Additive number theory: syllabus

Spring 2024

Time and location: Wednesdays 11:40 AM - 1:30 PM, in 528 Mathematics
Overall course instructor: Prof. Alisa Knizel
Section instructor: Avi Zeff, email cbz2106@columbia.edu
Office hours: TBD

The topic of this section is an introduction to additive number theory. Broadly speaking, as the name suggests this subject studies the additive structures of various kinds of numbers. A typical example is Lagrange's theorem: every positive integer can be written as the sum of four squares, e.g. $14 = 3^2 + 2^2 + 1^2 + 0^2$. More complicated problems may intertwine additive structures with multiplicative structures: for example, Goldbach's conjecture states that every even integer greater than 2 can be written as the sum of two primes, for example $20 = 13 + 7$.

Rather than focus on any one result, our goal is to study some of the major tools used in this area, and give example problems and results as applications of these tools.

This course has two major goals: to introduce students to the broader mathematical world in terms of topics not typically encountered in the undergraduate curriculum, and to develop students' mathematical communication skills. In service of the latter goal, and unlike in typical courses, students will present most of the material.

Summary

In the first part of the course, we will develop some analytic tools, building towards the Hardy–Littlewood circle method. The main source of applications will be Waring's problem and related results: these generalize Lagrange's theorem above about representing integers as sums of squares to more general questions about sums of powers.

This turns out to be a surprisingly deep field. One could ask about the analogue of Lagrange's theorem for cubes, and it turns out that the answer is that every positive integer can be written as the sum of nine (nonnegative) cubes. But in fact, one can very nearly do better: the only numbers that actually require nine cubes are $23 = 2^3 + 2^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$ and $239 = 5^3 + 3^3 + 3^3 + 3^3 + 2^3 + 2^3 + 2^3 + 2^3 + 1^3$! So we can write all *sufficiently large* integers as a sum of eight cubes. Can we do better? That is: what is the smallest integer n such that all *sufficiently large* integers can be written as the sum of n cubes?

It can be shown (and we may show in this course!) that $4 \leq n \leq 7$, i.e. we need at least 4 and at most 7 cubes for every integer. But the exact value of n is still unknown!

In the second part of the course, we'll start to incorporate some multiplicative structures, especially prime numbers, and see how they intertwine with our additive structures. The main type of tool we'll focus on here is sieve methods, the idea of which is to identify restrictions coming from divisibility by each prime, and then assemble this information together. For example, if we're looking for prime numbers within a given set, knowing that no primes (other than 2) are odd lets us rule out a large swath already; we can rule out more by observing that no primes (other than 3) are divisible by 3. But there's some overlap between these sets, namely numbers that are divisible by 6, so we have to be careful to count correctly.

This idea, broadly interpreted, turns out to be extremely useful. The simplest sieves by themselves don't give too much information; but by incorporating various tricks, including some analytic methods (e.g. the circle method!), one can carefully "sieve" sets of interest to get some powerful results. Some example applications include some problems approaching Goldbach's conjecture, such as Vinogradov's theorem (any sufficiently large *odd* integer can be written as the sum of *three* primes, e.g. $15 = 7 + 5 + 3$) and Chen's theorem (every sufficiently large even number can be written as the sum of a prime and a number with at most two prime factors) as well as a wide variety of other results on prime numbers of specific type, e.g. Brun's theorem (the set of twin primes is "small" in a precise sense, though it may still be infinite).

Prerequisites

There are no formal prerequisites for this section beyond a solid background in calculus (in particular the material of calculus 1-2 will be used freely). Some experience or at least interest in number theory may be useful, but we will not expect any formal background. However the material is not easy: in my opinion the crucial thing is an interest in the material and a willingness to struggle with it. If you are unsure if you have the background for the class, please feel free to discuss with me via email or in person.

For written assignments, the use of LaTeX is preferred but not required; we may also include some introduction to/resources for LaTeX in the class.

Textbook

We will use a variety of sources in the class, mostly to be determined by the interests of the class and the particular topics to be studied as they arise. One text which broadly shapes the curriculum is Melvyn Nathanson's *Additive number theory: the classical bases*, an electronic version of which can be found here.

