Applications of the Circle Method: Partition Counts

Julie Griswold

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1 Introduction

Throughout the course we were introduced to generating functions as a method of studying sequences, and to the Circle Method. The Circle Method is a way of recovering information about sequences from their generating functions, and provides a systematic approach to understanding the distribution of integers as sums of smaller integers. In today's talk, counting partitions, or the number of ways a positive integer can be expressed as a sum of smaller integers expressed as p_n. Using the Circle Method we can find a way to estimate p_n. This talk will also be in reference to this paper Circle Method.

2 The Partition Formula

A partition function p_n represents the number of possible partitions of a non negative integer n. Below are a few examples of p_n

$p_1 = 1 = 1$

$p_2 = 2 = 2 = 1 + 1$

$p_3 = 3 = 2 + 1 = 1 + 1 + 1$

$p_4 = 5 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$

$p_5 = 7 = 5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$

This way of manually finding partitions is not very efficient and very tedious. Need to find a way to compute p(n).

2.1 Generating Function of Partitions

$$f(z) = \sum_{n=0}^{\infty} P_n Z^n \tag{1}$$

We will derive Euler's generating function from the sequence of P_n. $(1 + x + x^2 + x^3...)((1 + x^2 + x^4 + x^6..)(1 + x^3 + x^6 + ...)$ By expanding this product we get $\sum_{n=0}^{\infty} P_n Z^n$ Relating this to partitions we have $1 + x^i + x^2 i + x^3 i...$ and i represents the number of times i will appear c_i times in the partition. Now we have x^n is just a way of writing $n = c_1 + 2c_2 + 3c_3 + ...$ This can be written as Euler's product

$E(x) = 1/(1-x) \cdot 1/(1-x^2) \cdot 1/(1-x^3) = 1/(1-x^n) = \sum_{n=0}^{\infty} P_n Z^n$

$Z^n = P^n$ so we have the product

$$f(z) = P(z) = \prod_{n=1}^{\infty} \frac{1}{1-Z^n} \tag{2}$$

2.2 Asymptotic Estimate

The Asymptotic estimate we want to prove is

$$P_n = \frac{e^(\pi\sqrt{2n/3})}{4n\sqrt{3}}(1 + O(n^-1/8) \tag{3}$$

as n goes to ∞

This is also an error term and not an optimal one. This is because the term $O(n^-\frac{1}{8})$ can be replaced by $O(n\frac{-1}{2})$. We will use the Circle Method to produce series representations for the partition function. It is a subexponential function and uses similar ways in the Euler's expansion in which we proved the generating function in order to relate it back to the function.

$f(z) = P(z) = \prod_{n=1}^{\infty} \frac{1}{1-Z^n}$

3 Integral

In reference to 2.1 in the handout it lays out how to compute an integral that is very technical. Here is the final product integral.

$$\int_0^{\infty} g(n) = \int_0^{\infty} \frac{1}{u(e^u - 1} - \frac{1}{u^2} + \frac{e^-u}{2u} du = -\frac{1}{2} log(2\pi) \tag{4}$$

What is important from this integral is the value $-\frac{1}{2}log(2\pi)$ and the fact that this integral converges.

4 Approximating f on the Major Arc

In this section which corresponds to 2.2 in the paper we will find the explicit function ϕ which approximates the generating function of the partition counts. To find the function ϕ which gives an approximation of f near 1. To do so we need to approximate the series $\sum_{n=0}^{\infty} g(nw)$ where the function g is a continuous at 0 and integrable on $[0, \infty]$ (How we get to this series in found in Lemma 3). Then the series is approximated by the integral of g over the ray $L_w = wt : t\epsilon[0, \infty)$. The ray is used in order to show that this is able to be applied onto the unit circle.

