Homework 1

Calculus II, section 3

Hard due date: 6:10 PM Wednesday January 26, 2022

+2 extra credit points if turned in by 6:10 PM Tuesday January 25

+5 extra credit points[1] if turned in by 6:10 PM Monday January 24

As usual, you may use any resources to solve these problems except where stated otherwise, with the exception of computational software and posting these problems anywhere to be answered by others. Collaboration is encouraged, but everyone should write their own solutions. Write the names of any collaborators or sources used at the top of your homework. If you did not use any sources, write “sources used: none.”

Any error in either the lecture notes or the homework is worth up to 5 points of extra credit to the first person to spot it, depending on the severity of the error; email me (cbz2106@columbia.edu) if you find one. (You do *not* lose points for incorrectly pointing out an error, so please do not hesitate!)

As on most math problems, the mathematics is the issue, not the answer: whether you have a correct method is more important then whether you get to the correct number at the end, so include your method!

Problem 1. Evaluate the following limits. Since some of these computations can be easily found online, please refrain from doing so unless you have exhausted your ingenuity—which should take quite a while—and for whatever reason you do not have access to other resources, such as office hours. (10 pts each.)

(a) $\lim_{x\to 0} \frac{4^x-1}{2^x-1}$

(b) $\lim_{x\to 0^+} x \log x$

(c) $\lim_{x\to a} \frac{\sqrt{f(x)}-2}{f(x)-4}$, where $f(x)$ is continuous at a and $f(a) = 4$

Problem 2. For each case, give an example of a function $f(x)$, defined on some subset of the real numbers, and a real number a satisfying the desired property. (10 pts each.)

(a) Neither $f(a)$ nor $\lim_{x\to a} f(x)$ is well-defined.

(b) Both $f(a)$ and $\lim_{x\to a} f(x)$ are well-defined, but they are different.

Problem 3. In class, we computed the derivative of x^x. Let’s now generalize that to the following setting. Let $f(x)$ be any function differentiable for all $x > 0$.

(a) Compute the derivative of $x^{f(x)}$, where $x > 0$, in terms of $f(x)$ and $f'(x)$. (15 pts)

[1] Total, not in addition to the +2 from turning in by Tuesday.

(b) Use part (a) together with our computation in class to find the derivative of $x^{(x^x)}$. (10 pts)

Problem 4. Compute the average value[2] of the function $f(x) = \frac{\log x}{(x \log x - x)^2} + \sin^3(\pi x)\cos(\pi x)$ between 3 and 5. (25 pts)

Optional. (No calculus needed! Just good mathematical fun.) The factorial $n!$ is the product of the positive integers up to n, e.g. $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$. Show that

(a) we have

$$n! \leq \frac{(n+1)^n}{2^n}$$

for every positive integer n;

(b) for any positive real number b, there exists a positive integer N such that $n! \geq b^n$ for every $n \geq N$.

Survey[3] (optional). Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			
Problem 4			
Optional problem			

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.

[2]Recall that the average value of a function between two points a and b, generalizing the usual idea of an average, is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

[3]Stolen from Drew Sutherland.