Generating Functions

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1 Introduction

Generating functions are a powerful tool in number theory that can be used to solve recurrence relations. In this talk, I will discuss the purpose of generating functions and how they are used. I will begin with a basic definition of generating functions, introduce four operations on generating functions, and explain how to find both generating and closed functions using the Fibonacci sequence. Lastly, I will talk about asymptotic approximations of the Fibonacci Sequence.

2 Generating Functions

Generating Functions transforms problems about sequences into problems about functions. They do this by coding the terms of a sequence of real numbers as coefficients of powers of a variable (for example, x) in a formal power series. This transformation allows us to compute different operations on the formal power series and solve problems related to the infinite sequence.

Let $a_0, a_1x, a_2x^2...$ be a sequence of real numbers. The generating function associated with this sequence is the formal power series:

$$[G(x) = \sum_{n=0}^{\infty} a_n x^n]$$

Here are some basic examples of infinite series and their corresponding generating function.

$$\langle 0,0,0,0,...\rangle \longleftrightarrow 0+0x+0x^2+0x^3+...=0$$
$$\langle 1,0,0,0,...\rangle \longleftrightarrow 1+0x+0x^2+0x^3+...=1$$
$$\langle 3,2,1,0,...\rangle \longleftrightarrow 3+2x+1x^2+0x^3+...=3+2x+x^2$$

Notice the pattern above.The i-th term in the sequence (indexing from 0) is the coefficient of x in the generating function.

Recall that the sum of an infinite geometric series is:

$$1 + x + z^2 + z^3 + ... = \frac{1}{1-z}$$

This equation does not hold when the absolute value of z $\geq$1.This formula gives closed-form generating functions for a whole range of sequences. For example:

$$\langle 1, 1, 1, 1, ... \rangle \longleftrightarrow 1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$
$$\langle 1, -1, 1, -1, ... \rangle \longleftrightarrow 1 - x + x^2 - x^3 + x^4 - ... = \frac{1}{1+x}$$
$$\langle 1, 0, 1, 0, 1, 0, ... \rangle \longleftrightarrow 1 + x^2 + x^4 + x^6 + ... = \frac{1}{1-x^2}$$

3 Operations on Generating Functions

We can carry out all sorts of manipulations on sequences by performing mathematical operations on their associated generating functions. Let's experiment with four operations and characterize their effects in terms of sequences

3.1 Scaling

Multiplying a generating function by a constant scales every term in the associated sequence by the same constant. For example, we noted above that:

$$\langle 1, 0, 1, 0, 1, 0, ... \rangle \longleftrightarrow 1 + x^2 + x^4 + x^6 + ... = \frac{1}{1-x^2}$$

Multiplying the generating function by 2 gives:

$$\frac{2}{1-x^2} = 2 + 2x^2 + 2x^4 + 2x^6 + ...$$

which generates the sequence:

$$\langle 2, 0, 2, 0, 2, 0, ... \rangle$$

In the example above, we followed the Scaling Rule:

$$If\ \langle a_0, a_1, a_2 ... \rangle \longleftrightarrow F(x),$$
$$\text{Then} \quad \langle ca_0, ca_1, ca_2 ... \rangle \longleftrightarrow c \cdot (F(x)),$$

Proof of Scaling Rule:

$$\langle ca_0, ca_1, ca_2 ... \rangle \longleftrightarrow ca_0 + ca_1 x, ca_2 x^2 + ...$$
$$= c \cdot (a_0 + a_1 x + a_2 x^2 + ...)$$
$$= c \cdot F(x)$$

3.2 Addition

Adding generating functions corresponds to adding the two sequences term by term. For example, adding two of our earlier examples gives:

$$
\begin{array}{r}
1,1,1,1,1,1,...\rangle \longleftrightarrow \frac{1}{1-x} \\
+\langle 1,-1,1,-1,1,-1,...\rangle \longleftrightarrow \frac{1}{1+x} \\
\hline
= \langle 2,0,2,0,2,0,...\rangle \longleftrightarrow \frac{1}{1-x} + \frac{1}{1+x} = \frac{2}{1-x^2}
\end{array}
$$

Proof of Addition Rule:
If:

$$\langle a_0, a_1, a_2...\rangle \longleftrightarrow F(x)$$
$$\langle g_0, g_1, g_2...\rangle \longleftrightarrow G(x)$$

