Tools and Easier Warings Problem Talk

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1 Introduction

Over the past few weeks we have discussed Waring's problem. Learning that for every natural number k, there exists a number $g = g(k)$ such that every number can be written as a sum of g k^{th} powers. We have discovered that k is the sum of at most 4 squares, 9 cubes, or $g(2) = 4$ and $g(3) = 9$. We used a series of bounding to determine that if g is finite for k, then it is also finite for $2k$, and we reveal that the set of k^{th} non-negative powers is a basis or finite order for every positive integer k. As Waring's problem is further investigated, this talk will provide some essential analytical tools, which will assist us in the understanding and application of Weyl's inequality and Hua's Lemma.

2 Notation

Fractional representations, if x is a real number:

$$[x] = \text{real portion} \quad \text{and} \quad \{x\} = \text{fractional portion}$$

$$[x] + \{x\} = x$$

3 Dirichlet's Theorem

Useful for approximating real numbers, as we can express them as rationals with small denominators. It shows the distribution of prime numbers, stating that for any pair of positive coprimes, we can find primes by observing their arithmetic progression. This is important because it shows that primes are not randomly distributed, but occur in patterns.

Let α and Q be real numbers, where $Q \geq 1$. There exists coprime integers a and q such that

$$1 \leq q \leq Q \quad \text{and} \quad \left|\alpha - \frac{a}{q}\right| < \frac{1}{qQ}$$

Let $N = [Q]$, q be a positive integer where $q \leq N$, $a = [q\alpha]$.

If we consider $\{q\alpha\}$ in the range or being more than (and including) 0 and less than $\frac{1}{(N+1)}$:

$$\{q\alpha\} \in \left[0, \frac{1}{(N+1)}\right)$$

We can write that:

$$0 \leq \{q\alpha\} = q\alpha - [q\alpha] = q\alpha - a < \frac{1}{N+1}$$

Then when we divide the last portion by q and recall that $N = [Q]$:

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q(N+1)} < \frac{1}{qQ} \leq \frac{1}{q^2}$$

If we consider $\{q\alpha\}$ in the range or being more than (and including) $\frac{N}{(N+1)}$ and 1:

$$\{q\alpha\} \in \left[\frac{N}{(N+1)}, 1\right)$$

And make $a = [q\alpha] + 1$, we can write that

$$\frac{N}{N+1} \leq \{q\alpha\} = q\alpha - a + 1 < 1$$

Which we can simplify to:

$$|q\alpha - a| \leq \frac{1}{N+1}$$

Then when we divide by q and recall that $N = [Q]$:

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q(N+1)} < \frac{1}{qQ} \leq \frac{1}{q^2}$$

Lastly, we can consider the range for $q\alpha$ to being more than (and including) $\frac{1}{N+1}$ and $\frac{N}{N+1}$:

$$\{q\alpha\} \in \left[\frac{1}{N+1}, \frac{N}{N+1}\right)$$

As q goes from $1, ..., N$, then each (there will be N different ones) real numbers $\{q\alpha\}$ are going to be in one of the $N-1$ intervals which we just simplified.

Dirichlet's Box Principle: there exists i integers $\in [1, N-1]$ and q_1, $q_2 \in [1, N]$ such that:

$$1 \leq q_1 < q_2 \leq N \quad \text{and} \quad \{q_2\alpha\}, \{q_2\alpha\} \in \left[\frac{i}{N+1}, \frac{i+1}{N+1}\right)$$

When we let $q = q_2 - q_1 \in [1, N-1]$ and $a = [q_2\alpha] - [q_1\alpha]$ then:

$$|q\alpha - a| = |(q_2\alpha - [q_2\alpha]) - (q_1\alpha - [q_1\alpha])| = |q_2\alpha - q_1\alpha| < \frac{1}{N+1} < \frac{1}{Q}$$

Which we can condense to:

$$|q\alpha - a| < \frac{1}{Q}$$

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{qQ}$$

4 Difference Operators

We will first go over difference operators and how they can be computed, and then prove five lemmas that will assist us in understanding and using them. Difference operators are linear operators that are defined on a particular function.

