Lecture 1: review

Calculus II, section 3
January 19, 2022

1. Syllabus

The syllabus can be found here. We'll take some time to go over it in class.

2. Basic concepts

Next, let's move into a review of calculus 1. What is calculus about? There's various poetic descriptions like "the science of change," but in more concrete terms it's really about tools to study functions. The most fundamental tool in calculus is the limit: if we have some function $f(x)$, we might care about $\lim_{x\to a} f(x)$ for some real number a, or $\lim_{x\to\pm\infty} f(x)$.

For the former kind of limit, the best case is when we can just plug in a, i.e. $f(a)$ is well-defined and $\lim_{x\to a} f(x) = f(a)$. In this case we say that f is continuous at a.

In other cases, this doesn't work as well. For example, there are cases where nothing is well-defined: $\lim_{x\to 0} \frac{1}{x}$ does not exist, since the limits from the left and right are different (and neither goes to a real number), and $\frac{1}{0}$ is not well-defined. We could also have one defined, but not the other: $\lim_{x\to 0} x \log|x| = 0$ (though this is not obvious if you haven't seen it before - see the homework), while $0 \cdot \log 0$ is not defined; on the other hand if $\operatorname{sign}(x)$ is the sign function sending positive numbers to 1, negative numbers to -1, and zero to 0, then $\lim_{x\to 0} \operatorname{sign}(x)$ is not well-defined, but $\operatorname{sign}(0) = 0$. You can also imagine (see the homework) a situation where neither the function nor the limit is defined at a given point, or where both are defined but they are different.

Let's think about the function $f(x) = \frac{\sin x}{x}$. This is a perfectly good function everywhere except $x = 0$, so it's natural to ask if we can extend it to 0 by continuity: that is, if $\lim_{x\to 0} f(x)$ exists, we could just think of $f(0)$ as this limit, so that f would now be a function on all real numbers.

If we know about the small-angle approximation $\sin x \approx x$, then we'd guess that this limit should be 1, which is correct. A slightly more sophisticated version of this observation is that this limit is

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0} = \sin'(0) = \cos(0) = 1.$$

To see if this gets us anywhere, let's talk a bit about differentiation.

3. Differentiation

The way we're going to think about differentiation is as first-order approximation. In other words, we have some function $f(x)$ and a point a; let's imagine we know $f(a)$ and all 'local' information around a, i.e. if we zoom in on a we know how $f(x)$ behaves around

$x = a$. The line tangent to the graph of $y = f(x)$ at a, if it exists, is given by an equation $y = \ell(x) = mx + b$ for some m and b; it satisfies $\ell(a) = ma + b = f(a)$, and for x near a we expect to have $\ell(x) = mx + b \approx f(x)$. If we solve these 'equations' we get $b = f(a) - ma = f(x) - mx$, so $m = \frac{f(x)-f(a)}{x-a}$. This is only 'approximately' true for any given x near a, but we expect it to become closer and closer to true as $x \to a$, and so we can find m by taking a limit; this is the derivative

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

whenever it exists. Thus what we hope to have is an approximation $f(x) \approx f'(a)x + f(a) - f'(a)a = f(a) + f'(a)(x-a)$. (We'll come back to this formula towards the end of this course, and add some further terms to make the approximation an equality.)

This makes it easy to compute limits of the type above, when we know the derivatives:

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

whenever f is differentiable at a. It actually gives us much more, though, including for example L'Hopital's rule: $\lim_{x\to a} \frac{f(x)}{g(x)}$ is $\frac{f(a)}{g(a)}$ if that is defined; if it is not, then that means $g(a) = 0$. If $f(a)$ is nonzero this goes to infinity, and if $f(a) = 0$ then our first-order approximation gives

$$\lim_{x\to a} \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Now that we've decided derivatives are useful, let's talk about computing them. In some cases, we can do this by directly computing the limits: for example, if $f(x) = x^2$, we have

$$f'(a) = \lim_{x\to a} \frac{x^2 - a^2}{x - a} = \lim_{x\to a} \frac{(x - a)(x + a)}{x - a} = \lim_{x\to a} x + a = 2a.$$

In general, this gets complicated, and so we use a few rules to simplify things.