Then:

$$\langle a_0+g_0, a_1+g_1, a_2+g_2...\rangle \longleftrightarrow F(x)+G(x)$$

Proof.

$$\langle a_0+g_0, a_1+g_1, a_2+g_2...\rangle \longleftrightarrow \sum_{n=0}^{\infty}(a_n \cdot g_n)x^n$$
$$= \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} g_n x^n$$
$$=F(x)+G(x)$$

3.3 Right Shifting

Let's use a simple sequence and its corresponding generating function:

$$\langle 1,1,1,1,...\rangle \longleftrightarrow \frac{1}{1-x}$$

Now let's right-shift the sequence by adding k leading zeros:

$$= \langle 0,0,...,0,1,1,1,1,...\rangle \longleftrightarrow x^k + x^{k+1} + x^{k+2} + x^{k+3} + ...$$
$$= \mathrm{x}^k \cdot (1+x+x^2+x^3+...)$$
$$=\mathrm{x}^k \frac{}{1-x}$$

Evidently, adding k leading zeros to the sequence corresponds to multiplying the generating function by x^k. This holds true in general, as seen by the Right-Shift Rule:
If

$$\langle a_0, a_1, a_2, ...\rangle \longleftrightarrow F(x),$$

Then

$$\langle 0,0,...,0,a_0,a_1,a_2,...\rangle \longleftrightarrow x^k \cdot F(x)$$

Proof of Right-Shift Rule:

$$\langle 0,0,...,0,a_0,a_1,a_2,...\rangle \longleftrightarrow a_0x^k + a_1x^{k+1} + a_2x^{k+2} + ...$$
$$=\mathrm{x}^k \cdot (a_0 + a_1x + a_2x^2 + a_3x^3 + ...)$$
$$=\mathrm{x}^k \cdot F(x)$$

3.4 Differentiation

What happens when we take the derivative of a generating function? Let's differentiate the generating function for an infinite sequence of 1's (as seen earlier):

$$\frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + ...) = \frac{d}{dx}(\frac{1}{1-x})$$
$$1 + 2x + 3x^2 + 4x^3 + ... = \frac{1}{(1-x)^2}$$
$$\langle 1, 2, 3, 4\rangle \longleftrightarrow \frac{1}{(1-x)^2}$$

Above, we found the generating function for the sequence $\langle 1, 2, 3, 4, ...\rangle$. In general, differentiating a generating function has two effects on the corresponding sequence: each term is multiplied by its index and the entire sequence is shifted left one place.
Derivative Rule:
If

$$\langle a_0, a_1, a_2, a_3, ...\rangle \longleftrightarrow F(x)$$

Then

$$\langle a_1, 2a_2, 3a_3, ...\rangle \longleftrightarrow F'(x)$$

Proof of Derivative Rule:

$$\langle a_1, 2a_2, 3a_3, ...\rangle \longleftrightarrow a_1 + 2a_2x + 3a_3x^2 + ...$$
$$= \frac{d}{dx}(a_0, a_1x, a_2x^2 + a_3x^3 + ...)$$
$$= \frac{d}{dx}F(x)$$

The Derivative Rule is very useful. In fact, there is a frequent, independent need for each of differentiation's two effects, multiplying terms by their index and left-shifting one place. For example, let's try to find the generating function for the sequence of squares $\langle 0, 1, 4, 9, 16, ...\rangle$. If we could start with the sequence $\langle 1, 1, 1, 1, 1, ...\rangle$ and multiply each term by its index two times, then we'd have the desired result. A challenge is that differentiation not only multiplies each term by its index but also shifts the whole sequence left one place . However, we can cancel this unwanted left shift using the following steps: Differentiate, multiply by x, and then differentiate and multiply by x once more.

4 Fibonacci Sequence

Sometimes, we can find nice generating functions for more complicated sequences. For example, the Fibonacci numbers:

$$\langle 0, 1, 1, 2, 3, 5, 8, 13, 21, ...\rangle \longleftrightarrow \frac{x}{1-x-x^2}$$

We're going to derive this generating function and then use it to find a closed form for the n=Fibonacci number. The definition of a closed-form expression is a mathematical process that can be completed in a finite number of operations.

