Additive Number Theory Notes

Weyl's Lemma and Hua's Inequality (adapted from Nathanson's *Additive Number Theory*)

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1 Introduction

The material I'm presenting today is generally about the behavior of large sums of functions. We'll build up to the concept of Weyl sums, and show that they are of an order smaller than you'd initially expect. This leads us to Weyl's inequality, which is an essential step towards our use of the circle method for Waring's problem. Firstly, I'll introduce some notation for this talk. This symbol is commonly referred to as Vinogradov notation: ¡¡ To say x ¡¡ y is to say that, using Big-O notation, that x = O(y). Aloud, this is said "x is of order y" or "x is less than less than y." We will use this notation throughout the talk, and it is the argument of most of today's claims. Throughout the talk, $e(t) = e^{2\pi it}$. Note that $\overline{e(t)} = e(-t)$.

2 Preliminary Lemmas

Let α be a real number. We can split α up into two parts:

$[\alpha]$ is the integer part of α, and $\{\alpha\}$ is its fractional part.

We express the distance from α to the nearest integer as:

$$||\alpha|| = min(n - \alpha : n \in \mathbb{Z})$$

which equals

$$inf(\{\alpha\}, 1 - \{\alpha\})$$

That tells us that $||\alpha|| \in [0, 1/2]$. Then, $\alpha = n \pm ||\alpha||$. Then, we have that

$$|sin\pi\alpha| = sin\pi||\alpha||.$$

Lemma 4.6 If $0 < \alpha < 1/2$,
then $2\alpha < sin\pi\alpha < \pi\alpha$.

Lemma 4.7 For all $\alpha \in \mathbb{R}$, all $N_1 < N_2 \in \mathbb{Z}$,

$$\sum_{N_1+1}^{N_2} e(\alpha n) << min(N_2 - N_1, ||\alpha||^{-1}).$$

Proof. $|e(\alpha n)| = 1$ for all $n \in \mathbb{Z}$, which implies that

$$|| \sum_{n=N_1+1}^{N_2} e(\alpha n)|| \leq \sum_{n=N_1+1}^{N_2} 1 = N_2 - N_1.$$

This is a geometric progression, so we use that fact and simplify:

$$| \sum_{n=N_1+1}^{N_2} e(\alpha n)| = |e(\alpha(N_1 + 1)) \sum_{n=0}^{N_2-N_1-1} e(\alpha)^n$$

$$= |\frac{e(\alpha(N_2 - N_1)) - 1}{e(\alpha - 1) - 1}|$$

$$\leq \frac{2}{|e(\alpha) - 1|}$$

$$= \frac{2}{|e(\alpha/2) - e(-\alpha/2)|}$$

$$= \frac{2}{|2isin\pi\alpha|}$$

$$= \frac{1}{|sin\pi\alpha|}$$

$$= \frac{1}{sin(\pi||\alpha||)}$$

$$\leq \frac{1}{2||\alpha||}$$

That's the end of the proof.

Lemma 4.8 Let $alpha \in \mathbb{R}$. $q, a \in \mathbb{Z}$. q, a coprime.
If $|\alpha - \frac{q}{a}| \leq \frac{1}{q^2}$, then

$$\sum_{1\leq r \leq \frac{q}{2}} e(\alpha n) << qlogq.$$

This is proven in Nathanson's textbook using several facts from chapter 4.3.
This Lemma is important for proofs of upcoming lemmas 4.10 and 4.11.

Lemma 4.9 Let $\alpha \in \mathbb{R}$. $q, a \in \mathbb{Z}$. q, a coprime with $|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}$.
Then for any $V \in \mathbb{R}+$ and $h \in \mathbb{Z}$, we have

$$\sum_{r=1}^{q} min(V, \frac{1}{||\alpha(hq + r)||}) << V + qlogq.$$

The proof of this lemma is quite challenging and long, and I recommend reading over it fully in Nathanson chapter 4.4. We will find that in our proofs for lemmas 4.10 and 4.11, it is useful to bound this sum.

3 Building up to Weyl's Inequality

Lemmas 4.10 and 4.11 These two lemmas are very similar. I will only prove lemma 4.11 here, and the proof for lemma 4.10 is in our textbook.

4.10 Let $alpha \in \mathbb{R}$. $q, a \in \mathbb{Z}$. q, a coprime with $|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}$.
Then for any $U \geq 1 \in \mathbb{R}$, $n \in \mathbb{Z}+$, we have

$$\sum_{1 \leq k \leq U} min(\frac{n}{k}, \frac{1}{||\alpha k||}) << (\frac{n}{q} + U + q)log2qU.$$

4.11 Let $alpha \in \mathbb{R}$. $q, a \in \mathbb{Z}$. q, a coprime with $|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}$.
Then for any $U, n \in \mathbb{R}$, we have

$$\sum_{1 \leq k \leq U} min(n, \frac{1}{||\alpha k||}) << (q + U + n + \frac{Un}{q})max\{1, logq\}.$$

Proof. call

$$S = \sum_{1 \leq k \leq U} min(n, \frac{1}{||\alpha k||})$$
$$\leq \sum_{0 \leq h \leq U/q} \sum_{1 \leq r \leq q} min(n, \frac{1}{||\alpha(hq + r)||})$$

Note the resemblance above to Lemma 4.8. Then,

$$\leq qlogq + \sum_{0 \leq h \leq U/q} (n + \sum \frac{q}{s}$$
$$<< qlogq + \sum(n + qlogq)$$
$$<< qlogq + (\frac{U}{q} + 1)(n + qlogq)$$
$$<< qlogq + Ulogq + n + \frac{Un}{q}$$
$$<< (q + U + \frac{Un}{q})max\{1, logq\},$$

which is our desired result, and the proof is complete. The proof for 4.10 is almost identical.

