Lecture 10: parametric curves

Calculus II, section 3
March 2, 2022

Let's return to parametric curves. As we've seen, the idea of parametric curves is very simple: instead of specifying y as a function of x (or x as a function of y), we give both x and y as functions of some parameter t: $x = x(t)$, $y = y(t)$. This includes graphs of the form $y = f(x)$, by just setting $x = t$ and $y(t) = f(t) = f(x)$.

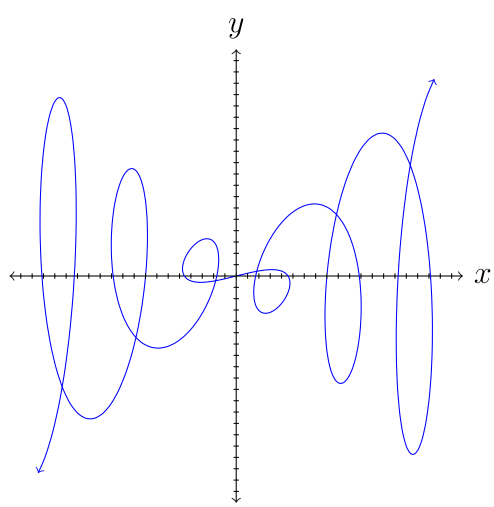
More generally, we can try to use a certain strategy: in order to better understand a parametric curve, we can try to "eliminate the parameter" to get a direct relationship between x and y, which is the kind of thing we're more used to seeing. For example, suppose we have the parametric curve $y = t^3$, $x = t^2$. We can solve the second equation for t to get $t = \sqrt{x}$, so $y = t^3 = \sqrt{x}^3 = x^{3/2}$. This no longer involves t, and may be easier to understand.

Note, however, that we've lost something. If t is positive, then $x = t^2$ and $t = \sqrt{x}$ are really the same thing, and so this is fine. However, if t is negative then it is not true that $t = \sqrt{x}$ if $x = t^2$, since $\sqrt{x}$ is positive. Thus this relationship between x and y is only valid for the part of the curve with $t \geq 0$, whereas the initial parametric formulation is valid for all t. This restriction is necessary in order to get anything of the form $y = f(x)$, since the initial curve is not described by y as a function of x.

What we could do, in this case, is instead describe it as a function of y. That is, instead of solving to get something of the form $y = f(x)$, we look for something of the form $x = f(y)$. To do so, we solve the first equation for t instead to get $t = \sqrt[3]{y} = y^{1/3}$, so $x = t^2 = y^{2/3}$. This is valid for all real numbers y (and t), and so we don't lose anything compared to the parametric version, although that version is a slightly simpler formulation.

The upshot here is that given a parametric curve, we may or may not be able to rewrite it as a usual rectangular graph; sometimes it may be easier to write it as one of $y = f(x)$ or $x = f(y)$, and it is worth playing around with to see what makes the most sense.

Sometimes this will work very badly. Consider a parametric curve like $x = t + 3\sin t$, $y = t\cos t$.



Not only is it impossible to solve these equations for t, but it is clear that any resulting relationship between x and y will be very far from one as a function of the other.

Sometimes we may be interested in going the other way: we have some relationship between x and y, and we want to turn it into a parametric equation.

Typically here we will not have y as a function of x or vice versa, since then there is no need to parametrize (and parametrization is very easy, just set $x = t$ and $y = f(t)$ as above). Instead, we might have something like $f(x, y) = 0$, and we want to find $x(t)$, $y(t)$ such that $f(x(t), y(t)) = 0$ for every t.

Solving this sort of thing in general is a multivariable calculus problem, and not an easy one. However there are many practical situations in which it is something we can do. One we've seen before is a circle, which we can parametrize using trigonometric functions. Others arise from problems we can think of as somehow happening over time. For example, consider a cycloid, the shape traced by the location of a point on a rolling ball. Directly finding the relationship between x and y at this point seems quite difficult. However, it is certainly a function of time, and from that point of view it is actually simpler. From physics, it is natural to put ourselves in the moving reference frame of the center of the ball, which is moving at a constant rate, let's say v. From this reference frame, the total position of the ball is constant and all that is happening is that a point is revolving around its rim, which we know how to parametrize; if our ball is rolling to the right, this point is going counter-clockwise, so we negate the parameter to get $x = r\cos t$, $y = -r\sin t$. Adjusting back out of the reference frame, we add back in the motion of the reference frame, i.e. we first move to the center of the ball, by radius r in the y-direction, and second x is at an additional vt, so in total we get $x = vt + r\cos t$, $y = -r\sin t$. Finally since the ball is rolling, it completes one revolution after time 2π, in which time the center has moved $2\pi v$; since this is the same distance the point on the edge has covered, namely $2\pi r$, we conclude that $v = r$, so in total we have $x = rt + r\cos t$ and $y = r - r\sin t$.

Let's get back to some more proper calculus. Suppose we have a parametric curve $x = x(t)$, $y = y(t)$. What is its slope $\frac{dx}{dy}$ at time t?

Well, we can do our linear approximation trick and say that dx and dy are infinitesimal changes in x and y relative to an infinitesimal change in t. This is not very rigorous, though.

Another option in the case where it is possible to write y as a function of x is to say $y = f(x) = f(x(t))$, so by the chain rule $y'(t) = f'(x(t))x'(t)$, i.e. $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, which we can think of as canceling the dt's, and so $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$. But we know it is not always true that we can write our parametric curve as $y = f(x)$.

The secret is that if we zoom in on a point close enough, this is always locally true unless the line becomes vertical, in which case the slope $\frac{dy}{dx}$ should not exist anyway. Therefore this really does always work.

We can take our fantastically swirly example $x = t + 3\sin t$, $y = t\cos t$ from above to find that it has slope

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\cos t - t\sin t}{1 + 3\cos t}.$$

Thus for example we see that the tangent line becomes vertical when $1 + 3\cos t = 0$, i.e. at

$2\pi n \pm \cos^{-1}(-\frac{1}{3})$. Note that we can define $\frac{dx}{dy}$ at any t (although periodically it blows up), but if we tried to define it directly by some relationship we would have points at which this curve is not smooth, and so the slope is not well-defined! It is only well-defined if we keep track of what t we are at.

We can also do integral calculus for parametric curves. To do Riemann integration, we need to get back to rectangular coordinates; we can think of a standard integral in terms of x as $\int_a^b y\,dx$ (and we've also done things of the form $\int_a^b x\,dy$ before). If y is a function of x, we know how to do this; but if both are a function of t, we can do it in the obvious way, i.e. by writing $y = y(t)$, $x = x(t)$, and so $dx = x'(t)\,dt$ to get $\int_a^b y(t)x'(t)\,dt$. To integrate with respect to y, we could reverse these coordinates.

For example, consider the parametric curve $x = e^t \sin t$, $y = t^2 + t$, with $0 \leq t \leq \pi$. Let's compute the area bounded by this curve and the y-axis. In principle, we could find t in terms of y and integrate that way, but it would be very messy; instead, let's use this method, integrating with respect to y, to get

$$\int_0^\pi e^t \sin t(2t+1)\,dt.$$

This is an integral you can do; it's a decent exercise in integration by parts if anyone is looking for practice problems, but let's not bother to do it now; the answer should come out to be $\pi e^\pi - \frac{1}{2}e^\pi - \frac{1}{2} \approx 60.6$.