Homework 11

Calculus II, section 3

Hard due date: 6:10 PM Wednesday April 27, 2022

+2 extra credit points if turned in by 6:10 PM Tuesday April 26

+5 extra credit points if turned in by 6:10 PM Monday April 25

As usual, you may use any resources to solve these problems except where stated otherwise, with the exception of computational software and posting these problems anywhere to be answered by others. Collaboration is encouraged, but everyone should write their own solutions. Write the names of any collaborators or sources used at the top of your homework. If you did not use any sources, write "sources used: none."

Any error in either the lecture notes or the homework is worth up to 5 points of extra credit to the first person to spot it, depending on the severity of the error; email me (cbz2106@columbia.edu) if you find one. (You do *not* lose points for incorrectly pointing out an error, so please do not hesitate!)

As on most math problems, the mathematics is the issue, not the answer: whether you have a correct method is more important then whether you get to the correct number at the end, so include your method!

Problem 1. Show that $\sin x$ is analytic around $x = 0$. (25 pts)

Problem 2. Recall the Taylor series

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

(a) Find the Taylor series of $f(x) = -\frac{1}{2}\log(1+x^2)$. (10 pts)

(b) Consider the power series

$$g(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots .$$

Compute $g'(x)$ as a Taylor series, and show that it is equal to $\frac{1}{x^2+1}$ for $|x| < 1$. (10 pts)

(c) Show that therefore $g(x) = \tan^{-1}(x)$, and use this to find an infinite series which converges to π. (10 pts)

(d) Prove the identity

$$\log(1+ix) = \frac{1}{2}\log(1+x^2) + i\tan^{-1}(x)$$

for $|x| < 1$ from the above Taylor series, where we take the complex logarithm to be defined by the above series (for $|x|$ sufficiently small). (10 pts)

(e) In polar coordinates, if $z = re^{i\theta}$ then $\log z = \log r + i\theta$, i.e. the logarithm of a complex number can be viewed as having real part corresponding to (the logarithm of) $r = |z|$ and imaginary part corresponding to the angle θ. In light of this interpretation, does the above identity make sense? Would you expect it to be true for all real x, in addition to those with $|x| < 1$? (5 pts)

Problem 3. Let $f(x) = (\log x)^2$.

(a) Compute the degree 4 Taylor polynomial approximating $f(x)$ around the point $x = 1$. (15 pts)

(b) Use the Lagrange error bound to bound the error in approximating $(\log(1.1))^2$ by the value of this degree 4 polynomial at $x = 1.1$. Compute the actual error by plugging in 1.1 to both the original function $(\log x)^2$ and your polynomial. How does the error compare to the bound? (15 pts)

Optional problem. Consider the function $f(x) = e^{-1/x^2}$, extended to 0 by continuity (i.e. we set $f(0) = \lim_{x\to 0} e^{-1/x^2} = 0$).

(a) Show that the nth derivative of $f(x)$ is a sum of terms of the form $Ce^{-1/x^2}x^{-k}$ for some integer k. (Hint: induction!)

(b) Similarly extend each term from part (a) to 0 by continuity, so that we can define $f^{(n)}(0)$ as $\lim_{x\to 0} f^{(n)}(x)$. Compute $f^{(n)}(0)$ for every nonnegative integer n.

(c) Is $f(x)$ analytic at 0, i.e. does its Taylor series about 0 exist and agree with $f(x)$ on some positive radius of convergence? If so, what is the radius of convergence?

Survey (optional). Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			
Optional Problem			

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.