Lemma 4.12 We now apply a similar type of evaluation to functions, and specifically their delta-sums from the lecture before mine. Let $N_1, N_2, N \in \mathbb{Z}. N_1 < N_2, 0 < N_2 - N_1 < N$. Let $f(n)$ be a real-valued arithmetic function, meaning that it takes integers as its domain. Let

$$S(f) = \sum_{N_1+1}^{N_2} e(f(n)).$$

Then

$$|S(f)|^2 = \sum_{|d|<N} S_d(f),$$

where

$$S_d(f) = \sum_{n\in I(d)} e(\Delta_d(f)(n))$$

and $I(d)$ is an interval of consecutive integers contained in $[N_1, N_2]$.

Proof.

$$S_d(f) = S(f) * \overline{S(f)}.$$

We can then expand that out into

$$\sum_{N_1+1}^{N_2} e(f(m)) \sum_{N_1+1}^{N_2} \overline{e(f(n))}$$

$$= \sum \sum e(f(m) = f(n))$$

Rearranging things gives us

$$\sum \sum e(f(n+d) - f(n))$$
$$= \sum \sum e(\Delta_d(f(n)))$$
$$= \sum_{|d|<N} S_d(f)$$

This completes the proof.

Lemma 4.13 This Lemma is very similar to Lemma 4.12.
Let $N_1, N_2, N \in \mathbb{Z}. N_1 < N_2, 0 < N_2 - N_1 < N$. and let $l \in \mathbb{Z} \leq 1$. Let $f(n)$ be a real-valued arithmetic function, meaning that it takes integers as its domain. Let

$$S(f) = \sum_{N_1+1}^{N_2} e(f(n)).$$

Then

$$|S(f)|^{2^l} = \sum_{|d_1|<N} \cdots \sum_{|d_1|<N} S_{d_l \ldots d_1}(f)$$

Proof Sketch. This proof is done by induction on l.
The case $l = 1$ is the same as Lemma 4.12.
To go from l to $l + 1$, we square everything, then apply the Cauchy-Schwarz inequality.
Then we use 4.12 to estimate each term to get the result.

Lemma 4.14 Let $k \leq 1, k = 2^{k-1}, \epsilon > 0$. Let $f(x) = \alpha x^k + ...$ be some polynomial of degree k with real-valued coefficients. If

$$S(f) = \sum_{1}^{N} e(f(n)),$$

Then

$$|S(f)|^k << N^{k-1} + N^{K-k+\epsilon} \sum_{m=1}^{k!N^{k-1}} min(N, ||m\alpha||^{-1}).$$

Proof Sketch This proof uses lemma 4.13 to decompose $|S(f)|^k$ into pieces, and use material from previous talks to bound each of these pieces. The full proof is in Nathanson's textbook chapter 4.

4 Weyl's Inequality and Hua's Lemma

Weyl's Inequality (Theorem 4.3) Let $k \leq 1, k = 2^{k-1}, \epsilon > 0$. Let $f(x) = \alpha x^k + ...$ be some polynomial of degree $k \geq 2$ with real-valued coefficients. Let $q, a \in \mathbb{Z}$ such that $|\alpha - \frac{q}{a}| \leq \frac{1}{q^2}$. Let $K = 2^{k-1}$ and $\epsilon > 0$. Let $S(f) = \sum_1^N e(f(n))$. Then

$$S(f) << N^{1+\epsilon}(\frac{1}{N} + \frac{1}{q} + \frac{k}{Nq})^{\frac{1}{K}}.$$

Proof Sketch We can assume $q < N^k$. (EXPLAOIN WHY) So$logq < log(N^K) = klogN$, so k is constant and we can ignore it up to order and say that

$$logq << logN$$

and

$$logq << logN << N^\epsilon$$

Lemma 4.14 tells us that $|s(f)|^k << N^{k-1} + N^{k-k+\epsilon} \sum^{k!N^{k-1}} min(N, ||m\alpha||^{-1})$.
The following two theorems are applications of Weyl's inqeuality. These proofs are very direct and short, and can be found in the textbook.

Theorem 4.4 Let $k \geq 2$, $\frac{a}{q}$ rational, with $q \geq 1, q, a$ coprime. Then

$$S(a, q) = \sum_{1}^{q} e(\alpha x^k/q) << q^{1-\frac{1}{k+\epsilon}}$$

Theorem 4.5 Let $k \geq 2, \frac{a}{q}$ rational, with $q \geq 1, q, a$ coprime. Let $\delta > 0$ with the following property: if $N \geq 2$ and $a/q \in \mathbb{Q}$ such that $N^1/2 \leq q \leq N^{k-1/2}$. Then,

$$\sum_{n=1}^{N} e(jan^k/q) << N^{1-\delta}.$$

This result comes from a direct application of Weyl's inequality.

Hua's Lemma For $k \geq 2$, let

$$T(\alpha) = \sum_{n=1}^{N} e(\alpha n^k).$$

Then,

$$\int_0^1 \|T(\alpha)\|^{2^k} d\alpha << M^{2^k - k + \epsilon}.$$

Proof Sketch. This proof is done by induction on k. Note that for $k = 1$ the base case matches the hypothesis for Lemma 4.13. This allows us to decompose

$$|T(\alpha)|^{2^j} \leq (2N)^{2j-2-1} \sum_{|d_1|<N} \dots \sum_{|d_j|<N} \sum_{n \in I(d_1,\dots d_j)} e(\Delta_{d_j,\dots,d_1})(f(n)).$$

Then, using tools from last talk we can rewrite the difference operation Δ as a polynomial. After this, we rearrange things in terms of the number of representations of d_n. After that, we can bound these using induction and some more formerly introduced properties of Δ_d.