Waring's Problem for Cubes

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1 Introduction

Waring's Problem refers to Edward Waring's statement in 1770 that for any integer greater than or equal to zero, it can be written as the sum of four squares, nine cubes, and nineteen fourth powers. For the case of cubes, the problem is to calculate g(3), which is the notation referring to the maximum number of cubes that can be added together in order to produce any nonnegative integer.

Although mathematicians Wieferich and Kempner proved that $g(3) = 9$, Landau found that only finitely many integers cannot be written as a sum of seven cubes. G(3) denotes the smallest number of cubes such that any sufficiently large nonnegative integer can be written as a sum of cubes. The exact value of G(3) is an unsolved problem, however it has been found that $4 \leq G(3) \leq 7$, but we will only prove that $4 \leq G(3)$.

2 Proof of g(3)=9

The Wieferich-Kempner Theorem, which proves that g(3)=9, first requires the proof of three lemmas, as we will now prove. We will be proving the theorem for integers $N > 8^{10}$.

2.1 Lemma 1

Claim: Let A and m be nonnegative integers such that $m \leq A^2$ and m is the sum of three squares. Then, $6A(A^2 + m)$ is the sum of six nonnegative cubes.

Proof: Let $m = m_1^2+m_2^2+m_3^2$. Thus, $6A(A^2+m) = 6A(A^2+m_1^2+m_2^2+m_3^2) =$

$6A^3 + 6Am_1^2 + 6Am_2^2 + 6Am_3^2 = \sum_{i=1}^{3} 2A(A^2 + 3m_i^2) = \sum_{i=1}^{3} 2A^3 + 6Am_i^2 + m_i^3 - m_i^3 + 3A^2m_i - 3A^2m_i = \sum_{i=1}^{3} (A+m_i)^3 + (A-m_i)^3$. We have thus proved that $6A(A^2+m)$ is the sum of six cubes.

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2.2 Lemma 2

Claim: Let $t \geq 1$. For every odd integer w, there is an odd integer b such that $w \equiv b^3 \bmod 2^t$.

Proof: Let b_1 and b_2 be two odd integers such that $b_1^3 \equiv b_2^3 \bmod 2^t$. Then, $b_1^3 - b_2^3 = (b_1 - b_2)(b_1^2 + b_1b_2 + b_2^2)$. This computation is easy to check. As 2^t divides $b_1^3 - b_2^3$ but 2^t does not divide $(b_1^2 + b_1b_2 + b_2^2)$ as it is odd, it must be that 2^t divides $(b_1 - b_2)$. This implies that $b_1 \equiv b_2 \bmod 2^t$. If $b_1 < b_2$, then $b_1^3 \not\equiv b_2^3 \bmod 2^t$.

Let f be a function from the set of odd integers mod2^t to itself, defined by $x \mapsto x^3$. Then, if $b_1 \neq b_2$, $f(b_1) \neq f(b_2)$ which implies that f is injective.

Thus, the domain of f is a subset of the codomain. Let X denote the domain and Y denote the codomain. There are thus $|X|$ elements in the image of f, and $|Y| - |X|$ elements not in the image of f. However, $|Y| = |X|$ which implies $|Y| - |X| = |X| - |X| = 0 \Rightarrow Y = Im(f)$ which implies that f is surjective. Thus, for all $w \in Y \exists b \in X$ such that $f(b) = b^3 = w$. Thus, For every odd integer w, there is an odd integer b such that $w \equiv b^3 \bmod 2^t$.

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2.3 Lemma 3

Claim: If $r \geq 10648 = 22^3$ then there exists an integer $d \in [0, 22]$ and an integer m that is a sum of three squares such that $r = d^3 + 6m$ and $m = a^2 + b^2 + c^2$ for some integers a, b, and c.

For the sake of contradiction assume that m cannot be written as the sum of three squares. By Legendre's Three Square Theorem, m can then be written in the form $4^a(7 + 8b)$. Then there are nonnegative integers a such that $m = 4^a(8b + 7)$ such that $6m = 6 * 4^a(8b + 7) \equiv 0, 72, 42, 90 \bmod 96$.

However m is the sum of threee squares so $6m \equiv h \bmod 96$ such that $h \in H = [0, 12, 18, 24, 30, 36, 45, 54, 60, 66, 78, 84]$ and $d \in [0, 22]$. Thus, $d^3 + h \bmod 96$

gives every congruence class modulo 96. Thus, for $r \geq 22^2, r - d^3 \geq 0$ and $r - d^3 \equiv h \bmod 96$ implies that $r - d^3 \equiv 6m$ such that m is the sum of three squares. Thus we have proved our claim.

