Shnirelman-Goldbach Theorem

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1 Introduction

The Goldbach Conjecture which states that every positive even integer greater than two can be written as the sum of two primes is an unsolved problem that holds true for integers up to $4 * 10^{11}$. What we will prove is the Shnirelman theorem - particularly that every integer greater than one is the sum of a bounded number of primes. First, we will introduce the Shnirelman density and show that for a nonempty set containing 0 with positive Shnirelman density is a basis of finite order, meaning that any non-negative integer can be written as a finite sum of elements of A. Then, we will prove that every integer greater than 1 is a sum of a bounded number of primes.

2 Shnirelman Density

Let A be a set of integers. $A(x)$ denotes the number of positive elements of A not exceeding x such that

$$A(x) = \sum_{a \in A} 1$$

Then, for $x \geq 0$,

$$0 \leq A(x) \leq x \implies 0 \leq \frac{A(x)}{x} \leq 1$$

We define the Shnirelman density of a set A of integers to be $\sigma(A)$ such that

$$\sigma(A) = \inf \frac{A(n)}{n}$$

for $n = 1, 2, 3...$

As n gets larger, $\frac{A(n)}{n}$ gets smaller, which implies that

$$0 \leq \sigma(A) \leq 1$$

We say that $A(n) \geq \sigma(A)$ which implies that if $\sigma(A) = a$,

$$A(n) \geq an$$

for all n. Say $1 \notin A$. Then, $A(1) = 0$ and thus $\sigma(A) = 0$. Let A contain every positive integer. Then, $A(n) = n$ and $\sigma(A) = 1$ which implies that if $\sigma(A) = 1$ if and only if A contains every positive integer.

Now, we will define a sumset A + B for two sets of integers A and B such that

$$A + B = \{a + b : a \in A, b \in B\}$$

Let $A_1, ..., A_h$ be sets of integers such that

$$A_1 + A_2 + ... + A_h = \{a_1 + a_2 + ... + a_h : a_i \in A_i, 1 \leq i \leq n\}$$

Then, let $A_i = A$ for $i = 1, 2, ...h$ so that $hA = A + ... + A$. We say that A is a basis of finite order h if every nonnegative integer is contained in hA and $h \geq 1$.

2.0.1 Theorem 1

Let A and B be sets of integers such that $0 \in A, 0 \in B$. If $n \geq 0$ and $A(n) + B(n) \geq n$, then $n \in A + B$.

Proof: If $n \in A, n + 0 = n \in A + B$. Similarly, if $n \in B, 0 + n = n \in A + B$.

Suppose $n \notin A, n \notin B$. Let A' and B' be two sets such that

$$A' = \{n - a : a \in A, 1 \leq a \leq n - 1\}$$

and

$$B' = \{b : b \in B, 1 \leq b \leq n - 1\}$$

Thus, $|A'| = A(n)$ and $|B'| = B(n)$. We know that

$$|A'| + |B'| = A(n) + B(n) \geq n$$

which means that there is overlap between A' and B'. This implies that there exists $b \in B$ such that $n - a = b$ for some $a \in A$ and $b \in B \implies n = a + b \in A + B$.

2.1 Theorem 2

Let A and B be sets of integers such that $0 \in A, 0 \in B$. If $\sigma(A) + \sigma(B) \geq 1$ then $n \in A + B$ for all nonnegative integers n.

Proof: Suppose $\sigma(A) = a$ and $\sigma(B) = b$. Then, if $n \geq 0, A(n) \geq an$ and $B(n) \geq bn \implies A(n) + B(n) \geq an + bn = (a + b)n \geq n$.

2.2 Theorem 3

Let A be a set of integers such that $0 \in A$ and $\sigma(A) \geq \frac{1}{2}$. Then, A is a basis of order 2.

Proof: Let A = B. Then, $\sigma(A) + \sigma(A) \geq 2 * \frac{1}{2} = 1$ which implies that $n \in A + B$ for every nonnegative integer n.

2.3 Theorem 4

Let A and B be sets of integers such that $0 \in A, 0 \in B$. Let $\sigma(A) = a$ and $\sigma(B) = b$.. Then,

$$\sigma(A + B) \geq a + B - ab$$

.

Proof: The full proof is on pages 193-194 of the textbook. The theorem also proves that $1 - \sigma(A + B) \leq (1 - \sigma(A))(1 - \sigma(B))$.

2.4 Theorem 5

Let $h \geq 1$, and let $A_1, ..., A_h$ be sets of integers such that $0 \in A_i$ for $i = 1, ..., h$. Then,

$$1 - \sigma(A_1 + ... + A_h) \geq \prod_{i=1}^{h}(1 - \sigma(A_i))$$

.

