Homework 3

Calculus II, section 3

Hard due date: 6:10 PM Wednesday February 9, 2022

+2 extra credit points if turned in by 6:10 PM Tuesday February 8

+5 extra credit points if turned in by 6:10 PM Monday February 7

As usual, you may use any resources to solve these problems except where stated otherwise, with the exception of computational software and posting these problems anywhere to be answered by others. Collaboration is encouraged, but everyone should write their own solutions. Write the names of any collaborators or sources used at the top of your homework. If you did not use any sources, write "sources used: none."

Any error in either the lecture notes or the homework is worth up to 5 points of extra credit to the first person to spot it, depending on the severity of the error; email me (cbz2106@columbia.edu) if you find one. (You do *not* lose points for incorrectly pointing out an error, so please do not hesitate!)

As on most math problems, the mathematics is the issue, not the answer: whether you have a correct method is more important then whether you get to the correct number at the end, so include your method!

Problem 1. In class, we used integration by parts to compute $\int \cos^4(x)\, dx$, and mentioned that we could also compute it via the double angle formula $\cos(2x) = 2\cos^2(x) - 1$. Do this computation, i.e. find $\int \cos^4(x)\, dx$ via the double angle formula. Does your answer agree with our answer from class? (15 pts)

Problem 2. For any positive real number a, generalize our method from class to compute the following indefinite integrals (10 pts each):

(a) $\int \frac{1}{\sqrt{a^2-x^2}}\, dx$

(b) $\int \frac{1}{\sqrt{x^2-a^2}}\, dx$

(c) $\int (x^2+a^2)^{-3/2}\, dx$

Problem 3. In the field of harmonic analysis, we find the correlation between a given function $f(x)$ and the various trigonometric functions $\sin(nx)$ and $\cos(nx)$ for integers n by integrating them against each other: the nth Fourier coefficient of a function $f(x)$ is $\frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\, dx$ (with respect to sines), or $\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\, dx$ (with respect to cosines). The basis for the theory of Fourier series, which has many applications to physics, engineering, audio processing, within pure math itself, and generally almost every even remotely mathematical field imaginable, is that these functions themselves are not correlated: in other words, if m and n are any two integers, we have

$$\int_0^{2\pi} \sin(mx)\cos(nx)\, dx = 0,$$

and

$$\int_0^{2\pi} \sin(mx)\sin(nx)\,dx$$

and

$$\int_0^{2\pi} \cos(mx)\cos(nx)\,dx$$

are also 0 unless $m = n$, in which case they are both equal to π. Prove these formulas. (You may find the angle sum formulae

$$\sin((m+n)x) = \sin(mx)\cos(nx) + \sin(nx)\cos(mx)$$

and

$$\cos((m+n)x) = \cos(mx)\cos(nx) - \sin(mx)\sin(nx)$$

helpful, keeping in mind that we can also get the angle difference formulae from these by replacing n by $-n$.) (25 pts)

Problem 4. Use the method of partial fractions, potentially in combination with other tools, to compute the following indefinite integrals (15 pts each):

(a) $\int \frac{1}{x^4-1}\,dx$

(b) $\int \frac{x+1}{(2x-5)^2}\,dx$

Optional problem (not for extra credit, just for fun). The integration by parts method for integrating powers of sine and cosine tends to reduce to integrating lower powers, which we can repeat until we get to something we recognize. In this problem we'll use this method to find a recurrence relation which will allow us to give a formula for π as an infinite product.

(a) For any nonnegative integer n, write $I(n)$ for the integral

$$I(n) = \int_0^{\pi} \sin^n(x)\,dx.$$

Use integration by parts to show that $nI(n) = (n-1)I(n-2)$. Keep in mind that you may have to use additional tricks such as the identity $\sin^2(x) + \cos^2(x) = 1$ or solving the resulting equation for the original integral.

(b) Since we can compute $I(n)$ from $I(n-2)$, if we know $I(0)$ then we can compute any $I(n)$ for any even n, and similarly we can compute $I(n)$ for every odd n if we know $I(1)$. Compute $I(0)$ and $I(1)$.

(c) Using parts (a) and (b), show that if n is even (so we can write $n = 2k$) then we have

$$I(n) = I(2k) = \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \pi$$

and if n is odd (so $n = 2k+1$) then

$$I(n) = I(2k+1) = \frac{2k}{2k+1} \cdot \frac{2k-2}{2k-1} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 2.$$

(d) Show that $I(n)$ is a nonincreasing function of n, i.e. $I(n+1) \le I(n)$ for every n, and conclude that $\frac{I(2k)}{I(2k+1)}$ approaches 1 as $k \to \infty$.[1]

(e) Use parts (c) and (d) to show that

$$\pi = 2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots.$$

Survey (optional). Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			
Problem 4			
Problem 5			

Please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="way too slow", 10="way too fast", 5="just right") and the novelty of the material to you (1="old hat", 10="all new").

Date	Lecture Topic	Material	Presentation	Pace	Novelty
1/31	Trigonometric substitution and volume				
2/2	Partial fractions				

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.

[1] We haven't talked about limits of sequences yet, which is what this really is, but all it means is that $\frac{I(2k)}{I(2k+1)}$ gets closer and closer to 1 as k grows.