Lemma 7: Let $\lambda : [0, \infty) \to C$ continuous and integrable on $[0, \infty)$, then

$|\int_0^\infty \lambda(t)dt - \sum_{n=1}^\infty \lambda(n)| \leq 2V$ where $V = sup\{\sum_{y=0}^{k-1} |\lambda(t_j+1) - \lambda(t_j)| : 0 \leq t_0 < t_1 < ... < t_k\}$

Apply Lemma 7 to $\lambda : [0,\infty) \to C$: t $\to g(wt)$ which is continous and holomorphic on L_w and by Cauchy's theorem and the integral of g, this implies that

$$\|w\sum_{n=1}^{\infty} g(nw) + \frac{1}{2}log(2\pi)\| = \|w\sum_{n=1}^{\infty} g(nw) - \int_{L_w} g(t)dt\| \leq 2\|w\|V_w \quad (5)$$

Lemma 8 will approximate V_w Lemma 8: let $\lambda : [0,\infty) \to C$ differentiable and integrable on $[0,\infty)$ then the total variation of V of λ satisfies

$V \leq \int_0^\infty |\dot{\lambda(t)}|dt$

Apply Lemma 8 to $t \to g(wt)$ implies that

$$V_w \leq \int_0^\infty |wg'(wt)|\,dt = \int_{L_w} |g'(z)|\,|dz|.$$

This results that the V_m is bounded but depends on M(K) where there is constraints on K.

Through many different manipulations writing in terms of z we have the approximation of ϕ for f near 1. $f(z) = \phi(z)(1 + O(1-z))$

when $z \to 1 while |arg(w)| = |arg(-log(z))| < K$

5 Major Arc

In this section we want to relate the coefficients of $f(z)$ and $\phi(z)$ this is where we will apply the Circle Method.

$$P_n - Q_n = \frac{1}{2(\pi)i}\int_C \frac{f(z) - \phi(z)}{z^n + 1}dz \quad (6)$$

We are integrating around a Circle that is parametrized and has a radius that is less than one.

Radius of Circle C: $1 - v(n)$ this is the radius because we want it to be near the point 1 on the complex plane, so the major arc consists of points on Z on this circle of radius $1 - v(n)$.

We define the Major Arc as $M = \{zEC : |1-z| < s(n)\}$

We define the Minor Arc as $m = M/C$

Now we want to apply the estimate that we calculated before $f(z) = \phi(z)(1+O(1-z))$ onto the major arc so we need to fix K in order to be able to use this approximation.

Lemma 9: Let $w \in C$ with $|arg(w)| \leq \pi$ such that e^-wEM, then for a sufficiently large n $|arg(w)| \leq arccos\frac{v(n)}{2s(n)}$

Now we make choices of these functions $v(n)$ and $s(n)$

$v(n) = C_v n^- t$ $s(n) = C_s n^- t$

These two functions tend towards 0 as $n \to \infty$. Both of these functions equal zero at the rate of $\frac{1}{n^t}$ this is because of Lemma 9 where $\frac{v(n)}{2s(n)}$ has to be a simple function.

Now take the estimate of $f(z)$ and manipulate it. $f(z) = \phi(z) + \phi(z)O(1-z)$ and subtract $\phi(z)$ $f(z) = \phi(z) + \phi(z)O(1-z) - \phi(z)$ $f(z) = \phi(z)O(1-z)$

$\int_M \frac{f(z)-\phi(z)}{z^{n+1}} = O(\int_M (|z^{-(n+1)(1-z)\phi(z)|}|dz|)$

Now we have found the integral and we need to bound it using Lemma 10.