We can define a single difference operator on function f as:

$$\Delta_d(f)(x) = f(x+d) - f(x)$$

In order to do a calculate using the difference operator multiple times ex. ($l \geq 2$), we can use with iterated difference operator:

$$\Delta_{d_t, d_{t-1}, ..., d_1} = \Delta_t \circ \Delta_{d_{t-1}, ..., d_1} = \Delta_{d_t} \circ \Delta_{d_{t-1}} \circ \ldots \Delta_{d_t}$$

We can standardize this iteration and recognize patterns that occur when calculating multiple difference operators. If the number of differences to compute is 2:

$$\Delta_{d_1, d_2}(f)(x) = \Delta_{d_2}(\Delta_{d_1}(f))(x)$$

$$= (\Delta_{d_1}(f))(x + d_2) - (\Delta_{d_1}(f))(x)$$
$$= f(x + d_2 + d_1) - f(x + d_2) - f(x + d_1) + f(x)$$

When the number of differences to compute gets even larger, we can write it as a summation. The notation $\Delta^{(l)}$ can be uses as the iterated difference operator, which is $\Delta_{1,\ldots,l}$. Meaning that for a given l, we can just write $\Delta^{(l)}$ to mean the iterated difference operator from 1 to l. For example:

$$\Delta^{(2)}(f)(x) = f(x + 2) - 2f(x + 1) + f(x)$$

$$\Delta^{(3)}(f)(x) = f(x + 3) - 3f(x + 2) + 3f(x + 1) - f(x)$$

5 Important Lemmas

Now that we have an understanding and useful notation for difference operators, lets get to proving the five lemmas that we aim to cover in this talk. These lemmas may not seem to have exciting implications, but will be helpful tools in exploring Warings Problem further.

5.1 Lemma 4.1

Whenever l is 1 or more (so $l \geq 1$), we can write the difference operator as a summation:

$$\Delta^{(l)}(f)(x) = \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} f(x + j)$$

As we can see, the individual terms in the sum will alternate between positive in negative as j approaches l. We can prove by induction that if these summation equation works for l that it works for $l + 1$. For the entirety of this proof, the textbook shows all the individual steps for these calculations.

5.2 Lemma 4.2

Lets consider the power function $f(x) = x^k$ for Lemma 4.2 and Lemma 4.3
If we know that $k \geq 1$ and that the number of iterations we need (l), is between 1 and k, or $1 \leq l \leq k$, then we can write the iterated difference operator as:

$$\Delta_{d_l,\ldots,d_1}(x^k) = d_1, \ldots, d_l P_{k-l}(x)$$

Where the first term in $P_{k-l}(x)$ follows the form:

$$(k(k - 1) \ldots (k - l + 1)) \cdot (x^{k-l}) \cdots$$

Meaning that it is a polynomial of degree $k - l$ and its leading coefficient is $(k(k - 1) \ldots (k - l + 1))$. We also know that since $d_t, \ldots, d_1$ are integers, then $P_{k-l}(x)$ is a polynomial that has integer coefficients.

5.3 Lemma 4.3

We can expand on Lemma 4.2 for when $k \geq 2$ that:

$$\Delta_{d_{k-1},\ldots,d_1}(x^k) = d_1 \cdots d_{k-1} k! \left(x + \frac{d_1 + \cdots + d_{k-1}}{2} \right)$$

5.4 Lemma 4.4

Lets consider the function $f(x) = \alpha x^k + \cdots$ which is a polynomial of degree k. The degree of the resulting polynomial is dependent on the relationship between k and l since x^{l-k} is part of the first, and largest, term in $P_{k-l}(x)$

If we let the number of iterations be greater than or equal to 1, or $l \geq 1$ and use an iterated difference operator $\Delta_{d_l, d_{l-1}, \cdots, d_1}$.
If l is less than or equal to k or $(1 \leq l \leq k)$, then:

$$\Delta_{d_l, \cdots, d_1}(f)(x) = d_l, \cdots, d_1 \cdot (k(k-1)\cdots(k-l+1)\alpha x^{k-l} + \cdots)$$

And then if l is greater than k or $(l > k)$ then:

$$\Delta_{d_l, d_{l-1}, \cdots, d_1}(f)(x) = 0$$

And, in particular ,if $l = k-1$ and $d_1, \cdots, d_{k-1} \neq 0$ then:

$$\Delta_{d_{k-1}, \cdots, d_1}(f)(x) = d_1 \cdots d_{k-1} k! \alpha x + \beta = \text{polynomial of degree 1}$$

We can prove this relatively easily since:

$$\Delta_{d_l, \cdots, d_1}(f)(x) = d_1 \cdots d_l \left(\frac{k!}{(k-1)!} \alpha x^{k-l} + \cdots \right)$$

has x^{k-l} as the first term, and x^{k-l} will have a degree of 1 since $l - k = 1$.