1. Linearity: if we have two functions $f(x)$ and $g(x)$ which are both differentiable at a, then $(f + g)'(a) = f'(a) + g'(a)$, and if c is a real number then $(cf)'(a) = c \cdot f'(a)$.

2. Product rule: if $f(x)$ and $g(x)$ are differentiable at a, then $(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$.

3. Chain rule: if $g(x)$ is differentiable at a and $f(x)$ is differentiable at $g(a)$, then $(f \circ g)'(a) = f'(g(a))g'(a)$.

Example. Let's compute the derivative of $f(x) = x^x$. How can we possibly do this?

Well, since this has complicated exponentiation things happening, we could make it simpler by taking logarithms: $\log f(x) = \log(x^x) = x \log x$. We can then differentiate this:

$$\frac{d}{dx} \log f(x) = \frac{d}{dx} x \log x = \log x + \frac{x}{x} = \log x + 1$$

by the product rule and the fact that the derivative of $\log x$ is $\frac{1}{x}$. On the other hand, by the chain rule

$$\frac{d}{dx}\log f(x) = \frac{1}{f(x)}f'(x) = \frac{f'(x)}{f(x)},$$

so

$$f'(x) = f(x)\frac{d}{dx}\log f(x) = x^x(\log x + 1).$$

Once we have this new and powerful tool, we can also apply it to other problems. For example, the maxima and minima of a differentiable function are at points where $f'(a) = 0$, and so we can solve optimization problems by first differentiating and then solving an equation. We won't spend time on this now, but it'll come up later in the course.

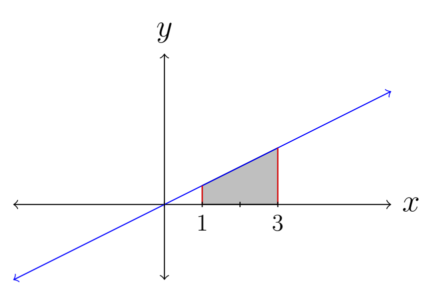
Once we know how to differentiate, the natural next question is if we can do it backwards: given a function $f(x)$, can we find $F(x)$ such that $F'(x) = f(x)$? The answer, as with all things, is 'sometimes,' but in general this is a much harder problem than differentiating. For example, it is easy to check that $\frac{x^{n+1}}{n+1}$ is an antiderivative of x^n, but for example although $f(x) = e^{x^2}$ does have an antiderivative $F(x)$ there is no way to describe this antiderivative through familiar functions like polynomials, exponentials, or trigonometric functions.

Before doing more computations, let's introduce a geometric interpretation of antiderivatives, as for derivatives.

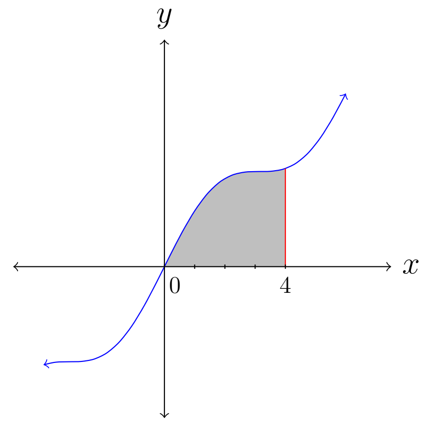
4. Integration

The geometric picture here is this: as for a derivative, we are looking at the graph of some function $y = f(x)$. As a sort of reverse of the derivative (suggestively), instead of looking at at the local rate of change of the height of the function now we are looking at the total cumulative height, i.e. the area under the curve defined by our function between two fixed points. We can define this rigorously by a limit process (Riemann sum), as for the derivative, but we'll rarely use that definition in practice so let's not bother with it here. (This is also called a definite integral, since the endpoints are specified and so it is well-defined, while an antiderivative is sometimes called an indefinite integral, since it is defined only up to an additive constant.)

