = u(x)v(y)$ by differentiating: $V'(y)y' = U'(x)$ is the same thing as $\frac{y'}{v(y)} = u(x)$, which recovers our original equation.

Another example is an equation like

$$yy' = x.$$

We could write this as $y' = x \cdot \frac{1}{y}$ and apply the method above, or just write $y' = \frac{dy}{dx}$ and move the differentials around:

$$y\,dy = x\,dx.$$

Integrating gives

$$\int y\,dy = \frac{y^2}{2} = \int x\,dx = \frac{x^2}{2} + C,$$

and so

$$y = \pm\sqrt{x^2 + C}$$

where we absorb the factor of 2 into the constant C.

So far we've only talked about ways of solving differential equations analytically, and this is after all a pure math-focused course. However there's at least one numerical method worth talking about, if only because it gives a good intuition for what differential equations are really doing. This is Newton's method.

Given a differential equation, say $y' = F(x, y)$, and an initial condition that $y = y_0$ at $x = x_0$, if we start at that point (x_0, y_0) the differential equation tells us the slope of y at that point, namely $y'(x_0) = F(x_0, y_0)$. This is the "direction" that y is going at this point. From there, we can do a linear approximation out to some nearby point $x_1 = x_0 + h$, i.e. $y(x_1) = y(x_0) + hy'(x_0) = y_0 + hF(x_0, y_0)$. Now that we know x_1 and $y_1 := y(x_1)$, we can plug these back in to $F(x, y)$ to get the slope at x_1, and repeat the process. If our differential equation is relatively well-behaved and h is chosen sufficiently small, this usually ends up giving a pretty good approximation to the true function y defined by the differential equation $y' = F(x, y)$.