Lemma 10: Let t $\in (0, 1)$ and a $\in$ R then $(1 - an^{-t}) \leq e^{an^{1-t}}$ as $n \to \infty$ Proof. The statement follows exponention of $-nlog(1 - an^{-t}) \leq an^{1-t}$ Since $z = 1 - v(n) = 1 - C_v n^{-t}$ We use Lemma 10 to say $|z|^{-n} = O(e^{c_v n^{1-t}})$ which is the bound of the integral over the major arc of

$$\int_M \frac{f(z) - \phi(z)}{z^{n+1}} = O(\int_M (|z^{-(n+1)(1-z)\phi(z)|}|dz|) \tag{7}$$

using this integral, $|z| = 1 - v(n) = 1 - c_v n^- t$ and the fact that Lemma 10 implies $|z| =^- n = O(e^{c_v n^{1-t}}$ we get the following integral. $\int_M \frac{f(z)-\phi(z)}{z^{n+1}} = O(\int_M |1 - z|^(3/2) exp(c_v n^(1 - t) + \frac{\pi^2}{6(1-z)}|dz|$

We can then use $|1-z| < s(n) = c_n^(- t)$ and $|1-z|^- 1 \leq (1-|z|)^- 1 = v(n)^- 1$ and the length of the Major Arc is $O(n^- t)$, and so we get

$\int_M \frac{f(z)-\phi(z)}{z^{n+1}} = O(n^{5t/2} exp(c_v n^(1 - t) + \frac{\pi^2 n^t}{6c_v})$

When looking at this bound the term inside the exp that is the largest is the one we are concerned with. Each term is a constant time $n^(1 - t)$ and n^t so we want to choose whichever t makes $max(1 - t, t)$ the smallest. This is going to be the one that them equalvalent. So we have $1 - t = t$ so $t = 1/2$ so both terms will be a constant times $n^1/2$ and $t = 1/2$ is the constant we plug into the bound. So now our bound looks like.

$O(n^{5/4} exp(c_v n^(1 - t) + \frac{\pi^2 n^t}{6c_v})$

Now we are free to choose c_v so we choose whichever value makes the coeffiecient $c_v + \frac{\pi^2}{6c_v}$ the smallest this will be the minimum of the function $f(x) = x + \frac{\pi^2}{6x}$ plugging in C_v for x. Then we can verify that $c_v = x = \frac{\pi}{\sqrt{6}}$ so this is the value that we use for this bound. So the total bound is.

$\int_M \frac{f(z)-\phi(z)}{z^{n+1}} = O(n^(- 5/4) e^{\pi\sqrt{2n/3}}$

6 Minor Arc

In this section we want to find the inequality of the minor arc and verify it against the Major Arc.

Using the inequality.

$$|f(z)| < exp(\frac{1}{|1-z|} + \frac{1}{1-|z|}) \tag{8}$$

$|\int_M \frac{f(z)-\phi(z)}{z^{n+1}} dz| < \int_m |z|^{-(n+1)} exp(\frac{1}{|1-z|} + \frac{1}{1-|z|}) + |\phi(z)|dz$

The definition of m is $|1-z| \geq s(n) = \frac{c_s}{\sqrt{n}}$
so $1 - |z| = \frac{\pi}{\sqrt{6n}}$
So we end up with
$< \int_m |z|^{-(n+1)} exp(\frac{\sqrt{n}}{c_s} + \frac{\sqrt{6n}}{\pi}) + |\phi(z)|dz$
So we use the definition of $|z|^(-n) = O(e^{c_v n^(1-t)}$ and the fact that $|1-z|^(-1) \leq (1-|z|)^-1 = \frac{\sqrt{6n}}{\pi}$
Giving us:
$\int_M \frac{f(z)-\phi(z)}{z^{n+1}} dz = O(exp(\pi\sqrt{n/6} + \frac{\sqrt{n}}{c_s} + \frac{\sqrt{6n}}{n}) + exp(\pi\sqrt{n/6} + \sqrt{6n}/\pi))$
To make sure the major arc is dominant over the minor arc is dominant over the minor arc it remains to check.
$\frac{\pi}{\sqrt{6}} + \frac{\sqrt{6}}{\pi} < \pi\sqrt{2/3}$ and choose c_s. Above we have choosen c_v and so now we choose c_s. The bound for the minor arc depends on c_s. The condition of $arccos\frac{v(n)}{2s(n)} = arccos\frac{c_v}{2c_s}$ cannot be too big.