In simple cases, we can find this area geometrically: for example, $\int_1^3 \frac{1}{2}x\,dx$ is the area of a certain trapezoid



which we can compute either by the formula for the area of a trapezoid or more simply as the difference of the area of triangles: $\frac{1}{2}(\frac{3}{2} \cdot 3) - \frac{1}{2}(\frac{1}{2} \cdot 1) = 2$. In other cases, this is essentially impossible: how would you approach $\int_0^4 x + \sin x \, dx$?



In some cases it is possible to directly compute the limits involved, but typically this is quite hard. We would rather be able to apply our more powerful tool, derivatives.

Fortunately, this is possible via the fundamental theorem of calculus, which connects integration and differentiation.

Theorem (Fundamental theorem of calculus). *Suppose that $F(x)$ is differentiable on $[a, b]$, with $F'(x) = f(x)$. Then*

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

In other words, if we know the antiderivative (or indefinite integral) F of f, we can compute the definite integrals.

With this in hand, we can compute a much larger class of integrals, though still not nearly all of them. For example, in the previous example we have $\frac{d}{dx}\frac{x^2}{2} = x$ and $\frac{d}{dx}(-\cos x) = \sin x$, so $\frac{x^2}{2} - \cos x$ is an antiderivative of $x + \sin x$; therefore

$$\int_0^4 x + \sin x\, dx = \frac{4^2}{2} - \cos 4 - \frac{0^2}{2} + \cos 0 = 9 - \cos 4 \approx 9.6536.$$

On the other hand, although we could numerically estimate $\int_{-2}^2 e^{x^2}\, dx$ (it's about 32.9) we have no idea how to find an exact formula for it, and it turns out that there basically isn't one.

There are more sophisticated methods for computing integrals, and we'll see some of them in the next couple weeks, but one that you probably saw in calculus 1 is the substitution rule. This is basically the chain rule run backwards: we know that the derivative of $f(g(x))$ is $f'(g(x))g'(x)$, so it follows that

$$\int f'(g(x))g'(x)\, dx = f(g(x)).$$

(Here we use the integral symbol without limits to denote the antiderivative or indefinite integral.) Now, we have the notation $\frac{d}{dx}g(x) = g'(x)$; formally, we can then write $dg(x) = g'(x)\,dx$. This does have some formal meaning, but for us it just means that this dx in the integral somehow corresponds to the dx in the denominator of the operator $\frac{d}{dx}$. In particular if we write $u = g(x)$, then the corresponding symbol is $du = dg(x) = g'(x)\,dx$, so in this case the above becomes

$$\int f'(u)\,du = f(u),$$

which is just the fact that f is the antiderivative of f'.

Example. Consider the definite integral

$$\int_2^7 x\cos(x^2)\,dx.$$

If we write $f(x) = \cos(x)$ and $g(x) = x^2$, then the integrand is $\frac{1}{2}f(g(x))g'(x)$, which is of close to the right form. In particular, if we write $u = g(x) = x^2$ then we have

$$\int x\cos(x^2)\,dx = \int \frac{1}{2}\cos(u)\,du,$$

and we know that the antiderivative of $\frac{1}{2}\cos(u)$ is $\frac{1}{2}\sin(u)$. Therefore as an indefinite integral we have

$$\int x\cos(x^2)\,dx = \frac{1}{2}\sin(u) = \frac{1}{2}\sin(x^2),$$

and we can apply the fundamental theorem of calculus to compute

$$\int_2^7 x\cos(x^2)\,dx = \frac{1}{2}\sin(49) - \frac{1}{2}\sin(4) \approx -0.098475.$$

Next time we'll talk about some more applications of integrals, especially more physical and spatial ones. This week's homework has a soft deadline of Monday (the 24th) and a hard deadline of Wednesday (the 26th); it will generally be focused on reviewing material we'll need from calculus 1.