4.1 Finding Generating Function

Let's begin with the definition of the Fibonacci numbers (using f):

$$f_0 = 0$$
$$\mathrm{f}_1 = 1$$
$$\mathrm{f}_n = f_{n-1} + f_{n-2} (for n \geq 2)$$

We can expand the final clause into an infinite sequence of equation. Thus, the Fibonacci numbers are defined by:

$$f_0 = 0$$
$$\mathrm{f}_1 = 1$$
$$\mathrm{f}_2 = f_1 + f_0$$
$$\mathrm{f}_3 = f_2 + f_1$$
$$.$$
$$.$$
$$.$$

We want to define a function F(x) that generates the sequence on the left side of the equality symbols, which are the Fibonacci numbers. Then, we can derive a function that generates the sequence on the right.
First lets define F(x):

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + x_4 x^4 + ...$$

Now we need to derive a generating function for the sequence:

$$\langle 0, 1, f_1 + f_0, f_2 + f_1, f_3 + f_2, ... \rangle$$

One approach is to break this into a sum of three sequences, which we can do using three different generating functions and applying the Addition Rule

After applying the addition rule, we get:

$$F(x) = x + xF(x) + x^2 F(x)$$
$$= F(x) = \frac{x}{1-x-x^2}$$

4.2 Finding a Closed form

If we can find a generating function for a sequence, then we can often find a closed form for the n-th coefficient (not always), which can be very useful. For example, a closed form for the coefficient of x^n in the power series for $\frac{x}{1-x-x^2}$ would be an explicit formula for the n-th Fibonacci number
There are several ways to extract coefficients from a generating function. For a generating function that is a ratio of polynomials, we can use the method of partial fractions. Let's try this with the generating function for Fibonacci numbers. First we factor the denominator:

$$1 - x - x^2 = (1 - \alpha_1 x)(1 - \alpha_2 x)$$

where $\alpha_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\alpha_2 = \frac{1}{2}(1 - \sqrt{5})$. Next, we find A_1 and A_2 which satisfy:

$$\frac{x}{1-x-x^2} = \frac{A_1}{1-\alpha_1 x} + \frac{A_2}{1-\alpha_2 x}$$

We do this by plugging in various values of x to generate linear equations in A_1 and A_2. We can find A_1 and A_2 by solving a linear system. This gives:

$$A_1 = \frac{1}{\alpha_1-\alpha_2} = \frac{1}{\sqrt{5}}$$
$$A_2 = \frac{-1}{\alpha_1-\alpha_2} = \frac{-1}{\sqrt{5}}$$

Substituting into the equation above gives the partial fractions expansion of $F(x)$:

$$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha_1 x} - \frac{1}{1-\alpha_2 x}\right)$$

Each term in the partial fractions expansion has a simple power series given by the geometric sum formula:

$$(\frac{1}{1-\alpha_1 x} = 1 + \alpha_1 x + \alpha_1^2 x^2 + ...$$
$$(\frac{1}{1-\alpha_2 x} = 1 + \alpha_2 x + \alpha_2^2 x^2 + ...$$

When we substitute the series from above into our generating function, we end with:

$$F(x) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

5 Asymptotic Approximations

Asymptotic approximations are a tool used to estimate the behavior of a function/sequence as a variable approaches a certain limit (usually infinity or 0). It's a way of examining the behavior of a function as it gets really close to a certain value without evaluating it at that point. The Taylor series of a function is a common example of how to obtain the asymptotic approximation.

For the Fibonacci numbers, we found that the generating series is $\frac{x}{1-x-x^2}$, and from here, we can derive the exact formula for the Fibonacci numbers. However, one could more easily make some analytic observations: $1-x-x^2$ has zeros at $\frac{sqrt5 \pm 1}{2}$, so the generating function F(x) has poles at those two points. From knowing about geometric series, we know that a simple function with a pole at point c is $\frac{1}{c-x} = \frac{1}{c} \cdot \frac{1}{1-x/c}$, which is the generating function of the sequence $(\frac{1}{c}, \frac{1}{c^2}, \frac{1}{c^3}...$ i.e. the sequence $a_n = c^{(-n)}$. This suggests that the nth Fibonacci number might be close to some combination of $(\frac{\sqrt{5}\pm 1}{2})^{-n}$. We only need to observe that one of these zeros will give an increasing sequence. The first (+1) tends to 0, while the second (-1) is increasing and will dominate, so we expect that asymptotically the n-th Fibonacci number should be about $(\frac{\sqrt{5}+1}{2})^n$ up to some scalar.

Note that $\frac{\sqrt{5}+1}{2}$ is the golden ratio.