Additive Number Theory Talk #8: The Circle Method

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Contents

1 Introduction/Motivation **1**

2 Proving the Lemma **2**

3 A “Simple” Application **3**

3.1 Set Up 3
3.2 Minor Arc 4
3.3 Major Arc 5
3.4 Bonus: A Fun Integral 6

4 The Partition Formula **7**

1. Introduction/Motivation

The circle method builds on Lily's discussion of generating series, and so our initial setup is the same: we have a sequence $(a_n)_{n\in\mathbb{N}}$ that we're interested in. To study this sequence, we consider the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. More specifically, we will be analyzing the size of a_n as $n \to \infty$ (in contrast to generating series, in which we derived a closed form expression for F_n). Also note our use of "z" rather than "x", as we'll be working with complex numbers now.

So how does the circle method help us study a_n's limiting behavior? A key fact (that is proven in the next chapter) is the following:

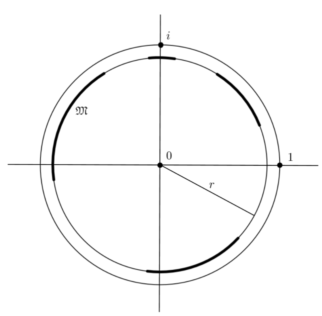
Lemma 1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be convergent on $D = \{z \in \mathbb{C} : |z| < 1\}$, and let C_r be the circle around the origin of radius $r \in (0, 1)$ oriented counterclockwise. Then

$$a_k = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z^{k+1}} dz.$$

(EXPLAIN CONTOUR INTEGRAL). The takeaway here is that the coefficients (which is what we're interested in) are equivalent to a type of integral. The circle method helps us approximate the integral, and thus helps us approximate the sequence we care about. (CONVERGENCE CONCERN)

How does the circle method help compute the integral? The main idea is that it breaks up the integral into "major arcs" and "minor arcs". We will partition the circle into $C_r = \mathcal{M} \cup m$ so that

- $|f(z)|$ for $z \in m$ is small compared to $|f(z)|$ for $z \in \mathcal{M}$
- The integral over the major arcs is easier to compute than the original integral



2. Proving the Lemma

Lemma 1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be convergent on $D = \{z \in \mathbb{C} : |z| < 1\}$, and let C_r be the circle around the origin of radius $r \in (0, 1)$ oriented counterclockwise. Then

$$a_k = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z^{k+1}} dz.$$

Proof. Recall the definition of the function $e : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, $e(x) = e^{i\cdot 2\pi x}$. For any $k \in \mathbb{Z}$, we have

$$\int_0^1 e(kt)dt = \begin{cases} 1 & \text{if } k = 0 \quad \text{(integrating 1)} \\ 0 & \text{otherwise} \quad \text{(complex numbers along circle cancel out)} \end{cases}$$

A fact from complex analysis (uniform convergence, if you've taken analysis) allows us to do the following manipulation:

$$\begin{aligned} \int_0^1 \frac{1}{(re^{i\cdot 2\pi t})^k} \cdot f\left(re^{i\cdot 2\pi t}\right) dt &= \int_0^1 \frac{1}{(re^{i\cdot 2\pi t})^k} \sum_{n=0}^{\infty} a_n (re^{i\cdot 2\pi t})^n dt \quad \text{plug in defn of } f \\ &= \sum_{n=0}^{\infty} r^{n-k} a_n \int_0^1 e^{2\pi i(n-k)t} dt \quad \text{switch integral and sum (analysis)} \\ &= a_k \quad \text{integral is zero except when } n = k \end{aligned}$$

Now, this may not look exactly the same as the stated lemma, but it's equivalent by change of variables.

$$z = re^{2\pi i t} \implies dz = 2\pi i \cdot re^{2\pi i t} dt \implies dz = 2\pi i \cdot z \cdot dt \implies \frac{dt}{dz} = \frac{1}{2\pi i} \cdot \frac{1}{z}. \quad \square$$

3. A "Simple" Application

3.1 Set Up

Goal: In this chapter, we apply the circle method to a simple example to demonstrate how it works. The result we'll obtain could be achieved in a simpler ways (which also provides a more precise result), but this work will show how the circle method works generally.

The problem we'll consider is the number of ways to write an integer N as the sum of s non-negative integers. Denote this value with $R_s(N)$; we'll think of these as coefficients of a particular polynomial.

Recall the definition of the function $e : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, $e(x) = e^{i \cdot 2\pi x}$. For any $\alpha \in \mathbb{R}$, let

$$f_N(\alpha) := \sum_{n=0}^{N} e(n\alpha)$$

Don't worry about the transition from infinite sum to finite sum, as we will consider $R(N)$ when N gets really large. The finite sum helps us not think about convergence issues, and stopping at the N-th term doesn't impact the coefficient of x^N.