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2.4 Wieferich Kempner Theorem

Now we have proved all the lemmas required to prove that g(3) = 9. We will be proving this for the case of integers $N > 8^{10}$. Let $n = [N^{1/3}]$. Let $2^{10} \leq n \leq 2^{3k+4} = 8^{k+1}$ such that $8^{10} = (2^{10})^{1/3}$ and $k \geq 3$. Thus, $2^{9k+3} = (2^{3k+1})^{1/3} = 8 * 8^{3k} < N \leq 2^{9k+12} = (2^{3k+4})^{1/3} = 8^{3k+1}$. Now, let $i = 1, ..., n$ and $N_i = N - i^3$. We say that $d_i = N_{i-1} - N_i = (N - (i-1)^3) - N - i^3 = N - i^3 + 3i^2 - 3i + 1 - N = 3i^2 - 3i + 1$. Induction on i shows that $-3i + 1 < 0$, which means that $d_i < 3i^2 \leq 3N^{2/3} \leq 2^{6k+8}$. We can show this by proving that $N \leq 2^{6k+8}$. Multiplying $3N^{2/3}$ and 2^{6k+8} by 1/3 and raising both to the power of 3/2 we find that $(2^{6k+8})^{3/2} = 2^{9k+25/2} \geq 2^{9k+12}$, which is greater than or equal to N.

Let us choose i such that $N_{i+1} < 2^{9k+3} \leq N_i$. We know that $N_{i+1} < N_i$ as $(i+3)^3 > (i^3)$ which implies that $N - (i+1)^3 < N - (i)^3$. Let $N_n = N - n^3$. We say that $N_n = N - n^3 \leq (n+1)^3 - n^3 - 1$ as $N - n^3 \leq 2^{9k+12} - (2^{3k+4})^3 = 2^{9k+12} - 2^{9k+12} = 0$ and $(n+1)^3 - n^3 - 1 \geq (2^{10}+1)^3 - (2^{10})^3 - 1 > 0$. We can further simplify such that $(n+1)^3 - n^3 - 1 = 3n^2 + 2n \Rightarrow N_n \leq 3n^2 + 3n < 6n^2$ as this inequality simplifies to $n + 1 < 2n$ which is true as n is greater than 1. $6n^2 \leq 6 * (8^{k+1})^2 = 3 * 2^{6k+8} \leq 3 * 8^{2k+3} = 3 * 2^{6k+9}$. We say that $N_i < N_{i-1} = (N_{i-1} - N_i) + (N_i - N_i - 1) = d_i + d_{i+1} + N_{i+1}$. Thus, $d_i + d_{i+1} + N_{i+1} < 3 * 8^{2k+3} + 2^{9k+3} \leq 11 * 8^{3k}$.

We say that $d_i = N_{i-1} - N_i$ is odd implies that exactly one of the integers N_i and N_{i-1} is odd, as subtracting an odd integer from an odd integer produces an even integer. Choose an a such that $a = i-1$ or $a = i$ such that $N_a = N - a^3$ is odd. By lemma 2, there is an odd integer $b \in [1, 8^k - 1]$ such that $N - a^3 \equiv b^3 \bmod 8^k$. We say that $7 * 8^{3k} = 8 * 8^{3k} - 8^{3k} < N - a^3 - b^3 < N_a < 11 * 8^{3k} \Rightarrow N - a^3 - b^3 = 8^k q$ as $N - a^3 - b^3 \equiv 0 \bmod 8^k \Rightarrow 7 * 8^{2k} < q < 11 * 8^{2k} \Rightarrow N - a^3$ either equals $N - i^3 = d_i$ or $N - (i-1)^3 = d_{i-1}$. In either case, $d_i < d_{i-1} < d_i + d_{i+1} + N_{i+1} < 11 * 8^{3k} \Rightarrow N - a^3 - b^3 < N_a < 11 * 8^{3k}$.

Let $r = q - 6 * 8^{2k}$. We know that $22^3 < 8^6 < 8^{2k} < r < (11.8^{2k} - 6 * 8^{2k}) = 5 * 8^{2k}$. By the third lemma, r can be written such that $r = d^3 + 6m, d \in [0, 22]$ and m is the sum of three squares. Now let $A = 8^k \Rightarrow m \leq r/6 < (5 * 8^{2k})/6 < A^2 = 8^{2k}$. Now let $c = (2^k)d$. We say that $N = b^3 + a^3 + 8^k q \Rightarrow N = a^3 + b^3 + 8^k(6*8^{2k}+r) = a^3+b^3+8^k(6*8^{2k}+d^3+6m) = a^3+b^3+(2^k d)^3+8^k(6*8^{2k}+6m) = a^3 + b^3 + c^3 + 6A(A^2 + m)$. Lemma 1 states that $6A(A^2 + m)$ is a sum of six nonnegative cubes, thus N is the sum of nine nonnegative cubes.

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2.5 Proof of G(3) is greater or equal to 4

Claim: If $N \equiv \pm 4 \bmod 9$ then N is not the sum of three integer cubes, which means $G(3) \geq 4$.

Proof: One can check that every integer is congruent to either $0, 1, -1$ by cubing 0, 1, 2, 3, 4, 5, 6, 7, 8. Thus, the sum of three cubes must belong to the one of the congruence classes $0, \pm 1, \pm 2, \pm 3$. Thus, for any integer N such that $N \equiv 4 \bmod 9$, this implies that N is not a sum of three cubes as it does not belong to any of the previously mentioned congruence classes, and because the three cubes consisting of 0, 1, -1 cannot add up to 4. Thus, $G(3) \geq 4$.

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