Proof: The proof follows from induction on h. Let $\sigma(A_i) = a_i$ for $i = 1, ...h$. For h = 1, $1 - \sigma(A_1) = 1 - a_1$. If h = 2, $1 - \sigma(A_1 + A_2) \leq (1 - \sigma(A_1))(1 - \sigma(A_2)) = \prod_{i=1}^{2}(1 - A_i)$.

Now, let $h \geq 3$ and assume the theorem holds for integers less than or equal to h. Let $A_1, ..., A_h$ be sets of integers such that $0 \in A_i \forall i$. Let $B = A_2 + ... + A_n$ such that

$$(1 - \sigma(B)) = 1 - \sigma(A_2 + ... + A_n) \leq \prod_{i=2}^{h}(1 - \sigma(A_i))$$

$$\implies 1 - \sigma(A_1 + .. + A_h) = 1 - \sigma(A_1 + B)$$

$$\leq (1 - \sigma(A_1))(1 - \sigma(B))$$

$$\leq (1 - \sigma(A_1)) \prod_{i=2}^{h})1 - \sigma(A_i)) = \prod_{i=1}^{h}(1 - \sigma(A_i))$$

2.5 Shnirelman Theorem

Let A be a set of integers such that $0 \in A$ and $\sigma(A) > 0$. Then A is a basis of finite order.

Proof: Let $\sigma(A) = a$ such that a is greater than 0. Then, we know that $0 \leq 1 - a < 1$ so that

$$0 \leq (1-a)^d \leq 1/2$$

for an integer d greater than 1. Theorem 5 shows that

$$1 - \sigma(dA) \leq (1 - \sigma(A))^d = (1-a)^d \leq 1/2$$

such that $\sigma(dA) \geq 1/2$. Let $h = 2d.Theorem4showsthatthesetdAisabasisoforder2, soAisabasisoforderh, meaningthatAisabasisoffiniteorde$

3 Shnirelman-Goldbach Theorem

Now, we will prove that every integer greater than one is a sum of a bounded number of primes.

3.1 Lemma 1

Let $r(N)$ denote the number of representations of the integer N as the sum of two primes. Then,

$$\sum_{N \leq x} r(N) << \frac{x^2}{(logx)^2}$$

Proof: Let p and q be two primes such that $p, q \leq \frac{x}{2} \implies p + q \leq 2(\frac{x}{2}) = x$. By Chebyshev's theorem,

$$\sum_{N \leq x} r(N) \geq \pi(x/2)^2 >> \frac{(x/2)^2}{(log(x/2))^2} >> \frac{x^2}{(logx)^2}$$

.

3.2 Lemma 2

Let $r(N)$ denote the number of representations of N as the sum of two primes. Then

$$\sum_{N \leq x} r(N)^2 << \frac{x^3}{(logx)^4}$$

Proof: The proof is on pages 196-197 of the textbook.

3.3 Theorem 6

The set

$$A = \{0, 1\} \cup \{p + q\}$$

for p, q primes has positive Shnirelman density.

Proof: The Cauchy-Schwarz inequality implies that

$$(\sum_{N \leq x} r(N))^2 \leq \sum_{r(N) \geq 1} 1 \sum_{N \leq x} r(N)^2 \leq A(x) \sum_{N \leq x} r(N)^2$$

. Lemma 1 and 2 imply that

$$\frac{A(x)}{x} \geq \frac{1}{x} \frac{(\sum_{N \leq x} r(N))^2}{\sum_{N \leq x} r(N)^2}$$

$$>> \frac{1}{x} \frac{\frac{x^4}{(logx)^4}}{\frac{x^3}{(logx)^4}} >> 1$$

This implies that there exists an integers a_1, a_2 such that $A(x) \geq a_1 x$ and $A(x) \geq a_2 x$ as 1 is contained in the set A of integers. Therefore, $\forall x \geq 1, A(x) \geq min(a_1, a_2)x$ which implies the Shnirelman density of A is positive.

3.4 Shnirelman-Goldbach Theorem

Every integer greater than one is the sum of a bounded number of primes.

Proof: We know that for some integer h, a set of integers A is a basis of order h. Say $N \geq 2 \implies N - 2 \geq 0$ so for integers k, d such that $k + d \leq h$ there exist d pairs of primes p_i, q_i such that

$$N - 2 = 1 + ... + 1 + (p_1 + q_1) + ... + (p_d + q_d)$$

for k copies of 1. Let $k = 2m + r$ such that r equals 0 or 1. Let r = 0.

$$\implies N = 1 + ... + 1 + 2 + (p_1 + q_1) + ... + (p_d + q_d) = 2 + ... + 2 + (p_1 + q_1) + ... + (p_d + q_d)$$

for m + 1 copies of 2.

Let r = 1, so $k = 2m + 1$. Then,

$$N = 2 + ... + 2 + 3 + (p_1 + q_1) + ... + (p_d + q_d)$$

for m copies of 2. Either way, N is a sum of

$$2d + m + 1$$

primes.