I claim that $R_s(N)$, the number of ways to write an integer N as the sum of s non-negative integers, is given by the coefficient of α^N in $f_N(\alpha)^s$. Why is this the case? Consider when $s = 2$. Then

$$f_N(\alpha) = \sum_{n=0}^{N} e(n\alpha) = \sum_{n=0}^{N} \left(e^{2\pi i \alpha}\right)^n = \sum_{n=0}^{N} x^n,$$

so

$$\begin{aligned}(f_N(x))^2 &= \left(\sum_{n=0}^{N} x^n\right)^2 \\ &= (1 + x + x^2 + x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots) \\ &= 1 + 2x + 3x^2 + \cdots\end{aligned}$$

From this computation, it becomes clear that the coefficient of x^N in $(f_N(x))^s$ gives the number of ways to write N as the sum of s non-negative integers.

Therefore, as per Lemma 1, $R(N)$'s value is determined by the following integral:

$$R(N) = \int_{-1/2}^{1/2} f_N(\alpha)^s \ e(-N\alpha) d\alpha.$$

In case this isn't clear, let's break it down. Though it may look complicated, it follows directly from Lemma 1. In particular, if you look at the blue portion of Lemma 1's proof, you'll see the parallel. $f_N(\alpha)^s$ is the function we're considering, and to extract the coefficient of α^N from it, we take the contour integral. (The $e(-N\alpha)$ corresponds to $\frac{1}{(re^{i\cdot 2\pi t})^k}$.)

The subsequent work via the circle method will show us that $R(N)$ grows at a rate of $c \cdot N^{s-1}$, where c is a constant depending on s but not N.

Our major arc will be $\mathcal{M} = \left[-\frac{1}{N^{1-v}}, \frac{1}{N^{1-v}}\right]$, where $v > 0$ is small, and our minor arc will be $m = \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \mathcal{M}$. INSERT DRAWING OF THE CIRCLE.

As alluded to earlier, we can actually compute $R(N)$ exactly from a combinatorial argument:

$$R(N) = \binom{N+s-1}{s-1} \sim c \cdot N^{s-1},$$

where $c = \frac{1}{(s-1)!}$.

3.2 Minor Arc

In order to show that $\int_m f_N(\alpha)^s \ e(-N\alpha) d\alpha$ is "small," we first make a geometric argument to bound $|f_N(\alpha)|$. By definition, $f_N(\alpha)$ is the sum of evenly spaced points on the unit circle, and so this sum lies on a line with angle given by the average of all angle arguments. (DRAW PICTURE.) Therefore,

$$f_N(\alpha) \cdot e\left(-(\text{avg of } 0, \alpha, 2\alpha, \cdots, n\alpha)\right) = f_N(\alpha) \cdot e\left(-N\alpha/2\right) \in \mathbb{R} \tag{3.2.1}$$

The above logic tells us that $f_{N+1}(\alpha)$ and $f_N(\alpha)$ have an angle of $2\pi\left(\frac{(N+1)\alpha}{2} - \frac{N\alpha}{2}\right) = \pi\alpha$. Further, $|f_{N+1}(\alpha) - f_N(\alpha)| = 1$. By a simple geometric argument using these two pieces of information, we get $|\sin(\pi\alpha) f_N(\alpha)| \leq 1$.

Next, $|\sin(\pi\alpha)| \geq 2|\alpha|$ (proven for $|\alpha| \leq \frac{1}{2}$ in previous talk), so $|f_N(\alpha)| \leq \frac{1}{2|\alpha|}$. We will use this bound on $|f_N(\alpha)|$ to bound the integral along the minor arc. (Note that this is a good bound on m, but a bad bound on $\mathcal{M} = \left[-\frac{1}{N^{1-v}}, \frac{1}{N^{1-v}}\right]$.)

Now let's apply this bound to the minor arc integral.

$$\left|\int_m f_N(\alpha)^s e(-N\alpha) d\alpha\right| \leq \sup_{\alpha \in m} |f_N(\alpha)|^{s-2} \int_m |f_N(\alpha)|^2 d\alpha \leq \left(\frac{N^{1-v}}{2}\right)^{s-2} \int_m |f_N(\alpha)|^2 d\alpha$$

To deal with the remaining integral, we use Parseval's identity, which states that

$$\int_{-1/2}^{1/2} \left| \sum_{k=1}^{N} c_k e(\alpha k) \right|^2 = \sum_{k=1}^{N} |c_k|^2.$$

For our purposes, we have $c_i = 1$, so $\int_m |f_N(\alpha)|^2 d\alpha = N + 1$.
Finally, we are able to show that the integral along the minor arc is negligible compared to N^{s-1}:

$$\left| \int_m f_N(\alpha)^s e(-N\alpha) d\alpha \right| \leq \left(\frac{N^{1-v}}{2} \right)^{s-2} (N+1) \leq 2^{3-s} N^{s-1-v(s-2)}$$

3.3 Major Arc

For the major arc integral, we must show that its value is asymptotic to $c \cdot N^{s-1}$, where the constant c may depend on s but not N. As in the previous section, we begin with a discussion of $f_N(\alpha)$, and then use discovered approximations of $f_N(\alpha)$ to approximate the major integral. Recall equation 3.2.1 which said $f_N(\alpha) \cdot e(-N\alpha/2)) \in \mathbb{R}$. Expanding out the summation definition of $f_N(\alpha)$, we get

$$f_N(\alpha) \cdot e(-N\alpha/2)) = \sum_{n=0}^{N} e\left(\alpha \left(n - \frac{N}{2} \right) \right) \in \mathbb{R}$$