Format

We meet once weekly for about two hours. Classes will mostly consist of student presentations, together with occasional lectures and discussion; depending on our pace and the number of students, each student will most likely present 2-3 times, though we may choose to instead have a larger number of shorter talks. In addition to preparing your talks, there will be some homework, focused on understanding, synthesizing, and effectively communicating the material, and a written final project on a topic of your choice. The final project may be done either individually or in small groups (depending on the size of the section), and we may choose to have some group presentations as well, although group presentations and projects will be expected to have commensurately larger scope. Finally, you will be expected to give your peers feedback on their presentations and other work.

Presentations

Each presentation should (barring alteration) be around 45-50 minutes, so a typical class session will consist of two presentations plus time for setup, questions, comments, and a brief break. You may choose to give either "slide talks" using a pre-made slide deck or "chalk talks" just on the blackboard; you may also choose to use handouts, if desired. We will discuss pros and cons of each option.

Before your presentation, you should meet with me to discuss what material you'll cover, sources, etc. (this can also be done by email if necessary), and give a practice presentation to me before your presentation to the class to receive feedback which you should then incorporate into your presentation.

Final projects

Your final project will be a paper on some topic related to the subject of the seminar, applying some of the tools we've learned. Its focus should be on investigating an idea beyond what we've strictly covered in class, and clearly communicating the results of your investigation to your peers. More precise details and expectations for the projects will appear later in the semester.

Grading

Grading will be based on the following:

- presentations;
- homework;
- final projects;
- peer feedback.

For the first three items, I am looking for correctness and quality of exposition; for peer feedback (that you will give your classmates on presentations, some homework, and final projects), I am looking for completion. You will receive feedback from both myself and your peers on each of the first three items, and do some self-assessment. You'll also have opportunities to revise errors on homework in response to feedback.

In addition to feedback, each assignment (presentations, homeworks, and the final project) will receive an overall mark of either E (excellent), S (satisfactory), or N (not yet satisfactory). Your grade will be the highest row in the following table for which you've achieved *all* of the requirements:

Grade	Presentations	Homeworks	Final project
A	All at least S, at least one E	All at least S	E
B	All at least S	All at least S	At least S
C	At least one S	At most two N's	At least S
D	At least one S	At least one S	No requirement

Plus or minus modifiers will be used for cases where you are almost at the next grade level or significantly above the requirements for one grade but not yet reaching the next one: for example, if you have all the requirements for an A except for an E on a presentation, you might get an A-; if you have all the requirements for a C but with only one N on homework as well as an E, you might get a C+. Work not meeting the requirements for a D will receive an F.

If you miss giving peer feedback without prior excuse, I will notify you and give you the chance to make it up; if it is not completed within a few days it will lower your grade by a modifier (A to A-, A- to B+, etc.) for every two missing feedbacks.

Course policies

Attendance

Attendance is a crucial part of the participation in the seminar. Students will spend more time as an audience member than giving their own talks. Students are expected to attend all meetings and, in the event of a virtual meeting, they are expected to have their cameras on. Absences must be excused in advance (barring exceptional circumstances). If you are unable to come to class in person but are available virtually, we may be able to arrange a Zoom option. If you regularly need to arrive late or leave early, you should arrange a way to still give full feedback with me.

Deadlines and extensions

Due to the peer evaluation process, it is very important to turn in homework on time. If you are absolutely unable to do so, please notify me in advance if at all possible, preferably with at least 24 hours' notice, and we may be able to work out an extension; you should expect unexcused or repeatedly late homework to count against you.

If you are scheduled to give a presentation and cannot (e.g. due to illness or another emergency), please give me as much notice as possible—preferably at least 48 hours, ideally more—and we will see if we can rearrange things at all.

Joining late

If you are joining the class after the first meeting, you will be expected to make up any homeworks you missed, but not peer feedback. You will need to give the same number of presentations as everyone else, so they may be condensed into a shorter amount of time.

COVID-19 policies

Classes will be in person, but please do not come to class if you are feeling sick or test positive—illness is always a valid reason to miss class. It may be possible to stream classes on Zoom as needed.

Academic Honesty Policy

Please read (and follow) the Columbia University Undergraduate Guide to Academic Integrity. If you are ever unsure whether something is allowed, please ask me first—you will never be penalized for asking.

Accessibility and accommodations

Please let me know if there is anything I can do to make this course more accessible to you, or if aspects of the course are excluding you, and we can work together to develop strategies to improve the class. If you think you may need official accommodations, I encourage you to contact the Office of Disability Services for an accommodation letter.