So we choose the value of $c_s = 2c_v = \frac{\pi\sqrt{6}}{}$. Now we just need to find some value that works to make the minor arc smaller than the major arc. Now we get $K = arccos(1/4) < \pi/2$
ending with

$$P_n = q_n + O(n^{-5/4} \dot{e}^{\pi\sqrt{2n/3}} \tag{9}$$

7 Approximating

We have related P_n to Q_n and now we need to approximate q_n.

7.1

First we need to write $\phi(z)$ as an integral. $\phi(z) = \frac{e^{-\pi^2/12}}{\pi\sqrt{2}}(1-z)\int_{-\infty}^{\infty} e^{-(1-z)x^2+\pi\sqrt{2/3x}}dx$

This integral only depends on z by this term and we can expand it using Taylor expansion. By rearranging the sum and the integral we get
$\int_{-\infty}^{\infty} e^{-(1-z)x^2+\pi\sqrt{2/3x}}dx = \int_{-\infty}^{\infty} e^{-x^2+\pi\sqrt{2/3x}e^{zx^2}} dx$
$\int_{-\infty}^{\infty} e^{-(1-z)x^2+\pi\sqrt{2/3x}}dx = \int_{-\infty}^{\infty} e^{-x^2+\pi\sqrt{2/3x}} \sum_{n=0}^{\infty} \frac{X^2 n}{n!} z^n dx$
Because q_n is z^n in the power series expansion of ϕ the integral
$q_n = \frac{e^{-\pi^2/12}}{\pi\sqrt{2}} \int_{-\infty}^{\infty} e^{-(1-z)x^2+\pi\sqrt{2/3x}}(\frac{x^{2n}}{n!} - \frac{x^{2n-3}}{(n-1)!})dx$

7.2

Because the expansion came from the Taylor Series and invovles the $\frac{1}{n!}$ term, there are a bunch of complicated manipulations in order to get the integral in the form that we can use Stirling's approximation.

Stirling's aproximation

$$\Gamma(z-1) = \sqrt{2\pi(z)}(\frac{z}{e})^z \dot{e}^{O(-z)} = \sqrt{2\pi(z)}(\frac{z}{e})^z \dot{(}1 + O(z^{-1})) \tag{10}$$

to $\frac{n^n e^{-n}}{n!}$ to see that

$q_n = \frac{e^{\pi\sqrt{2n/3}-\pi^2/12}}{\pi^3/2(2n)}(1+O(n^{-1}))\int_{-\infty}^{\infty}(x)(1+\frac{x}{\sqrt{n}})^{2n-2}(2+\frac{x}{\sqrt{n}})e^{\pi\sqrt{2/3}x-x^2-2x\sqrt{n}}dx$

This is just an analytic function that invloves function of x and n. In basic terms

$q_n =$ (explicit function of n)(integral of some function $s_(n)(x)$ (function that depends on x) and a small error.

7.3

use Lemma 12:

$\int_{-\infty}^{\infty} s_n(x)e^{\pi\sqrt{2/3}x-x^2}dx = (1+O(n^{-1/8}))\int_{-\infty}^{\infty} xe^{\pi\sqrt{2/3}x-x^2}dx$

The integral tends toward something independent of n up to some reasonale error.

We can conclude:

$q_n = (1+O(n^{-1}))\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\sqrt{2/\pi}\int_{-\infty}^{\infty} xe^{\pi\sqrt{2/3}x-2x^2}dx$

In simple terms this is equal to

q_n= (explicit function of n)(integral that does not depend on n)

Now we evaluate this integral explicitly since it doesn't depend on n, this is a constant and we end up with:

$q_n = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}(1+O(n^{-1/8}))$

This in simple terms is an explicit function of n and a small error term.

Finally we get: $p_n = q_n + O(n^{-5/4}e^{\pi\sqrt{2n/3}}$ $p_n = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}(1+O(n^{-1/8}))$

Which is the value for our function p_n