5.5 Lemma 4.5

We take a closer look at the scenario where $1 \leq l \leq k$. Lets let the difference operator be $\Delta_{d_l, d_{l-1}, \cdots, d_1}$. We are examining the constant P.
If we have that:

$$-P \leq d_1 \cdots d_l, x \leq P \quad \text{then} \quad \Delta_{d_l, d_{l-1}}(x^k) \ll P^k$$

Then it is shown that $\Delta_{d_l, d_{l-1}}(x^k)$ is significantly less than P^k, which is dependent only on k.

6 Easier Waring's Problems

We can apply the lemmas we just established to solved an easier Waring's problem: Is it true that every integer can be written as the sum or difference of a bounded number of kth powers? In order to solve this, we won't be proving the existence $g(k)$, but rather will be proving the existence of $v(k)$
If this is true, then there exists a smallest integer $v(k)$ such that:

$$n = \pm x_1^k \pm x_2^x \cdots \pm x_{v(k)}^k$$

So that the equation has a solution in integers for every integer n.

6.1 Theorem for Easier Waring's Problem

If we let $k \geq 2$ and say that $v(k)$ exists, then:

$$v(k) \leq 2^{k-1} + \frac{k!}{2}$$

6.2 Proof

We can use a difference operator, recall Section 4 of the talk where we defined a difference operator for $l = 1$ and we can use Lemma 4.1 to get an equation for the summation and use Lemma 4.3 since this is a scenario where $k \geq 2$. We apply the $(k-1)$-st difference operator:

$$\Delta^{(k-1)}(x^k) = k!x + (k-1)!\binom{k}{2} = \sum_{l=0}^{k-1}(-1)^{k-1-l}\binom{k-1}{l}(x+l)^k$$

Let's assign m = $(k-1)!\binom{k}{2}$, so we can say that integers in the form $k!x + m$ can be written as the sum or difference of at most:

$$\sum_{l=0}^{k-1}\binom{k-1}{l} = 2^{k-1}\text{kth powers}$$

Then we can say that for any integer n, we can introduce integers q and r such that:

$$n - m = k!q + r \quad \text{where} \quad -\frac{k!}{2} < r \leq \frac{k!}{2}$$

It's been established that r is either the sum or difference of exactly $|r|$ kth powers 1^k . Because of this:

$$n \leq \sum \quad \text{of at most} \quad 2^{k-1} + \frac{k!}{2} \quad \text{integers of the form} \quad \pm x^k$$

7 Conclusion

In this talk, we explored the distribution of prime numbers through our analysis and proof of Dirichlet's theorem. Our investigation revealed patterns in the occurrence of prime numbers, providing valuable insights into their distribution.

Additionally, we delved into the concept of difference operators, which are linear operators used to compute the difference between function values at different points. The single difference operator, denoted as Δ_d, calculates the difference between function values at $x + d$ and x.

Iterated difference operators, such as $\Delta_{d_t, d_{t-1}, \ldots, d_1}$, involve applying the single difference operator with different intervals sequentially.

During our exploration, we introduced several important lemmas:

- **Lemma 4.1: Summation Representation**: The iterated difference operator can be expressed as a summation of terms involving the function evaluated at different points.

- **Lemma 4.2 and Lemma 4.3: Polynomial Representation**: These lemmas discuss how the iterated difference operator can be written as a polynomial with integer coefficients for power functions.

- **Lemma 4.4: Degree of Resulting Polynomial**: The degree of the resulting polynomial when applying the iterated difference operator depends on the relationship between the number of iterations and the degree of the original polynomial.

- **Lemma 4.5: Difference Operator Bounds**: This lemma provides conditions under which the resulting difference is significantly smaller than a bound P^k, where P is a constant.

These lemmas provide valuable insights into the behavior of difference operators and their applications, particularly in tackling Waring's problem. By leveraging these analytical tools, we can establish bounds on the number of terms required to represent an integer as a sum of k^{th} powers.

Our analysis of Dirichlet's theorem, coupled with the insights from the lemmas regarding difference operators, equips us with essential analytical tools for further investigations into Waring's problem. Moreover, these tools enhance our understanding and application of related concepts such as Weyl's inequality and Hua's Lemma.