Because the sum is real, the imaginary component is zero, meaning that in $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we're only left with the cosine. So we can write the sum as

$$\sum_{n=0}^{N} e\left(\alpha \left(n - \frac{N}{2} \right) \right) = \sum_{n=0}^{N} \cos\left(2\pi\alpha \left(n - \frac{N}{2} \right) \right).$$

Next, we convert the sum into an integral, up to an error term, by splitting $\left[-\frac{N}{2}, \frac{N}{2}\right]$ into $O(\alpha N)$ subintervals s.t. $\cos(2\pi\alpha x)$ is monotone.

$$\sum_{n=0}^{N} \cos\left(2\pi\alpha \left(n - \frac{N}{2} \right) \right) = \int_{-N/2}^{N/2} \cos(2\pi\alpha x) dx + O(\alpha N)$$

On the major arc, we have $|\alpha| \leq \frac{1}{N^{1-v}}$, so

$$f_N(\alpha) = e(N\alpha/2) \int_{-N/2}^{N/2} \cos(2\pi\alpha x) dx + O(N^v) = \frac{\sin(N\pi\alpha)}{\pi\alpha} e(N\alpha/2) + O(N^v)$$

At this point, we can now use $g(\alpha) := \frac{\sin(N\pi\alpha)}{\pi\alpha} e(N\alpha/2)$ rather than the original $f_N(\alpha)$, which makes computing the major arc integral an easier job. I claim the following:

$$\int_{\mathcal{M}} \frac{\sin(N\pi\alpha)^s}{(\pi\alpha)^s} e\left(N\alpha(s-2)/2\right) d\alpha = \int_{\mathcal{M}} f_N(\alpha)^s e(-N\alpha) d\alpha + O(N^{s-2+v})$$

In this, there are two claims at play:

- The integrals on both sides have the same overall behavior
- The error term is negligible compared to N^{s-1}

We'll explain why the main portion (not error) is true, and leave the error as an exercise to the reader (it's not too hard).

$$\begin{aligned} \int_{\mathcal{M}} \frac{\sin(N\pi\alpha)^s}{(\pi\alpha)^s} e\left(N\alpha(s-2)/2\right) d\alpha &= \int_{\mathcal{M}} \frac{\sin(N\pi\alpha)^s}{(\pi\alpha)^s} e\left(N\alpha/2\right)^s e(-N\alpha) d\alpha \\ &= \int_{\mathcal{M}} \left(f_N(\alpha) + O(N^v)\right)^s e(-N\alpha) d\alpha \end{aligned}$$

Now we may focus on the LHS integral. Using a change of variables $x = N\pi\alpha$, we get

$$I := \int_{-\frac{1}{N^{1-v}}}^{\frac{1}{N^{1-v}}} \frac{\sin(N\pi\alpha)^s}{(\pi\alpha)^s} e\left(N\alpha(s-2)/2\right) d\alpha = \frac{N^{s-1}}{\pi} \int_{-\pi N^v}^{\pi N^v} \frac{\sin(x)^s}{x^s} e^{i\cdot x(s-2)} dx$$

Note that $\sin(x)/x$ is even, so that the imaginary component of the integrated is odd (and the real component is even), so the imaginary component cancels out over the integral, leaving us with

$$I = N^{s-1}\frac{2}{\pi} \int_0^{\pi N^v} \frac{\sin(x)^s}{x^s} \cos(x(s-2)) dx = N^{s-1}\frac{2}{\pi} \int_0^{\infty} \frac{\sin(x)^s}{x^s} \cos(x(s-2)) dx + O\left(N^{(1-v)(s-1)}\right).$$

Thus, we get

$$R(N) = cN^{s-1} + O\left(N^{s-1-v(s-2)}\right) + O(N^{s-2+v}) \sim cN^{s-1},$$

as desired.

3.4 Bonus: A Fun Integral

Since we knew $R(N) \sim \frac{N^{s-1}}{(s-1)!}$ beforehand, we can use this to derive the value of the weird integral above. That is,

$$\int_0^{\infty} \frac{\sin(x)^s}{x^s} \cos(x(s-2)) dx = \frac{\pi}{2(s-1)!}$$

4. The Partition Formula

The circle method can be used to analyze more complicated sequences as well. Take, for example, p_n to be the number ways to write n as a sum of positive integers where different orderings do not count count as different sums. Here are the first few values of p_n:

- $p_1 = 1$
  - 1
- $p_2 = 2$
  - $2 = 1 + 1$
- $p_3 = 3$
  - $3 = 2 + 1 = 1 + 1 + 1$
- $p_4 = 5$
  - $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$
- $p_5 = 7$
  - $5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$

The circle method can be used to show

$$p_n \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$$